Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 23, 2004, 383–386

ADDENDA AND CORRIGENDA TO "A DIRECT TOPOLOGICAL DEFINITION OF THE FULLER INDEX FOR LOCAL SEMIFLOWS"

(TOPOL. METHODS NONLINEAR ANAL. 21 (2003), 195–209)

Christian C. Fenske

Abstract. We remedy an error and simplify an argument in the article mentioned in the title.

We retain the notation of [1]. Firstly, I want to correct a well-hidden mistake in that article. Secondly, I want to show that the manifold Z_k is orientable if M is orientable.

In the section "Construction of nonbounding cycles" I had started from an n-dimensional orientable manifold, and, with k a large odd number, I defined a set $Z^{(k)}$ and I claimed that $Z^{(k)}$ was a manifold. But if we take open sets U and V in Ω' such that $\{(g_k(x,t),t) \mid (x,t) \in U\} \cap \{(\zeta g_k(x,t),t) \mid (x,t) \in V\} \neq \emptyset$, then the intersection can consist only of periodic points. But the set of periodic points need not have interior points in $M \times [0,\infty)$. So in order to remedy this defect we have to choose a "fatter" set.

We start by choosing a metric d on M. Then we choose an $\varepsilon > 0$ such that, for $(x,t) \in P$ and $0 \leq s \leq 2t$, the sets $\overline{B}(\phi_s\phi_{it/k}x;\varepsilon)$ are disjoint for $i = 0,\ldots,k-1$. Since P is compact there is a $\rho > 0$ such that $d(\phi_s x, \phi_s y) < \varepsilon$ whenever $(x,t) \in P$, $0 \leq s \leq 2t$, and $d(x,y) < \rho$. For $\beta \in (0,\varepsilon)$ and $(x,t) \in P$

2000 *Mathematics Subject Classification.* Primary: 37C25; Secondary: 54H20, 55M25.
Key words and phrases. Fuller index, periodic orbit, local semiflow.

we define the *k-pseudo-orbit* $\mathcal{O}((x,t);\beta)$ to be the set of all $(x_0,\dots,x_{k-1})$ such that $x_i \in B(\phi_{it/k}x;\beta)$ for $i=0,\dots,k-1$. We then observe that

- $\mathcal{O}((x,t);\beta) \subset M^{(k)}$,
- $\zeta \cdot \mathcal{O}((x,t);\beta) = \mathcal{O}((\phi_{t/k}x,t);\beta)$ and
- $(\phi_s x_0,\dots,\phi_s x_{k-1}) \in \mathcal{O}((\phi_s x,t);\varepsilon)$ if $(x_0,\dots,x_{k-1}) \in \mathcal{O}((x,t);\rho)$ and $0 \le s \le 2t$.

We now define

$$\mathcal{O}^{(k)} := \left\{ (x_0,\dots,x_{k-1},s) \in \mathcal{O}((x,t);\varepsilon) \times \left(\frac{t}{k}-\delta, \frac{t}{k}+\delta\right) \;\middle|\; (x,t) \in P \right\}$$

where δ is taken from the Lemma. We then let $Z^{(k)} := \{(x_0,\dots,x_{k-1}) \in \mathcal{O}((x,t);\varepsilon) \mid (x,t) \in P\}$. Obviously, $Z^{(k)}$ is an open subset of $M^{(k)}$, hence a manifold (of dimension kn). Since M was assumed to be orientable, so will be $Z^{(k)}$. On the base space we let $\mathcal{O}_k := \{(q_k\xi, s) \mid (\xi,s) \in \mathcal{O}^{(k)}\}$ and $Z_k := q_k(Z^{(k)})$. Obviously $q_k\colon Z^{(k)} \to Z_k$ is a covering map, so Z_k is a manifold. There are obvious local semiflows $\Psi^{(k)}$ on $Z^{(k)}$ defined by $\psi_s^{(k)}(x_0,\dots,x_{k-1}) = (\phi_s x_0,\dots,\phi_s x_{k-1})$ where $(x_0,\dots,x_{k-1},s) \in \mathcal{O}^{(k)}$ and Ψ^k on Z_k defined by $\psi_s^k(q_k(x_0,\dots,x_{k-1})) = (q_k(\phi_s x_0,\dots,\phi_s x_{k-1}))$ if $(q_k(x_0,\dots,x_{k-1}),s) \in \mathcal{O}_k$. We will now show that Z_k is always orientable if M is. This will somewhat simplify the presentation in [1].

