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ON THE SOLUTION OF STOCHASTIC OSCILLATORY QUADRATIC NONLINEAR EQUATIONS USING DIFFERENT TECHNIQUES, A COMPARISON STUDY

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Abstract. In this paper, nonlinear oscillators under quadratic nonlinearity with stochastic inputs are considered. Different methods are used to obtain first order approximations, namely the WHEP technique, the perturbation method, the Pickard approximations, the Adomian decompositions and the homotopy perturbation method (HPM). Some statistical moments are computed for the different methods using Mathematica 5. Comparisons are illustrated by figures for different case-studies.

1. Introduction

Quadratic oscillation arises through many applied models in applied sciences and engineering when studying oscillatory systems [25]. These systems can be exposed to a lot of uncertainties through the external forces, the damping coefficient, the frequency and/or the initial or boundary conditions. These input uncertainties cause the output solution process to be also uncertain. For most of the cases, getting the probability density function (p.d.f.) of the solution process may be impossible. So, developing approximate techniques through which

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approximate statistical moments can be obtained, is an important and necessary work. There are many techniques which can be used to obtain statistical moments of such problems. The main goal of this paper is to compare some of these methods when applied to the problem with quadratic nonlinearity.

2. Problem formulation

In this paper, the following quadratic nonlinear oscillatory equation is considered as a comparison prototype equation for the application of the different solution techniques:

$$\ddot{x}(t;\omega) + 2w\xi\dot{x} + w^2x + \varepsilon w^2x^2 = F(\omega; t \in [0,T] \tag{2.1}$$

under stochastic excitation $F(t;\omega)$ with deterministic initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,$$

where

w — frequency of oscillation,
ξ — damping coefficient,
ε — deterministic nonlinearity scale,
$\omega \in (\Omega, \sigma, P)$ — a triple probability space with Ω as the sample space, σ is a σ-algebra of events in Ω and P is a probability measure.

Lemma 2.1. *The solution of equation* (2.1)*, if exists, then it is a power series of* ε*.*

Proof. Rewriting equation (2.1), it can take the following form

$$\ddot{x}(t;\omega) + 2w\xi\dot{x} + w^2x = F(t) - \varepsilon w^2x^2$$

Following Pickard approximation, the equation can be rewritten as

$$\ddot{x}_{n+1}(t) + 2w\xi\dot{x}_{n+1} + w^2x_1 = F(t) - \varepsilon w^2x_n^2, \quad n \geq 0$$

where the solution at $n = 0$, x_0, is corresponding for the simple linear case at $\varepsilon = 0$. At $n = 1$, the iteration takes the form:

$$\ddot{x}_1(t) + 2w\xi\dot{x}_1 + w^2x_1 = F(t) - \varepsilon W^2x_0^2,$$

which has the following general solution

$$x_1t) = \psi(t) - \varepsilon w^2\int_0^t h(t-s)x_0^2(s)\,ds,$$

or

$$x_1(t) = x_1^{(0)} + \varepsilon x_1^{(1)}.$$

At $n = 2$, the iteration takes the form:

$$\ddot{x}_2(t) + 2w\xi\dot{x}_2 + w^2x_2 = F(t) - \varepsilon w^2x_1^2,$$

which has the following general solution

$$x_2(t) = x_2^{(0)} + \varepsilon x_2^{(1)} + \varepsilon^2 x_2^{(2)} + \varepsilon^3 x_2^{(3)}.$$

Proceeding like this, one can get the following

$$x_n(t) = x_n^{(0)} + \varepsilon x_n^{(1)} + \varepsilon^2 x_n^{(2)} + \varepsilon^3 x_n^{(3)} + \cdots + \varepsilon^{n+m} x_n^{(n+m)}.$$

Assuming the solution exists, it will be

$$x(t) = \lim_{n\to\infty} x_n(t) = \sum_{j=0}^{\infty} \varepsilon^j x_j,$$

which is a power series of ε.

As a direct result of this lemma, it is expected that the average, the variance as well as the covariance are also power series of ε.

