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ATTRACTORS FOR SINGULARLY PERTURBED DAMPED WAVE EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. For an arbitrary unbounded domain $\Omega \subset \mathbb{R}^3$ and for $\varepsilon > 0$, we consider the damped hyperbolic equations

$$(\mathrm{H}_\varepsilon) \qquad \varepsilon u_{tt} + u_t + \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i} = f(x,u),$$

with Dirichlet boundary condition on $\partial\Omega$, and their singular limit as $\varepsilon \to 0$. Under suitable assumptions, (H_ε) possesses a compact global attractor $\mathcal{A}_\varepsilon$ in $H^1_0(\Omega)\times L^2(\Omega)$, while the limiting parabolic equation possesses a compact global attractor $\widetilde{\mathcal{A}_0}$ in $H^1_0(\Omega)$, which can be embedded into a compact set $\mathcal{A}_0 \subset H^1_0(\Omega) \times L^2(\Omega)$. We show that, as $\varepsilon \to 0$, the family $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,\infty[}$ is upper semicontinuous with respect to the topology of $H^1_0(\Omega) \times H^{-1}(\Omega)$.

1. Introduction

In their paper [13] Hale and Raugel considered the damped hyperbolic equations

$$\begin{aligned} \varepsilon u_{tt} + u_t - \Delta u &= f(u) + g(x), && x \in \Omega, \quad t \in [0,\infty[\,, \\ u(x,t) &= 0, && x \in \partial\Omega, \ t \in [0,\infty[\,. \end{aligned}$$

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and their singular limit as $\varepsilon \to 0$, i.e. the parabolic equation

$$\begin{aligned} u_t - \Delta u &= f(u) + g(x), \quad x \in \Omega, \quad t \in [0, \infty[\,, \\ u(x,t) &= 0, \qquad\qquad\qquad x \in \partial\Omega,\ t \in [0, \infty[\,. \end{aligned}$$

In [13] the set Ω is a bounded smooth domain or a convex polyhedron, ε is a positive constant, $g \in L^2(\Omega)$ and f is a C^2 function of subcritical growth such that

$$\limsup_{|u| \to \infty} \frac{f(u)}{u} \le 0.$$

Under these assumptions, for any fixed $\varepsilon > 0$ the corresponding hyperbolic equation generates a global semiflow which possesses a compact global attractor $\mathcal{A}_\varepsilon$ in the phase space $H^1_0(\Omega) \times L^2(\Omega)$ (see [2], [8], [12]). Moreover, the limiting parabolic equation generates a global semiflow which possesses a compact global attractor $\widetilde{\mathcal{A}_0}$ in the phase space $H^1_0(\Omega)$ (see [5], [12]). Due to the smoothing effect of parabolic equations, it turns out that $\widetilde{\mathcal{A}_0}$ is actually a compact subset of $H^2(\Omega)$. Hence one can define the set

$$\mathcal{A}_0 = \{(u, \Delta u + f(u) + g) \mid u \in \mathcal{A}_0\},$$

which is a compact subset of $H^1_0(\Omega) \times L^2(\Omega)$. Hale and Raugel proved that the family $(\mathcal{A}_\varepsilon)_{\varepsilon \in [0,\infty[}$ is upper semicontinuous with respect to the topology of $H^1_0(\Omega) \times L^2(\Omega)$, i.e.

$$\lim_{\varepsilon \to 0^+} \sup_{y \in \mathcal{A}_\varepsilon} \inf_{z \in \mathcal{A}_0} |y - z|_{H^1_0 \times L^2} = 0.$$

In this paper we extend the result of Hale and Raugel in three directions: firstly, we allow f to have critical growth; secondly, we let Ω be unbounded; thirdly, we replace $f(u) + g(x)$ by $f(x,u)$ and $-\Delta$ by $\beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i}$, without any smoothness assumption on $\partial\Omega$, $\beta(\,\cdot\,)$, $a_{ij}(\,\cdot\,)$ and $f(\,\cdot\,,u)$.

In [13] the proof of the main result relies on some uniform $(H^2 \times H^1)$-estimates for the attractors $\mathcal{A}_\varepsilon$, combined with the compactness of the Sobolev embedding $H^1_0(\Omega) \subset L^2(\Omega)$. The uniform $(H^2 \times H^1)$-estimates are obtained through a bootstrapping argument originally due to Haraux [14]. Such argument works only if f is subcritical, and if Ω is such that the domain of the $L^2(\Omega)$-realization of $-\Delta$ is $H^2(\Omega) \cap H^1_0(\Omega)$ (e.g. if Ω is a convex polyhedron).

A different bootstrapping argument was proposed by Grasselli and Pata in [10] and [11]. Their argument also works in the critical case, and is based on certain a-priori estimates that can be obtained "within an appropriate Galerkin approximation scheme". Here, "appropriate" means "on a basis of eigenfunctions of $-\Delta$". Therefore, their approach cannot be used in the case of an unbounded domain Ω. More recently, in [15] Pata and Zelik obtained $(H^2 \times H^1)$-estimates for $\mathcal{A}_\varepsilon$ without using bootstrapping arguments, but again their a-priori estimates are obtained "within an appropriate Galerkin approximation scheme". We point out that also in [10], [11], [15] Ω must have the property that the domain of the $L^2(\Omega)$-realization of $-\Delta$ is $H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, the Nemitski operator associated with f must be Lipschitz continuous from $H^2(\Omega) \cap H_0^1(\Omega)$ to $H^1(\Omega)$ in [15] and from $D((-\Delta)^{(\alpha+1)/2})$ to $D((-\Delta)^{\alpha/2})$ for all $0 \le \alpha \le 1$ in [10], [11]. Therefore, if one wants to replace $f(u) + g(x)$ by $f(x, u)$, one needs to impose severe smoothness conditions on $f(x, u)$ with respect to the space variable x.

If Ω is unbounded, the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is no longer compact, and this poses some additional difficulties even for the existence proof of the attractors $\mathcal{A}_\varepsilon$. In [6], [7], Feireisl circumvented these difficulties by decomposing any solution $u(t, x)$ into the sum $u_1(t, x)+u_2(t, x)$ of two functions, such that $u_1(t, \cdot\,)$ is asymptotically small, and $u_2(t, \cdot\,)$ has a compact support which propagates with speed $1/\varepsilon^2$. As $\varepsilon \to 0$, the speed of propagation tends to infinity, and, indeed, the estimates obtained by Feireisl are not uniform with respect to ε. It is therefore apparent that, if one wants to pass to the limit as $\varepsilon \to 0$, a different approach is needed.

In our previous paper [17] we proved the existence of compact global attractors for damped hyperbolic equations in unbounded domains using the method of tail-estimates (introduced by Wang in [19] for parabolic equations), combined with an argument due to Ball [3] and elaborated by Raugel in [18]. Here we exploit the same techniques to establish an upper semicontinuity result similar to that of Hale and Raugel, when Ω is an unbounded domain and f is critical. Our arguments do not rely on $(H^2 \times H^1)$-estimates for the attractors $\mathcal{A}_\varepsilon$. Therefore they also apply to the case of an open set Ω for which the domain of the $L^2(\Omega)$-realization of $-\Delta$ is not $H^2(\Omega) \cap H_0^1(\Omega)$ (e.g. if Ω is the exterior of a convex polyhedron).

Before we describe in detail our assumptions and our results, we need to introduce some notation. In this paper, $N = 3$ and Ω is an arbitrary open subset of $\mathbb{R}^N$, bounded or not. For a and $b \in \mathbb{Z}$ we write $[a\,.\,.\,b]$ to denote the set of all $m \in \mathbb{Z}$ with $a \le m \le b$. Given a subset S of $\mathbb{R}^N$ and a function $v\colon S \to \mathbb{R}$ we denote by $\widetilde{v}\colon \mathbb{R}^N \to \mathbb{R}$ the trivial extension of v defined by $\widetilde{v}(x) = 0$ for $x \in \mathbb{R}^N \setminus S$. Given a function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $\widehat{g}$ the Nemitski operator which associates with every function $u\colon \Omega \to \mathbb{R}$ the function $\widehat{g}(u)\colon \Omega \to \mathbb{R}$ defined by

$$\widehat{g}(u)(x) = g(x, u(x)), \quad x \in \Omega.$$

Unless specified otherwise, given $k \in \mathbb{N}$ and functions g, $h\colon \Omega \to \mathbb{R}^k$ we write

$$\langle g, h \rangle := \int_\Omega \sum_{m=1}^{k} g_m(x) h_m(x)\, dx,$$

whenever the integral on the right-hand side makes sense.

If $I \subset \mathbb{R}$, Y and X are normed spaces with $Y \subset X$ and if $u: I \to Y$ is a function which is differentiable as a function into X then we denote its X-valued derivative by $\partial(u; X)$. Similarly, if X is a Banach space and $u: I \to X$ is integrable as a function into X, then we denote its X-valued integral by $\int_I (u(t); X)\, dt$.

Assumption 1.1.