Claim. *Z_k is orientable.*

Proof. We start by choosing a covering $\mathcal{W}$ of Z_k by sets which are evenly covered by q_k. Since $Z^{(k)}$ is orientable we may choose an orientation $\tau \in H^{kn}(Z^{(k)} \times Z^{(k)}, Z^{(k)} \times Z^{(k)} \setminus \Delta)$ (we denote all diagonals indiscriminately by Δ). Let then $W \in \mathcal{W}$ with $q_k^{-1}(W) = \bigcup_{j=0}^{k-1} \zeta_1^j(V)$. Let $V_j = \zeta_1^j(V)$, denote by $i_j\colon V_j \to q_k^{-1}(W)$ the inclusion and call $\tau_j := i_j^*\tau$. We claim that $\zeta_1^*\tau_{j+1} = \tau_j$ for $j=0,\dots,k-1$ where we let $\tau_k := \tau_0$. In fact, since ζ_1 (being a covering transformation) is a homeomorphism mapping V_j onto V_{j+1} there is an $\alpha \in \{-1,1\}$ such that $\zeta_1^*\tau_{j+1} = \alpha\tau_j$. But $\zeta_1^k = \mathrm{id}$, so $\tau_0 = {\zeta_1^*}^k\tau_0 = \alpha^k\tau_0$ which implies $\alpha = 1$ since k was odd.

We now choose $\tau_W \in H^{kn}(W \times W, W \times W \setminus \Delta) \cong H^{kn}(Z_k \times W, Z_k \times W \setminus \Delta)$ such that $i_0^* q_k^* \tau_W = \tau_0$, and we claim that $(\tau_W)_{W \in \mathcal{W}}$ is a compatible family. This will then establish the orientability of Z_k (cf. [2, p. 294]). For $j=0,\dots,k-1$ we have that $\zeta_1^j i_0 \zeta_1^{-j} = i_j$. The definition of τ_W then implies that ${\zeta_1^{-j}}^* i_0^* {\zeta_1^j}^* q_k^* \tau_W = {\zeta_1^{-j}}^* i_0^* q_k^* \tau_W = {\zeta_1^{-j}}^* \tau_0 = \tau_j$ which means that $i_j^* q_k^* \tau_W = \tau_j$. So the definition of τ_W does not depend on the choice of the covering set V_j. Let then $W, W' \in \mathcal{W}$ and assume that $W \cap W' \neq \emptyset$. We have to show that τ_W and $\tau_{W'}$ restrict to the same class on $W'' = W \cap W'$. So denote the inclusions by $\iota\colon W'' \hookrightarrow W$ and $\iota'\colon W'' \hookrightarrow W'$. Suppose, we choose V and V' with $q_k(V) = W$, $q_k(V') = W'$. Then there is a $j \in \{0,\dots,k-1\}$ such that $(q_k \mid V')^{-1}(W'') = \zeta_1^j((q_k|V)^{-1}(W''))$. Again we have inclusions $I_0\colon U_0 := (q_k|V)^{-1}(W'') \hookrightarrow V$ and