3. WHEP technique

Since Meecham and his co-workers [3] developed a theory of turbulence involving a truncated Wiener–Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [4], [15], [20], [21], [30] and [35]. A lot of general applications in fluid mechanics were also studied in [2], [19], [23]. Scattering problems attracted the WHE applications through many authors [5], [8], [31], [33], [34]. The nonlinear oscillators were considered as an opened area for the applications of WHE as can be found in [1], [7], [9], [18], [26]–[28]. There are a lot of applications in boundary value problems [6], [32] and generally in different mathematical studies [22], [29].

The application of the WHE aims at finding a truncated series solution to the solution process of differential equations. The truncated series composes of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, there always exist difficulties of solving the resultant set of deterministic integro-differential equations received from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [9] using the perturbation technique to solve perturbed nonlinear problems.

The WHE method uses the Wiener–Hermite polynomials which are the elements of a complete set of statistically orthogonal random functions [16]. The

Wiener–Hermite polynomial $H^{(i)}(t_1, \dots, t_i)$ satisfies the following recurrence relation:

$$
\begin{aligned}
H^{(i)}(t_1, \dots, t_i) = H^{(i-1)}&(t_1, \dots, t_{i-1}) \cdot H^{(1)}(t_i) \\
&- \sum_{m=1}^{i-1} H^{i-2)}(T_{i_1}, \dots, t_{i_i-2}) \cdot \delta(t_{i-m} - t_i), \quad i \geq 2
\end{aligned}
$$

where

$$
\begin{aligned}
H^{(0)} &= 1, \\
H^{(1)}(t) &= n(t), \\
H^{(2)}(t_1, t_2) &= H^{(1)}(t_1) \cdot H^{(1)}(t_2) - \delta(t_1 - t_2), \\
H^{(3)}(t_1, t_2, t_3) &= H^{(2)}(t_1, t_2) \cdot H^{(1)}(t_3) \\
&\quad - H^{(1)}(t_1) \cdot \delta(t_2 - t_3) - H^{(1)}(T_2) \cdot \delta(t_1 - t_3), \\
H^{(4)}(t_1, t_2, t_3, t_4) &= H^{(3)}(t_1, t_2, t_3) \cdot H^{(1)}(T_4) - H^{(2)}(t_1, t_2) \cdot \delta(T_3 - t_4) \\
&\quad - H^{(2)}(t_1, t_3) \cdot \delta(t_2 - t_4) - H^{(2)}(T_2, t_3) \cdot \delta(t_1 - t_4),
\end{aligned}
$$

in which $n(t)$ is the white noise with the following statistical properties where $\delta(\,\cdot\,)$ is the Dirac delta function and E denotes the expectation. The Wiener–Hermite set is a statistically orthogonal set, i.e.

$$
\begin{aligned}
En(t) &= 0, \\
En(t_1) \cdot n(t_2) &= \delta(t_1 - t_2), \\
EH^{(i)} \cdot H^{(j)} &= 0 \quad \text{for all } i \neq j.
\end{aligned}
$$

The expectation of almost all H functions vanishes, particularly,

$$
EH^{(i)} = 0 \quad \text{for } i \geq 1.
$$

Due to the completeness of the Wiener–Hermite set, any random function $G(t;\omega)$ can be expanded as

$$
\begin{aligned}
G(t;\omega) = G^{(0)}(t) + &\int_{-\infty}^{\infty} G^{(1)}(t;t_1) H^{(1)}(t_1)\, dt_1 \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(2)}(t;t_1,t_2) H^{(2)}(t_1,t_2)\, dt_1\, dt_2 + \dots
\end{aligned}
$$

where the first two terms are the Gaussian part of $G(t;\omega)$. The rest of the terms in the expansion represent the non-Gaussian part of $G(t;\omega)$. The expectation of $G(t;\omega)$ is

$$
\mu_G = EG(t;\omega) = G^{(0)}(t).
$$

The covariance of $G(t;\omega)$ is

$$\begin{aligned}\operatorname{Cov}(G(t;\omega),G(\tau;\omega)) &= E(G(t;\omega)-\mu_G(t))(G(\tau;\omega)-\mu_G(\tau))\\ &= \int_{-\infty}^{\infty} G^{(1)}(t,t_1)G^{(1)}(\tau,t_1)\,dt_1\\ &\quad + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{(2)}(t;t_1,t_2)G^{(2)}(\tau,t_1,t_2)\,dt_1\,dt_2.\end{aligned}$$