(a) *a_0, $a_1 \in \,]0,\infty[$ are constants and $a_{ij}: \Omega \to \mathbb{R}$, $i, j \in [1..N]$ are functions in $L^\infty(\Omega)$ such that $a_{ij} = a_{ji}$, $i, j \in [1..N]$, and for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, $a_0|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \le a_1|\xi|^2$. $A(x) := (a_{ij}(x))_{i,j=1}^N$, $x \in \Omega$.*

(b) *$\beta: \Omega \to \mathbb{R}$ is a measurable function with the property that*
   (i) *for every $\overline{\varepsilon} \in \,]0,\infty[$ there is a $C_{\overline{\varepsilon}} \in [0,\infty[$ with $\big||\beta|^{1/2}u\big|_{L^2}^2 \le \overline{\varepsilon}|u|_{H^1}^2 + C_{\overline{\varepsilon}}|u|_{L^2}^2$ for all $u \in H_0^1(\Omega)$;*
   (ii) $\lambda_1 := \inf\{\, \langle A\nabla u, \nabla u\rangle + \langle \beta u, u\rangle \mid u \in H_0^1(\Omega),\ |u|_{L^2} = 1 \,\} > 0.$

Remark. In [17], [16] we gave conditions on β, ensuring that (b) is satisfied.

Assumption 1.2.

(a) *$f: \Omega \times \mathbb{R} \to \mathbb{R}$ is such that, for every $u \in \mathbb{R}$, $f(\,\cdot\,, u)$ is (Lebesgue-) measurable, $f(\,\cdot\,, 0) \in L^2(\Omega)$ and for a.e. $x \in \Omega$, $f(x, \cdot\,)$ is of class C^2 and such that $\partial_u f(\,\cdot\,, 0) \in L^\infty(\Omega)$ and $|\partial_{uu} f(x,u)| \le \overline{C}(1+|u|)$ for some constant $\overline{C} \in [0,\infty[$, every $u \in \mathbb{R}$ and a.e. $x \in \Omega$;*

(b) *$f(x,u)u - \overline{\mu}F(x,u) \le c(x)$ and $F(x,u) \le c(x)$ for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$. Here, $c \in L^2(\Omega)$ is a given function, $\overline{\mu} \in \,]0,\infty[$ is a constant and $F: \Omega \times \mathbb{R} \to \mathbb{R}$ is defined, for $(x,u) \in \Omega \times \mathbb{R}$, by*

$$F(x,u) = \int_0^u f(x,s)\, ds,$$

whenever $f(x, \cdot\,): \mathbb{R} \to \mathbb{R}$ is continuous and $F(x,u) = 0$ otherwise.

Note that Assumptions 1.1 and 1.2 imply the hypotheses of [17].

Let $D(\mathbf{B}_\varepsilon)$ be the set of all $(u,v) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $v \in H_0^1(\Omega)$ and $-\beta u + \sum_{ij}(a_{ij}u_{x_j})_{x_i}$ (in the distributional sense) lies in $L^2(\Omega)$. It turns out that the operator

$$\mathbf{B}_\varepsilon(u,v) = \bigg(-v, \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\beta u - \frac{1}{\varepsilon}\sum_{ij}(a_{ij}u_{x_j})_{x_i} \bigg), \quad (u,v) \in D(\mathbf{B}_\varepsilon)$$

is the generator of a (C_0)-semigroup $e^{-\mathbf{B}_\varepsilon t}$, $t \in [0,\infty[$ on $H_0^1(\Omega) \times L^2(\Omega)$. Moreover, the Nemitski operator $\widehat{f}$ is a Lipschitzian map of $H_0^1(\Omega)$ to $L^2(\Omega)$. Results in [4] then imply that the hyperbolic boundary value problem

$$\begin{aligned} \varepsilon u_{tt} + u_t + \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i} &= f(x,u), \quad x \in \Omega, \quad t \in [0,\infty[\,, \\ u(x,t) &= 0, \qquad\quad x \in \partial\Omega,\ t \in [0,\infty[\end{aligned}$$

with Cauchy data at $t = 0$ has a unique (mild) solution $z(t) = (u(t), v(t))$ in $H_0^1(\Omega) \times L^2(\Omega)$, given by the "variation-of-constants" formula

$$z(t) = e^{-\mathbf{B}_\varepsilon t} z(0) + \int_0^t e^{-\mathbf{B}_\varepsilon (t-s)} \left(0, \frac{1}{\varepsilon}\widehat{f}(u(s))\right) ds.$$

For $\varepsilon \in \,]0,\infty[$ we define π_ε to be the local semiflow on $H_0^1(\Omega) \times L^2(\Omega)$ generated by the (mild) solutions of this hyperbolic boundary value problem. We can summarize the results of [17] in the following:

Theorem 1.3. *Under Assumptions* 1.1 *and* 1.2, π_ε *is a global semiflow and it has a global attractor* $\mathcal{A}_\varepsilon$*.*

Analogously, consider the parabolic boundary value problem

$$\begin{aligned} u_t + \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i} &= f(x,u), \quad x \in \Omega, \quad t \in [0,\infty[\,, \\ u(x,t) &= 0, \qquad\quad x \in \partial\Omega,\ t \in [0,\infty[\end{aligned}$$

with Cauchy data at $t = 0$. Letting $\mathbf{A}$ denote the sectorial operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$, we have that $D(\mathbf{A})$ is the set of all $u \in H_0^1(\Omega)$ such that the distribution $\beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$ lies in $L^2(\Omega)$. Again, the Cauchy problem has a unique (mild) solution $u(t)$ in $H_0^1(\Omega)$, given by the "variation-of-constants" formula

$$u(t) = e^{-\mathbf{A}t}u(0) + \int_0^t e^{-\mathbf{A}(t-s)}\widehat{f}(u(s))\, ds.$$

Let $\widetilde{\pi}$ be the local semiflow on $H_0^1(\Omega)$ generated by the (mild) solutions of this parabolic boundary value problem. Results in [16] imply that $\widetilde{\pi}$ is a global semiflow and has a global attractor $\widetilde{\mathcal{A}}$ (see also [1]). Moreover, it is proved in [16] that $\widetilde{\mathcal{A}} \subset D(\mathbf{A})$ and $\widetilde{\mathcal{A}}$ is compact in $D(\mathbf{A})$ endowed with the graph norm.

Let $\Gamma\colon D(\mathbf{A}) \to H_0^1(\Omega) \times L^2(\Omega)$ be defined by $\Gamma(u) = (u, \mathbf{A}u + \widehat{f}(u))$. Set $\mathcal{A}_0 := \Gamma(\widetilde{\mathcal{A}})$. Then we have the following main result of this paper:

Theorem 1.4. *The family* $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,\infty[}$ *is upper semicontinuous at* $\varepsilon = 0$ *with respect to the topology of* $H_0^1(\Omega) \times H^{-1}(\Omega)$*, i.e.*

$$\lim_{\varepsilon\to 0^+} \sup_{y\in\mathcal{A}_\varepsilon} \inf_{z\in\mathcal{A}_0} |y - z|_{H_0^1\times H^{-1}} = 0.$$

Actually a stronger result is established in Theorem 3.9 below.

2. Preliminaries

In this section we collect a few preliminary results. We begin with an abstract lemma established in [16]:

Lemma 2.1. *Suppose $(Y, \langle \cdot\,,\cdot\rangle_Y)$ and $(X, \langle \cdot\,,\cdot\rangle_X)$ are (real or complex) Hilbert spaces such that $Y \subset X$, Y is dense in $(X, \langle \cdot\,,\cdot\rangle_X)$ and the inclusion $(Y, \langle \cdot\,,\cdot\rangle_Y) \to (X, \langle \cdot\,,\cdot\rangle_X)$ is continuous. Then for every $u \in X$ there exists a unique $w_u \in Y$ such that*

$$\langle v, w_u\rangle_Y = \langle v, u\rangle_X \quad \textit{for all } v \in Y.$$

The map $B: X \to X$, $u \mapsto w_u$ is linear, symmetric and positive. Let $B^{1/2}$ be a square root of B, i.e. $B^{1/2}: X \to X$ linear, symmetric and $B^{1/2} \circ B^{1/2} = B$. Then B and $B^{1/2}$ are injective and $R(B)$ is dense in Y. Set $X^{1/2} = X_B^{1/2} = R(B^{1/2})$ and $B^{-1/2}: X^{1/2} \to X$ be the inverse of $B^{1/2}$. On $X^{1/2}$ the assignment $\langle u, v\rangle_{1/2} := \langle B^{-1/2}u, B^{-1/2}v\rangle_X$ is a complete scalar product. We have $Y = X^{1/2}$ and $\langle \cdot\,,\cdot\rangle_Y = \langle \cdot\,,\cdot\rangle_{1/2}$.

Now let $\mathbf{A}$ be the sectorial operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$. Then $\mathbf{A}$ generates a family $X^\alpha = X^\alpha_{\mathbf{A}}$, $\alpha \in \mathbb{R}$, of fractional power spaces with $X^{-\alpha}$ being the dual of X^α for $\alpha \in \,]0,\infty[$. We write

$$H_\alpha = X^{\alpha/2}, \quad \alpha \in \mathbb{R}.$$

For $\alpha \in \mathbb{R}$ the operator $\mathbf{A}$ induces an operator $\mathbf{A}_\alpha: H_\alpha \to H_{\alpha-2}$. In particular, $H_0 = L^2(\Omega)$ and $\mathbf{A} = \mathbf{A}_2$.