$I_j\colon U_j := (q_k|V')^{-1}(W'') \hookrightarrow V'$. We then have that $I_0^* i_0^* q_k^* \iota^* \tau_W = I_0^* \tau_0$ and $I_j^* i_j^* q_k^* \iota'^* \tau_{W'} = I_j^* \tau_j$. Now we observe that $I_0^* \tau_0 = I_0^* {\zeta_1^j}^* \tau_j = {\zeta_1^j}^* I_j^* \tau_j$ and that $i_j I_j = \zeta_1^j i_0 I_0 \zeta_1^{-j}$ on U_j. But then

$$\begin{aligned} I_j^* i_j^* q_k^* \iota'^* \tau_{W'} &= I_j^* \tau_j, \\ (\zeta^{-j})^* I_0^* i_0^* {\zeta_1^j}^* q_k^* \iota'^* \tau_{W'} &= I_j^* \tau_j, \end{aligned}$$

and so

$$I_0^* i_0^* q_k^* \iota'^* \tau_{W'} = I_0^* {\zeta_1^j}^* \tau_j = I_0^* \tau_0 = I_0^* i_0^* q_k^* \iota^* \tau_W$$

which proves our claim since $q_k i_0 I_0$ is a homeomorphism. □

Thus, what we called the "orientable case" in [1] is just the case where M is orientable.

In the following text in the first instance we have to correct the dimensions (since the dimension of Z_k is now kn rather than $n+1$). In the proof of the normalization property we need two modifications: when we prove that ι'_k equals the fixed point index of the Poincaré mapping corresponding to γ we should first choose an $\varepsilon' > 0$ such that the ε'-neighbourhood of $|\gamma|$ is contained in V. Then we choose $\varepsilon > 0$ so small that

- $V \times (t_0 - \varepsilon, t_0 + \varepsilon) \subset \Omega''$,
- The sets $\overline{B}(\phi_s \phi_{it_0/k} x; \varepsilon)$ are disjoint for $i = 0, \dots, k-1$ whenever $x \in |\gamma|$ and $|s - t_0/k| < \delta$.

We then choose a $\rho' > 0$ such that $d(\phi_s x, \phi_s y) < \varepsilon$ whenever $x \in |\gamma|$, $|s - t_0/k| < \delta$, and $d(x,y) < 2\rho'$. Finally, we choose a $\rho > 0$ such that for any space Y any two maps $f, g\colon Y \to X$ which are 2ρ-near are ρ'-homotopic. The set W in [1] is then replaced with the set of all $(q_k(y_0, \dots, y_{k-1}), s)$ where $(y_0, \dots, y_{k-1}) \in \mathcal{O}((x, t_0); \rho)$ with $x \in |\gamma|$ and $|s - t_0/k| < \delta$. We then let $V_k := \{q_k(y_0, \dots, y_{k-1}) \mid (y_0, \dots, y_{k-1}) \in \mathcal{O}((x, t_0)); \rho)$ for some $x \in |\gamma|\}$. The homotopy Θ freezing the parameter t at t_0 is not needed anymore and Ψ'' is just the Ψ^k defined above. We then choose B, N, N_1, $\mathcal{O}$, and Σ as in [1] where we choose $\mathcal{O}$ so small that $\mathcal{O} \subset B(x_0; \rho)$ and we call $\ell := (q_k|N_1)^{-1}|V\colon N \cap V_k \to N_1$. If the multiplicity $m = 1$ we let $\sigma(y) := \eta(y)$ if $0 < \eta(y) < t_0/8$, $\sigma(y) := 0$ if $y \in \Sigma$, and $\sigma(y) := \max\{(t_0 - \eta(y))/2, 0\}$ if $\eta(y) > 7t_0/8$. Then $y \mapsto \phi_{t_0} \circ \phi_{\sigma(y)} y$ may serve as a Poincaré-mapping, so we let $\pi'(y) = \phi_{\sigma(y)} y$. If $m > 1$ we let again π' denote the Poincaré mapping for the period $(m-1)p(x_0)$ and we see that $\iota = \operatorname{ind}(X, \operatorname{pr}_1 \ell q_k i_k \phi_{t_0/k} \pi', \mathcal{O})$.