The variance of $G(t,\omega)$ is

$$\begin{aligned}\operatorname{Var} G(t;\omega) = E(G(t;\omega)-\mu_G(t))^2 = \int_{-\infty}^{\infty} &[G^{(1)}(t;t_1)]^2\,dt_1\\ &+ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[G^{(2)}(t;t_1,t_2)]^2\,dt_1\,dt_2.\end{aligned}$$

The WHEP technique can be applied to linear or nonlinear perturbed systems described by ordinary or partial differential equations. The solution can be modified in the sense that additional parts of the Wiener–Hermite expansion can always be taken into consideration and the required order of approximations can always be made depending on the computing tool. It can be even run through a package if it is coded in some sort of symbolic languages. The technique was successfully applied to several nonlinear stochastic equations, see [1], [6], [7] and [9].

3.1. Case study. The quadratic nonlinear oscillatory problem, equation (2.1), is solved using WHEP technique. The first order approximation of the solution process takes the following form:

$$x(t;\omega) = x^{(0)}(t) + \int_{-\infty}^{\infty} x^{(1)}(t;t_1)H^{(1)}(t_1)\,dt_1.$$

Applying the WHEP technique, the following equations in the deterministic kernels are obtained:

$$\begin{aligned}Lx^{(0)}(t) + \varepsilon w^2(x^{(0)}(t))^2 + \varepsilon w^2\int_{-\infty}^{\infty} x^{(1)}(t;t_1))^2\,dt_1 &= F^{(0)}(t),\\ Lx^{(1)}(t,t_1) + 2\varepsilon w^2 x^{(0)}(t)x^{(1)}(t,t_1) &= F^{(1)}(t,t_1).\end{aligned}$$

Let us take the simple case of evaluating the only Gaussian part (first order approximation) of the solution process. The expectation is

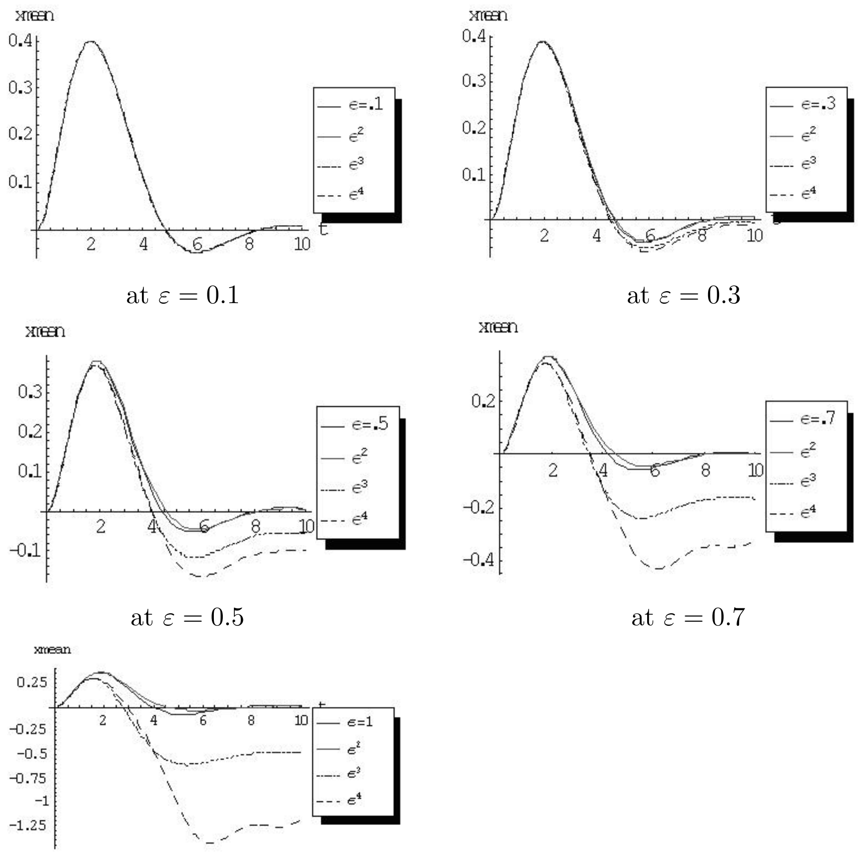
$$\mu_x(t) = x^{(0)}(t),$$

and the variance is

$$\sigma_x^2(t) = \int_{-\infty}^{\infty}[x^{(1)}(t;t_1)]^2\,dt_1.$$

The WHEP technique uses the following expansion for its deterministic kernels,

$$x^{(i)}(t) = x_0^{(i)} + \varepsilon x_1^{(i)} + \varepsilon^2 x_2^{(i)} + \varepsilon^3 x_3^{(i)} + \dots\,, \quad i = 0,1,$$