Note that, thanks to Assumption 1.1, the scalar product

$$\langle u, v\rangle_{H_0^1} = \langle A\nabla u, \nabla v\rangle + \langle \beta u, v\rangle, \quad u, v \in H_0^1(\Omega)$$

on $H_0^1(\Omega)$ is equivalent to the usual scalar product on $H_0^1(\Omega)$. Moreover,

$$\langle u, v\rangle_{H_0^1} = \langle \mathbf{A}_2 u, v\rangle, \quad u \in D(\mathbf{A}_2), v \in H_0^1(\Omega).$$

Corollary 2.2. *$H_1 = H_0^1(\Omega)$ with equivalent norms. Consequently $H_{-1} = H^{-1}(\Omega)$ with equivalent norms.*

Proof. Set $(X, \langle \cdot\,,\cdot\rangle_X) = (L^2(\Omega), \langle \cdot\,,\cdot\rangle)$, $(Y, \langle \cdot\,,\cdot\rangle_Y) = (H_0^1(\Omega), \langle \cdot\,,\cdot\rangle_{H_0^1})$. Then Y is dense in X and the inclusion $Y \to X$ is continuous. Let $B_2: X \to X$ be the inverse of $\mathbf{A}_2$. Then for all $u \in X$, $B_2 u \in Y$ and for all $v \in Y$

$$\langle v, u\rangle_X = \langle v, B_2 u\rangle_Y.$$

Thus $B_2 = B$ where B is as in Lemma 2.1. Now the lemma implies the corollary. $\square$

COROLLARY 2.3. *The linear operator* $\mathbf{A}_1 \colon H_1 \to X := H_{-1}$ *is self-adjoint hence sectorial on* X. *Let* X_1^α, $\alpha \in [0,\infty[$, *be the family of fractional powers generated by* $\mathbf{A}_1$. *Then* $X^{1/2} = L^2(\Omega)$ *with equivalent norms.*

PROOF. Set $(X, \langle \cdot\,,\,\cdot\rangle_X) = (H_{-1}, \langle \cdot\,,\,\cdot\rangle_{H_{-1}})$, $(Y, \langle \cdot\,,\,\cdot\rangle_Y) = (H_0, \langle \cdot\,,\,\cdot\rangle_{H_0})$. Then Y is dense in X and the inclusion $Y \to X$ is continuous. Let $B_1 \colon X \to X$ be the inverse of $\mathbf{A}_1$. Then for all $u \in X$, $B_1 u \in Y$ and for all $v \in Y$

$$\langle v, u\rangle_X = \langle B_1 v, B_1 u\rangle_{H_1} = \langle v, B_1 u\rangle_Y.$$

Thus $B_1 = B$ where B is as in Lemma 2.1. Now the lemma implies the corollary. $\square$

We end this section by quoting a result proved in [17], which can be used to rigorously justify formal differentiation of various functionals along (mild) solutions of semilinear evolution equations.

THEOREM 2.4. *Let* Z *be a Banach space and* $B \colon D(B) \subset Z \to Z$ *the infinitesimal generator of a* (C_0)*-semigroup of linear operators* e^{-Bt} *on* Z, $t \in [0,\infty[$. *Let* U *be open in* Z, Y *be a normed space and* $V \colon U \to Y$ *be a function which, as a map from* Z *to* Y, *is continuous at each point of* U *and Fréchet differentiable at each point of* $U \cap D(B)$. *Moreover, let* $W \colon U \times Z \to Y$ *be a function which, as a map from* $Z \times Z$ *to* Y, *is continuous and such that* $DV(z)(Bz + w) = W(z, w)$ *for* $z \in U \cap D(B)$ *and* $w \in Z$. *Let* $\tau \in]0,\infty[$ *and* $I := [0,\tau]$. *Let* $\overline{z} \in U$, $g \colon I \to Z$ *be continuous and* z *be a map from* I *to* U *such that*

$$z(t) = e^{-Bt}\overline{z} + \int_0^t e^{-B(t-s)} g(s)\, ds, \quad t \in I.$$

Then the map $V \circ z \colon I \to Y$ *is differentiable and*

$$(V \circ z)'(t) = W(z(t), g(t)), \quad t \in I.$$

3. Proof of the main result

In order to establish our main result we need uniform estimates for the attractors $\mathcal{A}_\varepsilon$ in $H_0^1(\Omega) \times L^2(\Omega)$.

LEMMA 3.1. *Let* f *be as in Assumption* 1.2. *Then there is a constant* $C \in [0,\infty[$ *such that for all* $u, v \in \mathbb{R}$ *and for a.e.* $x \in \Omega$,

$$\begin{aligned} |\partial_u f(x,u)| &\le C(1 + |u|^2), \\ |\partial_u f(x,v) - \partial_u f(x,u)| &\le C(1 + |u| + |v-u|)|v-u|, \\ |f(x,v) - f(x,u) - \partial_u f(x,u)(v-u)| &\le C(1 + |u| + |v-u|)|v-u|^2. \end{aligned}$$

Proof. For all u, $v \in \mathbb{R}$ and a.e. $x \in \Omega$ we have

$$\partial_u f(x,v) - \partial_u f(x,u) = \int_0^1 \partial_{uu} f(x, u + s(v-u))(v-u)\, ds$$

and

$$f(x,v) - f(x,u) - \partial_u f(x,u)(v-u) = (v-u)^2 \int_0^1 \theta \left[\int_0^1 \partial_{uu} f(x, u + r\theta(v-u))\, dr \right] d\theta.$$

This easily implies the assertions of the lemma. □

Proposition 3.2. *Let f and F be as in Assumption* 1.2. *Then, for every measurable function $v\colon \Omega \to \mathbb{R}$, both $\widehat{f}(v)$ and $\widehat{F}(v)$ are measurable and for all measurable functions u, $h\colon \Omega \to \mathbb{R}$*

$$|\widehat{f}(u)|_{L^2} \le |\widehat{f}(0)|_{L^2} + C(|u|_{L^2} + |u|^3_{L^6}), \tag{3.1}$$

$$|\widehat{f}(u+h) - \widehat{f}(u)|_{L^2} \le C|h|_{L^2} + C(|u|^2_{L^6} + |h|^2_{L^6})|h|_{L^6}, \tag{3.2}$$

$$|\widehat{F}(u)|_{L^1} \le C(|u|^2_{L^2}/2 + |u|^4_{L^4}/4) + |u|_{L^2}|\widehat{f}(0)|_{L^2}, \tag{3.3}$$

$$\begin{aligned}|\widehat{F}(u+h) - \widehat{F}(u)|_{L^1} \le {} & (|\widehat{f}(0)|_{L^2} + C(|u|_{L^2} + |h|_{L^2}) \\ & + 4C(|u|^3_{L^6} + |h|^3_{L^6}))|h|_{L^2},\end{aligned} \tag{3.4}$$

and

$$|\widehat{F}(u+h) - \widehat{F}(u) - \widehat{f}(u)h|_{L^1} \le (C|h|_{L^2} + C(|u|^2_{L^6} + |h|^2_{L^6})|h|_{L^6})|h|_{L^2}. \tag{3.5}$$

Finally, for every $r \in [3, \infty[$ there is a constant $C(r) \in [0, \infty[$ such that for all u, $h \in H^1_0(\Omega)$

$$|\widehat{f}(u+h) - \widehat{f}(u)|_{H^{-1}} \le C(r)|h|_{L^2} + C(r)(|u|^2_{L^6} + |h|^2_{L^6})|h|_{L^2}. \tag{3.6}$$

Proof. Lemma 3.1 implies that f satisfies the hypotheses of [17, Proposition 3.11], to which the reader is referred for details. □

For $s \in [2,6]$ we denote by $C_s \in [0, \infty[$ an imbedding constant of the inclusion induced map from H_1 to $L^s(\Omega)$.

Proposition 3.3. *Let f be as in Assumption* 1.2, *$I \subset \mathbb{R}$ be an interval, u be a continuous map from I to H_1 such that u is continuously differentiable into H_0 with $v := \partial(u; H_0)$. Then the composite map $\widehat{f} \circ u\colon I \to H_0$ is defined, $\widehat{f} \circ u$ is continuously differentiable into H_{-1} and $g := \partial(\widehat{f} \circ u; H_{-1}) = (\widehat{\partial_u f} \circ u) \cdot v$. Moreover, for every $t \in I$,*

$$|g(t)|_{H_{-1}} \le C(C_2 + C_6|u(t)|^2_{L^3})|v(t)|_{L^2} \le C(C_2 + C_6 C_3 |u(t)|^2_{H_1})|v(t)|_{L^2}. \tag{3.7}$$

Proof. It follows from Proposition 3.2 that for every $w \in H_1$, $\widehat{f}(w) \in H_0$. Thus $\widehat{f} \circ u$ is defined as a function from I to H_0. Moreover, for every $t \in I$

and $\zeta \in H_1$, the function $\widehat{\partial_u f}(u(t)) \cdot v(t) \cdot \zeta \colon \Omega \to \mathbb{R}$ is measurable and so by Lemma 3.1 and Hölder's inequality

$$|\widehat{\partial_u f}(u(t)) \cdot v(t) \cdot \zeta|_{L^1} \le C|v(t)|_{L^2}|\zeta|_{L^2} + C|\,|u(t)|^2\,|_{L^3}|v(t)|_{L^2}|\zeta|_{L^6}.$$

It follows that for every $t \in \mathbb{R}$, $g(t) = \widehat{\partial_u f}(u(t)) \cdot v(t) \in H_{-1}$ and (3.7) is satisfied.