Another modification is needed at the end of the proof. The commutativity property of the fixed point index gives

$$\operatorname{ind}(X, \operatorname{pr}_1 \ell q_k i_k \phi_{t_0/k} \pi', \mathcal{O}) = \operatorname{ind}(V_k, q_k i_k \phi_{t_0/k} \pi' \operatorname{pr}_1 \ell, q_k \operatorname{pr}_1^{-1}(\mathcal{O})),$$

and it is to be shown that the right hand side equals the fixed point index ι'_k. We then denote by Σ' the set of $(y_0, \dots, y_{k-1})$ (where $(y_0, \dots, y_{k-1} \in \mathcal{O}((x, t_0)); \varepsilon')$ for some $x \in |\gamma|$) such that $y_i \in \Sigma$ for some $i \in \{0, \dots, k-1\}$ (if such an i exists it is necessarily unique). Then we let $\Sigma_k := q_k(\Sigma')$. Let then $q_k(y_0, \dots, y_{k-1}) \in \Sigma_k$, choose i such that $y_i \in \Sigma$ and let $j = i - 1$ if $i > 0$ and $j = k - 1$ else. We then let $\tau'(q_k(y_0, \dots, y_{k-1})) := \eta(y_j)$. Then τ' will be continuous and Σ_k will be a section for Ψ^k at $q_k i_k x_0$. The corresponding Poincaré-mapping is easily computed: Let $(y_0, \dots, y_{k-1}) \in \mathrm{pr}_1^{-1}(\mathcal{O})$ and $\xi := q_k(y_0, \dots, y_{k-1})$. Then $\tau'_1(\xi) := \eta(y_{k-1})$, $\tau'_{j+1}(\xi) := \tau'_j(\xi) + \eta(\phi_{\tau_j(\xi)} y_{k-j-1})$ and the Poincaré-mapping for the multiplicity m is $\pi'' := \tau'_m$, so $\pi''(\xi) = q_k(\phi_{\tau'_m(\xi)} y_{k-m}, \dots, \phi_{\tau'_m(\xi)} y_{k-1}, \phi_{\tau'_m(\xi)} y_0, \dots, \phi_{\tau'_m(\xi)} y_{k-m-1})$. On the other hand we have that $q_k i_k \phi_{t_0/k} \mathrm{pr}_1 \ell(\xi) = q_k(\phi_{t_0} \pi' y_0, \phi_{t_0/k} \pi' y_0, \dots, \phi_{(k-1)t_0/k} \pi' y_0)$ and the arguments of both expressions are 2ρ-near, so we find ρ'-homotopies $h_0, \dots, h_{k-1}$ such that $h_0(\xi, 0) = \phi_{t_0} \pi' y_0$, $h_j(\xi, 0) = \phi_{j t_0/k} \pi' y_0$ for $j = 1, \dots, k-1$, $h_j(\xi, 1) = \phi_{\tau_m(\xi)} y_{k-m+j}$ if $j = 0, \dots, m-1$, and $h_j(\xi, 1) = \phi_{\tau_m(\xi)} y_{j-m}$ if $j = m, \dots, k-1$. So we let $h(\xi, \lambda) = q_k(h_0(\xi, \lambda), \dots, h_{k-1}(\xi, \lambda))$. Obviously, we cannot have $h(\xi, \lambda) = \xi$ for $\xi \in \partial V_k$ since this would give rise to a periodic point of Ψ^k on ∂V_k contradicting the fact that γ is isolated. So we finally have that $\iota'_k = \mathrm{ind}(V_k, q_k i_k \phi_{t_0/k} \pi' \mathrm{pr}_1 \ell, q_k \mathrm{pr}_1^{-1}(\mathcal{O}))$ which finishes the proof of the normalization property.

The case of a non-orientable manifold M is then handled by just embedding M as a neighbourhood retract in some $\mathbb{R}^n$. Then one argues as in the case of a simplicial complex.

References

[1] C. C. Fenske, *A direct topological definition of the Fuller index for local semiflows*, Topol. Methods Nonlinear Anal. **21** (2003), 195–209.

[2] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York et al., 1966.

Manuscript received August 5, 2003

Christian C. Fenske
Mathematisches Institut
der Justus-Liebig-Universität Gießen
Arndtstraße 2
D-35392 Gießen, GERMANY

E-mail address: christian.fenske@math.uni-giessen.de