at $\varepsilon = 0.1$ at $\varepsilon = 0.3$

at $\varepsilon = 0.5$ at $\varepsilon = 0.7$

at $\varepsilon = 1$

Figure 1. The first order approximation of the mean for different correction levels

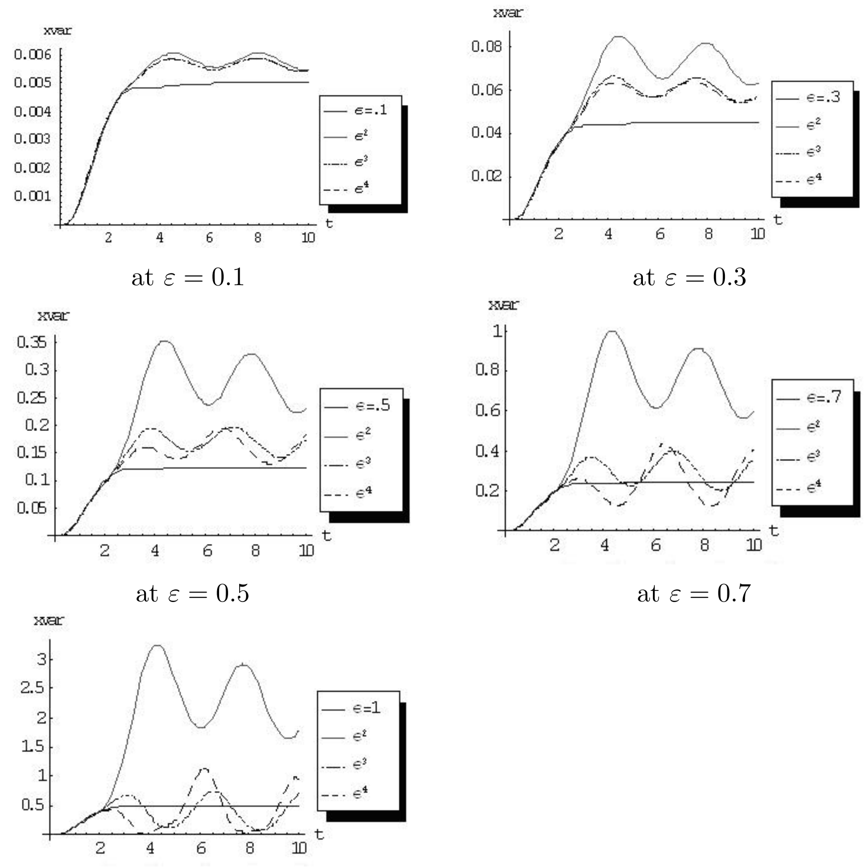
where the first two terms consider the first correction (up to ε), the first three terms represent the second correction (up to ε^2) and so on. This means that we have a lot of corrections possible within each order of approximation.

Example 3.1. Let us take $F(t;\omega) = e^{-i} + \varepsilon n(t;\omega)$, in the previous case-study and then solve it using the WHEP technique. The following results are obtained (see Figures 1–3):

4. The homotopy perturbation method (HPM)

In this technique [10]–[13], a parameter $p \in [0,1]$ is embedded in a homotopy function $v(r,p) \colon \phi \times [0,1] \to \Re$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0$$



at $\varepsilon = 0.1$ at $\varepsilon = 0.3$

at $\varepsilon = 0.5$ at $\varepsilon = 0.7$

at $\varepsilon = 1$

Figure 2. The first order approximation of the variance for different correction levels

where u_0 is an initial approximation to the solution of the equation

$$A(u) - f(r) = 0, \quad r \in \phi \tag{4.1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma$$

in which A is a nonlinear differential operator which can be decompose into a linear operator L and a nonlinear operator N, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of ϕ. The homotopy introduces a continuously deformed solution for the case of $p = 0$, $L(v) - L(u_0)$, to the case of $p = 1$, $A(v) - f(r) = 0$, which is the original equation (4.1). This is the basic idea of the homotopy method which is to continuously deform