Moreover, for s, $t \in I$,

$$\begin{aligned}
&|\widehat{\partial_u f}(u(t)) \cdot v(t) - \widehat{\partial_u f}(u(s)) \cdot v(s)|_{H_{-1}} \\
&\qquad = \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} |\widehat{\partial_u f}(u(t)) \cdot v(t) \cdot \zeta - \widehat{\partial_u f}(u(s)) \cdot v(s) \cdot \zeta|_{L^1} \\
&\qquad \le \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} T_1(t)(\zeta) + \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} T_2(t)(\zeta),
\end{aligned}$$

where

$$\begin{aligned}
T_1(t)(\zeta) &= |(\widehat{\partial_u f}(u(t)) - \widehat{\partial_u f}(u(s))) \cdot v(t) \cdot \zeta|_{L^1}, \\
T_2(t)(\zeta) &= |\widehat{\partial_u f}(u(s)) \cdot (v(t) \cdot \zeta - v(s) \cdot \zeta)|_{L^1}.
\end{aligned}$$

By Lemma 3.1 we obtain, for all $\zeta \in H_1$ with $|\zeta|_1 \le 1$,

$$\begin{aligned}
T_1(t)(\zeta) &\le C|(1 + |u(s)| + |u(t) - u(s)|) \cdot |u(t) - u(s)| \cdot \zeta|_{L^2}|v(t)|_{L^2} \\
&\le C|u(t) - u(s)|_{L^3}|v(t)|_{L^2}|\zeta|_{L^6} \\
&\quad + C|u(s)|_{L^6}|u(t) - u(s)|_{L^6}|v(t)|_{L^2}|\zeta|_{L^6} \\
&\quad + C|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^6}|v(t)|_{L^2}|\zeta|_{L^6} \\
&\le CC_3C_6|u(t) - u(s)|_{H_1}|v(t)|_{L^2} + CC_6^3|u(s)|_{H_1}|u(t) - u(s)|_{H_1}|v(t)|_{L^2} \\
&\quad + CC_6^3|u(t) - u(s)|_{H_1}^2|v(t)|_{L^2} \\
T_2(t)(\zeta) &\le C|(1 + |u(s)|^2) \cdot \zeta|_{L^2}|v(t) - v(s)|_{L^2} \\
&\le C(|\zeta|_{L^2} + |\,|u(s)|^2\,|_{L^3}|\zeta|_{L^6})|v(t) - v(s)|_{L^2} \\
&\le C(C_2 + C_6^3|u(s)|_{H_1}^2)|v(t) - v(s)|_{L^2}.
\end{aligned}$$

Since u is continuous into H_1 and v is continuous into $H_0 = L^2(\Omega)$ it follows that

$$\sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} T_1(t)(\zeta) + \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} T_2(t)(\zeta) \to 0 \quad \text{as } t \to s$$

so the map $(\widehat{\partial_u f} \circ u) \cdot v$ is continuous into H_{-1}.

Now, for s, $t \in I$, $t \ne s$,

$$\begin{aligned}
&(t - s)^{-1}|(\widehat{f} \circ u)(t) - (\widehat{f} \circ u)(s) - \widehat{\partial_u f}(u(s)) \cdot v(s)|_{H_{-1}} \\
&= \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} (t - s)^{-1}|(\widehat{f} \circ u)(t) \cdot \zeta - (\widehat{f} \circ u)(s) \cdot \zeta - \widehat{\partial_u f}(u(s)) \cdot v(s) \cdot \zeta|_{L^1} \\
&\le (t - s)^{-1} \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} T_3(t)(\zeta) + (t - s)^{-1} \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \le 1} T_4(t)(\zeta)
\end{aligned}$$

where $T_3(t)(\zeta) = |g_{t,\zeta}|_{L^1}$ with

$$g_{t,\zeta} = (\widehat{f} \circ u)(t) \cdot \zeta - (\widehat{f} \circ u)(s) \cdot \zeta - \widehat{\partial_u f}(u(s)) \cdot (u(t) - u(s)) \cdot \zeta$$

and

$$T_4(t)(\zeta) = |\widehat{\partial_u f}(u(s)) \cdot (u(t) - u(s) - v(s)) \cdot \zeta|_{L^1}.$$

Now, by Lemma 3.1, for all $\zeta \in H_1$ with $|\zeta|_{H_1} \leq 1$ and for a.e. $x \in \Omega$

$$|g_{t,\zeta}(x)| \leq C(1 + |u(s)(x)| + |u(t)(x) - u(s)(x)|)|u(t)(x) - u(s)(x)|^2|\zeta(x)|$$

so

$$\begin{aligned}
T_3(t)(\zeta) \leq{}& C(|u(t) - u(s)|_{L^3}|u(t) - u(s)|_{L^2}|\zeta|_{L^6}) \\
&+ C(|u(s)|_{L^6}|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^2}|\zeta|_{L^6}) \\
&+ C(|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^2}|\zeta|_{L^6}) \\
\leq{}& CC_6(C_3|u(t) - u(s)|_{H_1}|u(t) - u(s)|_{L^2}) \\
&+ CC_6(C_6^2|u(s)|_{H_1}|u(t) - u(s)|_{H_1}|u(t) - u(s)|_{L^2}) \\
&+ CC_6(C_6^2|u(t) - u(s)|_{H_1}^2|u(t) - u(s)|_{L^2}).
\end{aligned} \tag{3.8}$$

Since u is continuous into H_1 and locally Lipschitzian into $H_0 = L^2(\Omega)$ it follows from (3.8) that

$$(t - s)^{-1} \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \leq 1} T_3(t)(\zeta) \to 0 \quad \text{as } t \to s.$$

We also have

$$\begin{aligned}
T_4(t)(\zeta) &\leq C|(1 + |u(s)|^2) \cdot \zeta|_{L^2}|u(t) - u(s) - v(s)|_{L^2} \\
&\leq (C|\zeta|_{L^2} + C|\,|u(s)|^2\,|_{L^3}|\zeta|_{L^6})|u(t) - u(s) - v(s)|_{L^2} \\
&\leq C(C_2 + CC_6^3|u(s)|_{H_1}^2)|u(t) - u(s) - v(s)|_{L^2}.
\end{aligned}$$

Since $(t - s)^{-1}|u(t) - u(s) - v(s)|_{L^2} \to 0$ as $t \to s$ it follows that

$$(t - s)^{-1} \sup_{\zeta \in H_1,\, |\zeta|_{H_1} \leq 1} T_4(t)(\zeta) \to 0 \quad \text{as } t \to s.$$

It follows that $\widehat{f} \circ u$, as a map into H_{-1}, is differentiable at s and

$$\partial_u(\widehat{f} \circ u; H_{-1})(s) = (\widehat{\partial_u f} \circ u)(s) \cdot v(s). \qquad \square$$

Proposition 3.4. *Let $\varepsilon \in\,]0, \infty[$ be arbitrary. Define the function $\widetilde{V} = \widetilde{V}_\varepsilon \colon H_1 \times H_0 \to \mathbb{R}$ by*

$$\widetilde{V}(u, v) = \frac{1}{2}\langle u, u\rangle_{H_1} + \frac{1}{2}\varepsilon\langle v, v\rangle - \int_\Omega F(x, u(x))\, dx, \quad (u, v) \in H_1 \times H_0.$$

Let $z\colon\mathbb{R}\to H_1\times H_0$*,* $z(t)=(z_1(t),z_2(t))$*,* $t\in\mathbb{R}$*, be a solution of* π_ε*. Then* $\widetilde{V}\circ z$ *is differentiable and* $(\widetilde{V}\circ z)'(t)=-|z_2(t)|^2_{L^2}$ *for* $t\in\mathbb{R}$*.*

Proof. This is an application of Theorem 2.4 (for the details see [17, Proposition 4.1]). ☐

Proposition 3.5. *Let* $\varepsilon\in\,]0,\infty[$ *be arbitrary. Define the function* $V=V_\varepsilon\colon H_0\times H_{-1}\to\mathbb{R}$ *by*

$$V(v,w)=\frac{1}{2}\langle v,v\rangle+\frac{1}{2}\varepsilon\langle w,w\rangle_{H_{-1}},\quad (v,w)\in H_0\times H_{-1}.$$

Let $z\colon\mathbb{R}\to H_1\times H_0$*,* $z(t)=(z_1(t),z_2(t))$*,* $t\in\mathbb{R}$*, be a solution of* π_ε*. Then* $z=(z_1,z_2)$ *is differentiable as a map into* $H_0\times H_{-1}$ *with* $z_2=\partial(z_1;H_0)$*. Let* $u=z_1$*,* $v=z_2$*,* $w=\partial(v;H_{-1})$ *and* $g=(\widehat{\partial_u f}\circ u)\cdot v$*. Then the function* $\alpha\colon\mathbb{R}\to\mathbb{R}$*,* $t\mapsto V(v(t),w(t))$ *is differentiable and for every* $t\in\mathbb{R}$

$$\alpha'(t)=-\langle w(t),w(t)\rangle_{H_{-1}}+\langle g(t),w(t)\rangle_{H_{-1}}.$$

Proof. For $\varepsilon\in\,]0,\infty[$ and $\kappa\in\mathbb{R}$ let $\mathbf{B}_{\varepsilon,\kappa}\colon H_\kappa\times H_{\kappa-1}\to H_{\kappa-1}\times H_{\kappa-2}$ be defined by

$$\mathbf{B}_{\varepsilon,\kappa}(z)=\Big(-z_2,\frac{1}{\varepsilon}(z_2+\mathbf{A}_\kappa z_1)\Big),\quad z=(z_1,z_2)\in H_\kappa\times H_{\kappa-1}.$$

It follows that $-\mathbf{B}_{\varepsilon,\kappa}$ is m-dissipative on $H_{\kappa-1}\times H_{\kappa-2}$ (cf. [17, proof of Proposition 3.6]). Moreover, if $z\colon\mathbb{R}\to H_1\times H_0$ is a solution of π_ε, then

$$\begin{aligned}z(t)&=e^{-\mathbf{B}_{\varepsilon,2}(t-t_0)}z(t_0)+\int_{t_0}^t\Big(e^{-\mathbf{B}_{\varepsilon,2}(t-s)}\Big(0,\frac{1}{\varepsilon}\widehat{f}(z_1(s))\Big);H_1\times H_0\Big)\,ds\\&=e^{-\mathbf{B}_{\varepsilon,1}(t-t_0)}z(t_0)+\int_{t_0}^t\Big(e^{-\mathbf{B}_{\varepsilon,1}(t-s)}\Big(0,\frac{1}{\varepsilon}\widehat{f}(z_1(s))\Big);H_0\times H_{-1}\Big)\,ds,\end{aligned}$$

for $t,t_0\in\mathbb{R}$, $t_0\le t$. Since $z(t_0)\in D(\mathbf{B}_{\varepsilon,1})$ and $t\mapsto(0,(1/\varepsilon)\widehat{f}(z_1(s)))$ is continuous into $D(\mathbf{B}_{\varepsilon,1})$ it follows from [9, proof of Theorem II.1.3 (i)] that $z=(u,v)$ is differentiable as a map into $H_0\times H_{-1}$ with $v=\partial(u;H_0)$. Now, in H_{-1},

$$w=\partial(v;H_{-1})=\frac{1}{\varepsilon}(v-\mathbf{A}_1\circ u+\widehat{f}\circ u)=\frac{1}{\varepsilon}(v-\mathbf{A}_0\circ u+\widehat{f}\circ u).$$

It follows from Proposition 3.3 that w is differentiable into H_{-2} and

$$\partial(w;H_{-2})=\frac{1}{\varepsilon}(w-\mathbf{A}_0\circ v+g).$$

Again [9, proof of Theorem II.1.3 (i)] implies that

$$\begin{aligned}(v,w)(t)&=e^{-\mathbf{B}_{\varepsilon,-1}(t-t_0)}(v,w)(t_0)\\&\quad+\int_{t_0}^t\Big(e^{-\mathbf{B}_{\varepsilon,-1}(t-s)}\Big(0,\frac{1}{\varepsilon}g(s)\Big);H_{-2}\times H_{-3}\Big)\,ds\end{aligned}\tag{3.9}$$

$$\begin{aligned} &= e^{-\mathbf{B}_{\varepsilon,1}(t-t_0)}(v,w)(t_0) \\ &\quad + \int_{t_0}^{t} \left(e^{-\mathbf{B}_{\varepsilon,1}(t-s)}\left(0, \frac{1}{\varepsilon} g(s)\right); H_0 \times H_{-1} \right) ds, \end{aligned}$$

for $t, t_0 \in \mathbb{R}$, $t_0 \le t$. Now note that the function $V = V_\varepsilon$ is Fréchet differentiable and

$$DV(v,w)(\widetilde{v},\widetilde{w}) = \langle v,\widetilde{v}\rangle_{H_0} + \varepsilon\langle w,\widetilde{w}\rangle_{H_{-1}}.$$

Thus for $(u,v) \in D(-\mathbf{B}_{\varepsilon,1}) = H_1 \times H_0$ and $(\widetilde{v},\widetilde{w}) \in H_0 \times H_{-1}$

$$\begin{aligned} DV(v,w)(-\mathbf{B}_{\varepsilon,1}(v,w) + (\widetilde{v},\widetilde{w})) &= \langle v, w + \widetilde{v}\rangle_{H_0} + \varepsilon\left\langle w, -\frac{1}{\varepsilon}w - \frac{1}{\varepsilon}\mathbf{A}_1 v + \widetilde{w}\right\rangle_{H_{-1}} \\ &= \langle v,\widetilde{v}\rangle_{H_0} - \langle w,w\rangle_{H_{-1}} + \varepsilon\langle w,\widetilde{w}\rangle_{H_{-1}}. \end{aligned}$$

Here we have used the fact that

$$\langle w, \mathbf{A}_1 v\rangle_{H_{-1}} = \langle \mathbf{A}_1^{-1}w, \mathbf{A}_1^{-1}\mathbf{A}_1 v\rangle_{H_1} = \langle \mathbf{A}_1^{-1}w, v\rangle_{H_1} = \langle w, v\rangle_{H_0}$$

as $\mathbf{A}_1^{-1}w = \mathbf{A}_2^{-1}w \in H_2$. Defining $W\colon (H_0 \times H_{-1}) \times (H_0 \times H_{-1}) \to \mathbb{R}$ by

$$W((v,w),(\widetilde{v},\widetilde{w})) = \langle v,\widetilde{v}\rangle_{H_0} - \langle w,w\rangle_{H_{-1}} + \varepsilon\langle w,\widetilde{w}\rangle_{H_{-1}}$$

we see that W is continuous. Now it follows from (3.9) and Theorem 2.4 that $\alpha = V_\varepsilon \circ (v,w)$ is differentiable and

$$\alpha'(t) = -\langle w(t), w(t)\rangle_{H_{-1}} + \langle w(t), g(t)\rangle_{H_{-1}}, \quad t \in \mathbb{R}. \qquad \square$$

Proposition 3.6. *Let $\varepsilon_0 \in \,]0,\infty[$ be arbitrary. Then for every $r \in [0,\infty[$ there is a constant $C(r,\varepsilon_0) \in [0,\infty[$ such that whenever $\varepsilon \in \,]0,\varepsilon_0]$ and $z = (u,v)\colon \mathbb{R} \to H_1 \times H_0$ is a solution of π_ε with $\sup_{t\in\mathbb{R}}(|u(t)|^2_{H_1} + \varepsilon|v(t)|^2_{H_0}) \le r$ and $w := \partial(v; H_{-1})$, then*

$$\sup_{t\in\mathbb{R}}(|v(t)|^2_{H_0} + \varepsilon|w(t)|^2_{H_{-1}}) \le C(r,\varepsilon_0).$$

Proof. By $C_i(r) \in [0,\infty[$, resp. $C_i(r,\varepsilon_0) \in [0,\infty[$ we denote various constants depending only on r, resp. on r and ε_0, but independent of $\varepsilon \in \,]0,\varepsilon_0]$ and the choice of a solution z of π_ε with $\sup_{t\in\mathbb{R}}(|u(t)|^2_{H_1} + \varepsilon|v(t)|^2_{H_0}) \le r$.

Let $\varepsilon \in \,]0,\varepsilon_0]$ be arbitrary, $\alpha(t) = V_\varepsilon(v(t),w(t))$, $t \in \mathbb{R}$, and $g = (\widehat{\partial_u f} \circ u)\cdot v$. Using (3.7) we see that

$$|g(t)|_{H_{-1}} \le C(1 + C_6 C_3^2 r^2)|v(t)|_{H_0}, \quad t \in \mathbb{R}. \tag{3.10}$$

Proposition 3.5 implies that

$$\begin{aligned} \alpha'(t) &\le -|w(t)|^2_{H_{-1}} + \frac{1}{2}|g(t)|^2_{H_{-1}} + \frac{1}{2}|w(t)|^2_{H_{-1}} \\ &\le -\frac{1}{2}|w(t)|^2_{H_{-1}} + \frac{1}{2}C^2(1 + C_6C_3^2 r^2)^2|v(t)|^2_{H_0}, \end{aligned} \tag{3.11}$$

for $t \in \mathbb{R}$. Thus we obtain, for every $k \in]0, \infty[$,

$$\alpha'(t)+k\alpha(t) \le \left(-\frac{1}{2}+\frac{k\varepsilon}{2}\right)|w(t)|^2_{H_{-1}}+\left(\frac{1}{2}C^2(1+C_6C_3^2r^2)^2+\frac{k}{2}\right)|v(t)|^2_{H_0}, \quad t \in \mathbb{R}.$$

Choose $k = k(\varepsilon_0) \in]0, \infty[$ such that $(-(1/2) + (k\varepsilon_0/2)) < 0$. Hence we obtain

$$\alpha'(t) + k\alpha(t) \le C_1(r, \varepsilon_0)|v(t)|^2_{H_0} \quad t \in \mathbb{R}.$$