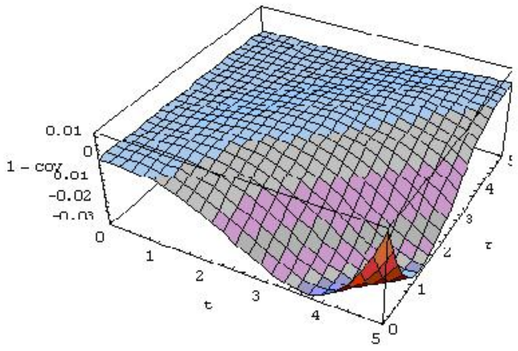


Figure 3. The first order approximation and first correction of the covariance at $\varepsilon = 0.1$

continuously a simple problem (and easy to solve) into the difficult problem under study [14].

The basic assumption of the HPM method is that the solution of the original equation (4.1) can be expanded as a power series in p as:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots$$

Now, setting $p = 1$, the approximate solution of equation (4.1) is obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots$$

The rate of convergence of the method depends greatly on the initial approximation v_0 which is considered as the main disadvantage of HPM.

The idea of the imbedded parameter can be utilized to solve nonlinear problems by imbedding this parameter to the problem and then forcing it to be unity in the obtained approximate solution if convergence can be assured. It is a simple technique which enables the extension of the applicability of the perturbation methods from small value applications to general ones.

Example 4.1. Considering the same previous example as in Subsection 3.1, one can get the following results w.r.t. homotopy perturbation:

$$\begin{aligned} A(x) &= L(x) + \varepsilon w^2 x^2, & L(x) &= \ddot{x} + 2w\xi\dot{x} + w^2 x, \\ N(x) &= \varepsilon x^2, & f(r) &= F(t;\omega). \end{aligned}$$

The homotopy function takes the following form:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0$$

or equivalently,

$$l(v) - L(u_0) + p[L(u_0)\varepsilon w^2 v^2 - F(t;\omega)] = 0. \tag{4.2}$$

Letting $v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots$, substituting in equation (4.1) and equating the equal powers of p in both sides of the equation, one can get the following results:

(a) $L(v_0) = L(Y_0)$, in which one may consider the following simple solution:

$$v_0 = y_0, \quad y_0(0) = x_0, \quad \dot{y}_0(0) = \dot{x}_0.$$

(b) $L(v_1) = F(t,\omega) - L(v_0) - \varepsilon w^2 v_0^2$, $v_1(0) = 0$, $\dot{v}_1(0) = 0$.
(c) $l(v_2) = -2\varepsilon w^2 v_0 v_1$, $v_2(0) = 0$, $\dot{v}_2(0) = 0$.
(d) $L(v_3 - \varepsilon w^2(v_1^2 + 2v_0 v_2)$, $v_3(0) = 0$, $\dot{v}_3(0) = 0$.

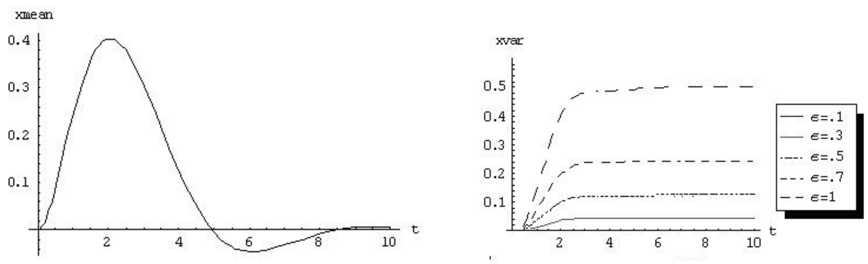
The approximate solution is

$$x(t;\omega) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots$$

One can notice that the algorithm of the solution is straightforward and that many flexibilities can be made. For example, we have many choices in guessing the initial approximation together with its initial conditions which greatly affects the consequent approximations. The following first order approximation expression is:

$$x(t;\omega) \cong x_1 = v_0 + v_1 = v_0 + \int_0^t h(t-s)(F(s;\omega) - L(v_0)(s) - \varepsilon w^2 v_0^2(s)\, ds$$