Using Propositions 3.4 and 3.2 we see that

$$\int_{t_0}^{t} |v(s)|^2_{H_0} \le C_2(r, \varepsilon_0), \quad t, t_0 \in \mathbb{R},\ t_0 \le t.$$

It follows that

$$\begin{aligned} \alpha(t) &= e^{-k(t-t_0)}\alpha(t_0) + C_1(r, \varepsilon_0) \int_{t_0}^{t} e^{-k(t-s)}|v(s)|^2_{H_0}\, ds \\ &\le e^{-k(t-t_0)}\alpha(t_0) + C_3(r, \varepsilon_0), \quad t, t_0 \in \mathbb{R},\ t_0 \le t. \end{aligned} \tag{3.12}$$

Using the definition of α we thus obtain from (3.12)

$$\begin{aligned} &\frac{1}{2}|v(t)|^2_{H_0} + \frac{1}{2}\varepsilon|w(t)|^2_{H_{-1}} \\ &\qquad \le e^{-k(t-t_0)}\left(\frac{1}{2}|v(t_0)|^2_{H_0} + \frac{1}{2}\varepsilon|w(t_0)|^2_{H_{-1}}\right) + C_3(r, \varepsilon_0), \end{aligned} \tag{3.13}$$

for $t, t_0 \in \mathbb{R}$, $t_0 \le t$. Since for $t \in \mathbb{R}$, $\varepsilon w(t) = -v(t) - \mathbf{A}_1 u(t) + \widehat{f}(u(t))$ in H_{-1}, it follows that

$$\begin{aligned} \varepsilon|w(t)|_{H_{-1}} &\le |v(t)|_{H_{-1}} + |u(t)|_{H_1} + |\widehat{f}(u(t))|_{H_{-1}} \\ &\le |v(t)|_{H_{-1}} + C_5(r) \le C_6(r)\varepsilon^{-1/2} + C_5(r), \quad t \in \mathbb{R}. \end{aligned}$$

Thus

$$\varepsilon|w(t_0)|^2_{H_{-1}} \le (1/\varepsilon)(C_6(r)\varepsilon^{-1/2} + C_5(r))^2. \tag{3.14}$$

Furthermore,

$$|v(t_0)|^2_{H_0} \le r/\varepsilon. \tag{3.15}$$

Inserting (3.14) and (3.15) into (3.13) and letting $t_0 \to -\infty$ we thus see that

$$|v(t)|^2_{H_0} + \varepsilon|w(t)|^2_{H_{-1}} \le 2C_3(r, \varepsilon_0), \quad t \in \mathbb{R}.\square$$

Fix a C^∞-function $\overline{\vartheta}\colon \mathbb{R} \to [0, 1]$ with $\overline{\vartheta}(s) = 0$ for $s \in]-\infty, 1]$ and $\overline{\vartheta}(s) = 1$ for $s \in [2, \infty[$. Let

$$\vartheta := \overline{\vartheta}^2.$$

For $k \in \mathbb{N}$ let the functions $\overline{\vartheta}_k\colon \mathbb{R}^N \to \mathbb{R}$ and $\vartheta_k\colon \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\overline{\vartheta}_k(x) = \overline{\vartheta}(|x|^2/k^2) \quad \text{and} \quad \vartheta_k(x) = \vartheta(|x|^2/k^2), \quad x \in \mathbb{R}^N.$$

The following theorem (actually a rephrasing of Theorem 4.4 in [17]) provides the "tail-estimates" mentioned in the Introduction:

Theorem 3.7. *Let Assumptions* 1.1 *and* 1.2 *be satisfied. Let* $\varepsilon_0 > 0$ *be fixed. Choose* δ *and* $\nu \in]0,\infty[$ *with*

$$\nu \le \min(1,\overline{\mu}/2), \quad \lambda_1 - \delta > 0 \quad \text{and} \quad 1 - 2\delta\varepsilon_0 \ge 0.$$

Under these hypotheses, there is a constant $c' \in [0,\infty[$ *and for every* $R \in [0,\infty[$ *there are constants* $M' = M'(R)$, $c_k = c_k(R) \in [0,\infty[$, $k \in \mathbb{N}$ *with* $c_k \to 0$ *for* $k \to \infty$, *such that for every* $\tau_0 \in [0,\infty[$, *every* ε, $0 < \varepsilon \le \varepsilon_0$ *and every solution* $z(\,\cdot\,)$ *of* π_ε *on* $I = [0,\tau_0]$ *with* $|z(0)|_Z \le R$

$$|z_1(t)|^2_{H_1} + \varepsilon|z_2(t)^2|_{H_0} \le c' + M'e^{-2\delta\nu t}, \quad t \in I.$$

If $|z(t)|_Z \le R$ *for* $t \in I$, *then*

$$|\vartheta_k z_1(t)|^2_{H_1} + \varepsilon|\vartheta_k z_2(t)^2|_{H_0} \le c_k + M'e^{-2\delta\nu t}, \quad k \in \mathbb{N},\ t \in I.$$

Now we can prove the following fundamental result:

Theorem 3.8. *Let* $(\varepsilon_n)_n$ *be a sequence of positive numbers converging to* 0. *For each* $n \in \mathbb{N}$ *let* $z_n = (u_n, v_n)\colon \mathbb{R} \to H_1 \times H_0$ *be a solution of* π_{ε_n} *such that*

$$\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} (|u_n(t)|^2_{H_1} + \varepsilon_n |v_n(t)|^2_{H_0}) \le r < \infty.$$

Then, for every $\alpha \in]0,1]$, *a subsequence of* $(z_n)_n$ *converges in* $H_1 \times H_{-\alpha}$, *uniformly on compact subsets of* $\mathbb{R}$, *to a function* $z\colon \mathbb{R} \to H_1 \times H_0$ *with* $z = (u,v)$, *where* u *is a solution of* $\widetilde{\pi}$ *and* $v = \partial(u;H_0)$.

Proof. We may assume that $\varepsilon_n \in]0,\varepsilon_0]$ for some $\varepsilon_0 \in]0,\infty[$ and all $n \in \mathbb{N}$. Write $u_n = z_{n,1}$ and $v_n = z_{n,2}$, and $n \in \mathbb{N}$. We claim that for every $t \in \mathbb{R}$, the set $\{u_n(t) \mid n \in \mathbb{N}\}$ is relatively compact in H_0. Let ϑ_k, $k \in \mathbb{N}$, be as above. Then, choosing $k \in \mathbb{N}$ large enough and using Theorem 3.7 we can make $\sup_{n\in\mathbb{N}} |\vartheta_k u_n(t)|_{H_1}$ as small as we wish. Therefore, by a Kuratowski measure of noncompactness argument, we only have to prove that for every $k \in \mathbb{N}$, the set $S_k = \{(1-\vartheta_k)u_n(t) \mid n \in \mathbb{N}\}$ is relatively compact in H_0. Let U be the ball in $\mathbb{R}^N$ with radius $2k$ centered at zero. Then $(1-\vartheta_k)|U \in C^1_0(U)$, so $(1-\vartheta_k)\widetilde{u}_n(t)|U \in H^1_0(U)$ for $n \in \mathbb{N}$. Since $H^1_0(U)$ imbeds compactly in $L^2(U)$ and $(1-\vartheta_k)\widetilde{u}_n(t)|(\mathbb{R}^N \setminus U) \equiv 0$, it follows that, indeed, S_k is relatively compact in H_0. This proves our claim.

Since, by Proposition 3.6, for each $n \in \mathbb{N}$, u_n is differentiable into H_0 and $v_n = \partial(u_n;H_0)$ is bounded in H_0 uniformly $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we may assume, using the above claim and Arzelà-Ascoli theorem, and taking subsequences if necessary, that $(u_n)_n$ converges in H_0, uniformly on compact subsets of $\mathbb{R}$, to a continuous function $u\colon \mathbb{R} \to H_0$. Moreover, since, for each $t \in \mathbb{R}$, $(u_n(t))_n$ has a subsequence that is weakly convergent in H_1, we see that u takes its values in H_1. Let $w_n = \partial(v; H_{-1})$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ and every $t \in \mathbb{R}$,

$$\varepsilon_n w_n(t) = -v_n(t) - \mathbf{A}_0 u_n(t) + \widehat{f}(u_n(t)) \tag{3.16}$$

in H_{-1}. Now, uniformly for t lying in compact subsets of $\mathbb{R}$, $\widehat{f}(u_n(t)) \to \widehat{f}(u(t))$ in H_{-1} (by Proposition 3.2), $\mathbf{A}_0 u_n(t) \to \mathbf{A}_0 u(t)$ in H_{-2} and $\varepsilon_n w_n(t) \to 0$ in H_{-1} (by Proposition 3.6). It follows from (3.16) that, uniformly for t in compact subsets of $\mathbb{R}$, $v_n(t) \to v(t)$ in H_{-2}, where $v\colon \mathbb{R} \to H_{-2}$ is a continuous map such that, for every $t \in \mathbb{R}$,

$$v(t) = -\mathbf{A}_0 u(t) + \widehat{f}(u(t))$$

in H_{-2}. It follows that u is differentiable into H_{-2} and $v = \partial(u; H_{-2})$. Then u is differentiable into H_{-3} and, for all $t \in \mathbb{R}$,

$$\partial(u; H_{-3})(t) = -\mathbf{A}_{-1} u(t) + \widehat{f}(u(t))$$

in H_{-3}. Since $\widehat{f} \circ u$ is continuous into $D(\mathbf{A}_{-1}) = H_{-1}$ it follows that

$$\begin{aligned} u(t) &= e^{-\mathbf{A}_{-1}(t-t_0)} u(t_0) + \int_{t_0}^{t} (e^{-\mathbf{A}_{-1}(t-s)} \widehat{f}(u(s)); H_{-3})\, ds \\ &= e^{-\mathbf{A}_{1}(t-t_0)} u(t_0) + \int_{t_0}^{t} (e^{-\mathbf{A}_{1}(t-s)} \widehat{f}(u(s)); H_{-1})\, ds, \end{aligned} \tag{3.17}$$

for $t, t_0 \in \mathbb{R}$, $t_0 \le t$. We claim that u is a solution of $\widetilde{\pi}$. To this end let $t_0 \in \mathbb{R}$ be arbitrary. Let $\widetilde{u}\colon [0, \infty[\to H_1$ be the solution of $\widetilde{\pi}$ with $\widetilde{u}(0) = u(t_0)$ ($\widetilde{u}$ exists by results in [16]). We must show that $\widetilde{u}(s) = u(s + t_0)$ for all $s \in [0, \infty[$. If not, then there is a $s_0 \ge 0$ with $\widetilde{u}(s_0) = u(s_0 + t_0)$ and $\widetilde{u}(s_n) \ne u(s_n + t_0)$ for all $n \in \mathbb{N}$, where $(s_n)_n$ is a sequence with $s_n \to s_0^+$ as $n \to \infty$. By Corollary 2.3 there is a constant $C \in [0, \infty[$ such that

$$|e^{-\mathbf{A}_1 t} w|_{H_0} \le C t^{-1/2} |w|_{H_{-1}}, \quad w \in H_{-1},\ t \in\,]0, \infty[\,.$$

Moreover, by Proposition 3.2, for every $b \in\,]0, \infty[$ there is an $L(b) \in\,]0, \infty[$ such that for all $w_i \in H_1$, $|w_i|_{H_1} \le b$, $i = 1$, 2,

$$|\widehat{f}(w_2) - \widehat{f}(w_1)|_{H_{-1}} \le L(b) |w_2 - w_1|_{H_0}.$$

There is an $\overline{s} \in\,]s_0, \infty[$ such that whenever $s \in [s_0, \overline{s}]$ then $|u(s + t_0)|_{H_1} < r + 1$ and $|\widetilde{u}(s)|_{H_1} < r + 1$. Let $L = L(b)$ where $b = r + 1$. Choosing $\overline{s}$ smaller, if necessary, we can assume that

$$C L (\overline{s} - s_0)^{1/2} / 2 < 1. \tag{3.18}$$

It follows that, for each $s \in [s_0, \overline{s}]$,

$$u(s + t_0) - \widetilde{u}(s) = \int_{s_0}^{s} e^{-\mathbf{A}_1 (s-r)} [\widehat{f}(u(r + t_0)) - \widehat{f}(\widetilde{u}(r))]\, dr$$

so

$$|u(s+t_0)-\widetilde{u}(s)|_{H_0} \le C\int_{s_0}^{s}(s-r)^{-1/2}L[|u(r+t_0)-\widetilde{u}(r)|_{H_0}]\,dr$$
$$\le CL(\overline{s}-s_0)^{1/2}/2 \sup_{r\in[s_0,\overline{s}]}|u(r+t_0)-\widetilde{u}(r)|_{H_0}.$$

In view of (3.18), we obtain that $u(s+t_0)=\widetilde{u}(s)$ for $s\in[s_0,\overline{s}]$, a contradiction, which proves our claim.

We now claim that $u_n(t)\to u(t)$ in H_1, uniformly for t lying in compact subsets of $\mathbb{R}$. If this claim is not true, then there is a strictly increasing sequence $(n_k)_n$ in $\mathbb{N}$ and a sequence $(t_k)_k$ in $\mathbb{R}$ converging to some $t_\infty\in\mathbb{R}$ such that

$$\inf_{k\in\mathbb{N}}|u_{n_k}(t_k)-u(t_\infty)|_{H_1}>0. \tag{3.19}$$

For $\varepsilon\in\,]0,\infty[$ define the function $\mathcal{F}_\varepsilon\colon H_1\times H_0\to\mathbb{R}$ by

$$\mathcal{F}_\varepsilon(z)=\frac{1}{2}\varepsilon\langle\delta z_1+z_2,\delta z_1+z_2\rangle+\frac{1}{2}\langle A\nabla z_1,\nabla z_1\rangle$$
$$+\frac{1}{2}\langle(\beta-\delta+\delta^2\varepsilon)z_1,z_1\rangle-\int_\Omega F(x,z_1(x))\,dx$$

where $\delta\in\,]0,\infty[$ is such that $\lambda_1-\delta>0$ and $1-2\delta\varepsilon_0>0$. Note that

$$\|u\|^2=\langle A\nabla u,\nabla u\rangle+\langle(\beta-\delta)u,u\rangle,\quad u\in H_1$$

defines a norm on H_1 equivalent to the usual norm on H_1. Let $\varepsilon\in\,]0,\varepsilon_0]$ and $\zeta=(\zeta_1,\zeta_2)\colon[0,\infty[\,\to Z$ be an arbitrary solution of π_ε. Using Theorem 2.4 (cf. [17, Proposition 4.1]) one can see that the function $\mathcal{F}_\varepsilon\circ\zeta$ is continuously differentiable and for every $t\in[0,\infty[$

$$(\mathcal{F}_\varepsilon\circ\zeta)'(t)+2\delta\mathcal{F}_\varepsilon(\zeta(t))=\int_\Omega(2\delta\varepsilon-1)(\delta\zeta_1(t)(x)+\zeta_2(t)(x))^2\,dx \tag{3.20}$$
$$+\int_\Omega\delta\zeta_1(t)(x)f(x,\zeta_1(t)(x))\,dx-2\delta\int_\Omega F(x,\zeta_1(t)(x))\,dx.$$

Moreover, define $\mathcal{F}_0\colon H_1\to\mathbb{R}$ by

$$\mathcal{F}_0(u)=\frac{1}{2}\langle A\nabla u,\nabla u\rangle+\frac{1}{2}\langle(\beta-\delta)u,u\rangle-\int_\Omega F(x,u(x))\,dx,\quad u\in H_1.$$

Every solution $\xi\colon\mathbb{R}\to H_1$ of $\widetilde{\pi}$ is differentiable into H_1 so the function $\mathcal{F}_0\circ\xi$ is differentiable and a simple computation shows that for $t\in\mathbb{R}$,

$$(\mathcal{F}_0\circ\xi)'(t)+2\delta(\mathcal{F}_0\circ\xi)(t)=-\langle\delta\xi(t)+\eta(t),\delta\xi(t)+\eta(t)\rangle \tag{3.21}$$
$$+\int_\Omega[\delta\xi(t)(x)f(x,\xi(t)(x))-2\delta F(x,\xi(t)(x))]\,dx$$

where $\eta(t)=-\mathbf{A}_1\xi(t)+\widehat{f}(\xi(t))$, $t\in\mathbb{R}$.

Fix $l \in \mathbb{N}$ and, for $k \in \mathbb{N}$, let $\zeta_k(t) = z_{n_k}(t_k - l + t)$ and $\zeta(t) = (u(t_\infty - l + t), v(t_\infty - l + t)$ for $t \in [0, \infty[$. Then (3.20) and (3.21) imply that

$$\begin{aligned}\mathcal{F}_{\varepsilon_{n_k}}(z_{n_k}(t_k)) &= e^{-2\delta l}\mathcal{F}_{\varepsilon_{n_k}}(z_{n_k}(t_k - l)) \\ &+ (2\delta\varepsilon_{n_k} - 1)\int_0^l e^{-2\delta(l-s)}\left(\int_\Omega (\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2\,dx\right) ds \\ &+ \int_0^l e^{-2\delta(l-s)}\rho_k(s)\,ds\end{aligned} \tag{3.22}$$

where

$$\rho_k(s) = \int_\Omega \delta\zeta_{k,1}(s)(x) f(x, \zeta_{k,1}(s)(x))\,dx - 2\delta\int_\Omega F(x, \zeta_{k,1}(s)(x))\,dx, \quad s \in [0, l]$$

and

$$\begin{aligned}\mathcal{F}_0(u(t_\infty)) = e^{-2\delta l}\mathcal{F}_0(u(t_\infty - l)) &\\ - \int_0^l e^{-2\delta(l-s)}&\left(\int_\Omega (\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2\,dx\right) ds \\ + \int_0^l e^{-2\delta(l-s)}&\left(\int_\Omega \delta\zeta_1(s)(x) f(x, \zeta_1(s)(x))\,dx\right. \\ &\left. - 2\delta\int_\Omega F(x, \zeta_1(s)(x))\,dx\right) ds.\end{aligned} \tag{3.23}$$

Since $\zeta_{k,1}(s) \to \zeta_1(s)$ in H_0, uniformly for s lying in compact subsets of $\mathbb{R}$, we obtain from Proposition 3.2 that

$$\begin{aligned}&\int_0^l e^{-2\delta(l-s)}\left(\int_\Omega \delta\zeta_{k,1}(s)(x) f(x, \zeta_{k,1}(s)(x))\,dx\right. \\ &\qquad\qquad\left. - 2\delta\int_\Omega F(x, \zeta_{k,1}(s)(x))\,dx\right) ds \\ &\to \int_0^l e^{-2\delta(l-s)}\left(\int_\Omega \delta\zeta_1(s)(x) f(x, \zeta_1(s)(x))\,dx\right. \\ &\qquad\qquad\left. - 2\delta\int_\Omega F(x, \zeta_1(s)(x))\,dx\right) ds\end{aligned} \tag{3.24}$$

as $k \to \infty$. We claim that

$$\begin{aligned}\limsup_{k\to\infty}(2\delta\varepsilon_{n_k} - 1)&\int_0^l e^{-2\delta(l-s)}\left(\int_\Omega (\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2\,dx\right) ds \\ &\le -\int_0^l e^{-2\delta(l-s)}\left(\int_\Omega (\delta\zeta_1(s)(x) + \zeta_2(s)(x))^2\,dx\right) ds.\end{aligned} \tag{3.25}$$

In fact, since $1 - 2\delta\varepsilon_{n_k} \ge 0$ for all $k \in \mathbb{N}$ we have by Fatou's lemma

$$\limsup_{k\to\infty}(2\delta\varepsilon_{n_k} - 1)\int_0^l e^{-2\delta(l-s)}\left(\int_\Omega (\delta\zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2\,dx\right) ds \tag{3.26}$$

$$
\begin{aligned}
&= -\liminf_{k\to\infty}(1-2\delta\varepsilon_{n_k})\int_0^l e^{-2\delta(l-s)}\bigg(\int_\Omega(\delta\zeta_{k,1}(s)(x)+\zeta_{k,2}(s)(x))^2\,dx\bigg)\,ds \\
&= -\liminf_{k\to\infty}\int_0^l e^{-2\delta(l-s)}\bigg(\int_\Omega(\delta\zeta_{k,1}(s)(x)+\zeta_{k,2}(s)(x))^2\,dx\bigg)\,ds \\
&\le -\int_0^l e^{-2\delta(l-s)}\liminf_{k\to\infty}\bigg(\int_\Omega(\delta\zeta_{k,1}(s)(x)+\zeta_{k,2}(s)(x))^2\,dx\bigg)\,ds.
\end{aligned}
$$

Let $s\in[0,l]$ be arbitrary. Since $((\zeta_{k,1}(s),\zeta_{k,2}(s)))_k$ converges to $(\zeta_1(s),\zeta_2(s))$ weakly in $H_1\times H_0$ it follows that $((\zeta_{k,1}(s),\delta\zeta_{k,1}(s)+\zeta_{k,2}(s)))_k$ converges to $(\zeta_1(s),\delta\zeta_1(s)+\zeta_2(s))$ weakly in $H_1\times H_0$. It follows that for every $v\in L^2(\Omega)$

$$
\langle v,\delta\zeta_{k,1}(s)+\zeta_{k,2}(s)\rangle\to\langle v,\delta\zeta_1(s)+\zeta_2(s)\rangle\quad\text{as } k\to\infty.
$$

Taking $v=(\delta\zeta_1(s)+\delta\zeta_2(s))$ we thus obtain

$$
\begin{aligned}
|(\delta\zeta_1(s)+\delta\zeta_2(s))|_{L^2}^2 &= \langle(\delta\zeta_1(s)+\delta\zeta_2(s)),(\delta\zeta_1(s)+\delta\zeta_2(s))\rangle \\
&= \lim_{k\to\infty}\langle(\delta\zeta_1(s)+\delta\zeta_2(s)),(\delta\zeta_{k,1}(s)+\delta\zeta_{k,2}(s))\rangle \\
&\le |(\delta\zeta_1(s)+\delta\zeta_2(s))|_{L^2}\liminf_{k\to\infty}|(\delta\zeta_{k,1}(s)+\delta\zeta_{k,2}(s))|_{L^2}
\end{aligned}
$$

and so

$$
\int_\Omega(\delta\zeta_1(s)(x)+\zeta_2(s)(x))^2\,dx\le\liminf_{k\to\infty}\int_\Omega(\delta\zeta_{k,1}(s)(x)+\zeta_{k,2}(s)(x))^2\,dx. \tag{3.27}
$$

Inequalities (3.27) and (3.26) prove (3.25). Since, by Proposition 3.2,

$$
\int_\Omega F(x,u_{n_k}(t_k)(x))\,dx\to\int_\Omega F(x,u(t_\infty)(x))\,dx
$$

we obtain, using Proposition 3.6, that

$$
\limsup_{k\to\infty}\mathcal{F}_{\varepsilon_{n_k}}(z_{n_k}(t_k))=(1/2)\limsup_{k\to\infty}\|u(t_k)\|^2-\int_\Omega F(x,u(t_\infty)(x))\,dx
$$

Moreover, there is a constant $C'\in\,]0,\infty[$ such that

$$
\sup_{k\in\mathbb{N}}\sup_{t\in\mathbb{R}}|\mathcal{F}_{\varepsilon_{n_k}}(z_{n_k}(t))|+\sup_{t\in\mathbb{R}}|\mathcal{F}_0(u(t))|\le C'.
$$

Thus

$$
\begin{aligned}
&\frac{1}{2}\limsup_{k\to\infty}\|u(t_k)\|^2-\int_\Omega F(x,u(t_\infty)(x))\,dx\le e^{-2\delta l}C' \\
&\quad-\int_0^l e^{-2\delta(l-s)}\bigg(\int_\Omega(\delta\zeta_1(s)(x)+\zeta_2(s)(x))^2\,dx\bigg)\,ds \\
&\quad+\int_0^l e^{-2\delta(l-s)}\bigg(\int_\Omega\delta\zeta_1(s)(x)f(x,\zeta_1(s)(x))\,dx-2\delta\int_\Omega F(x,\zeta_1(s)(x))\,dx\bigg)\,ds \\
&=e^{-2\delta l}C'+(1/2)\|u(t_\infty)\|^2-\int_\Omega F(x,u(t_\infty)(x))\,dx-e^{-2\delta l}\mathcal{F}_0(u(t_\infty-l))
\end{aligned}
$$

$$\leq 2e^{-2\delta l}C' + (1/2)\|u(t_\infty)\|^2 - \int_\Omega F(x, u(t_\infty)(x))\,dx.$$

Thus, for every $l \in \mathbb{N}$, $\limsup_{k\to\infty} \|u(t_k)\|^2 \leq 4e^{-2\delta l}C' + \|u(t_\infty)\|^2$ so

$$\limsup_{k\to\infty} \|u(t_k)\| \leq \|u(t_\infty)\|.$$

Since $(u_{n_k}(t_{n_k}))_k$ converges to $u(t_\infty)$ weakly in H_1 we have

$$\liminf_{k\to\infty} \|u_{n_k}(t_{n_k})\| \geq \|u(t_\infty)\|.$$

Altogether we obtain

$$\lim_{k\to\infty} \|u_{n_k}(t_{n_k})\| = \|u(t_\infty)\|.$$

This implies that $(u_{n_k}(t_{n_k}))_k$ converges to $u(t_\infty)$ strongly in H_1, a contradiction to (3.19). Thus, indeed, $u_n(t) \to u(t)$ in H_1, uniformly for t lying in compact subsets of $\mathbb{R}$.

Now (3.16) implies that $v_n(t) \to v(t)$ in H_{-1}, uniformly for t lying in compact subsets of $\mathbb{R}$. Since $(v_n)_n$ is bounded in H_0, interpolation between H_0 and H_{-1} (cf. [16]) now implies that $v_n(t) \to v(t)$ in $H_{-\alpha}$, uniformly for t lying in compact subsets of $\mathbb{R}$. The proof is complete. □

Now we obtain the main result of this paper.

THEOREM 3.9. *For every $\alpha \in]0, 1]$ the family $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,\infty[}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the topology of $H_1 \times H_{-\alpha}$, i.e.*

$$\lim_{\varepsilon\to 0^+} \sup_{y\in\mathcal{A}_\varepsilon} \inf_{z\in\mathcal{A}_0} |y - z|_{H_1\times H_{-\alpha}} = 0.$$

PROOF. Using the first part of Theorem 3.7, choosing $\varepsilon_0 \in]0, \infty[$ arbitrarily and $\delta \in]0, \infty[$ such that $\lambda_1 - \delta > 0$ and $1 - 2\delta\varepsilon_0 > 0$ and noting that the constant c' in that theorem is independent of $\varepsilon \in]0, \varepsilon_0]$, it follows that, for all $\varepsilon \in]0, \varepsilon_0]$ and all $(u, v) \in \mathcal{A}_\varepsilon$, $|u|^2_{H_1} + \varepsilon|v|^2_{H_0} \leq 2c'$. Now an obvious contradiction argument using Theorem 3.8 completes the proof of our main result. □

REMARK. Theorem 3.9 and Corollary 2.2 imply Theorem 1.4.

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