For zero initial conditions, we can choose $v_0 = 0$ which leads to the following results at $w = 1$ and $\xi = 0.5$ (see Figures 4 and 5):



(a) (b)

Figure 4. The first order approximation (a) of the mean; (b) of the variance at different values of ε

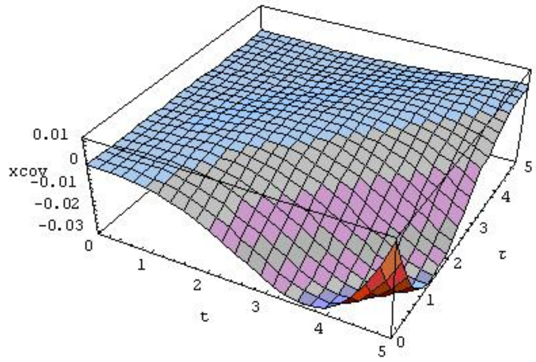
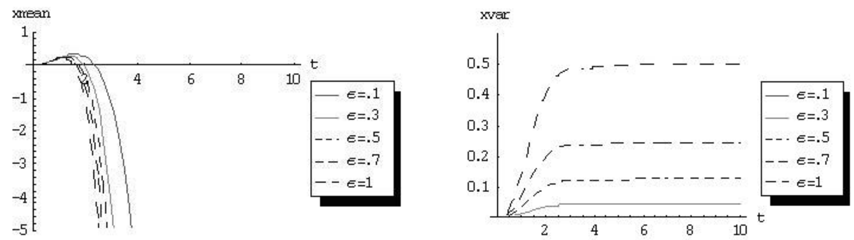


Figure 5. The first order approximation of the covariance at $\varepsilon = 0.1$



(a) (b)

Figure 6. The first order approximation (a) of the mean; (b) of the variance at different values of ε

We can choose $v_0 = t^2$ which leads to the following results at $w = 1$ and $\xi = 0.5$ (see Figure 6).

One can notice high deteriorations in the mean.

5. Pickard approximation

In this technique, the linear part of the differential operator is kept in the left hand side of the equation whereas the rest of the nonlinear terms are moved to the right side. The successive Pickard approximation are processed accordingly to let the L.H.S. as the $n+1$ approximations for the solution process depending on the n-th approximation in the R.H.S., $n \geq 0$. Let us illustrate the method by the following example.

Example 5.1. When solving the quadratic nonlinear oscillatory problem in equation (2.1) while using Pickard technique, the following successive approximations are obtained:

$$Lx_{n+1}(t;\omega) = F(t;\omega) - \varepsilon w^2 x_n^2(t;\omega)$$

which has the general iterative formula:

$$x_{n+1}(t;\omega) = x_{n+1}(0)\phi_1 + x_{n+1}(0)\phi_2 + \int_0^t h(t-s)F(s)\,ds - \varepsilon w^2 \int_0^t h(t-s)x_n^2(s)\,ds \tag{5.1}$$

If the convergence of the process is insured, one can obtain the solution as an ε series in stochastic terms. Following the iterative formula (5.1), the first approximation is

$$x_1(t;\omega) = x_1(0)\phi_1 + x_1(0)\phi_2 + \int_0^1 h(t-s)F(s)\,ds - \varepsilon w^2 \int_0^1 h(t-s)x_0^2(s)\,ds$$

where

$$x_0(t;\omega) = x_0(0)\phi_1 + \dot{x}_0(0)\phi_2 + \int_0^t h(t-s)F(s)\,ds.$$

The expectation is

$$Ex_1(t;\omega) = x_1(0)\phi_1 + x_1(0)\phi_2 + \int_0^t h(t-s)EF(s)\,ds - \varepsilon w^2 \int_0^t h(t-s)Ex_0^2(s)\,ds$$

The covariance is

$$\mathrm{Cov}(x_1(t), x_1(\tau)) \int_0^t \int_0^t h(t-s)h(\tau-z)\mathrm{Cov}(F(s), F(z)\,dz\,ds$$

The variance is

$$\mathrm{Var}\,(x_1(t)) \int_0^t \int_0^t h(t-s)h(t-z)\mathrm{Cov}(F(s), F(z))\,dz\,ds$$

The second approximation is obtained in a similar way.

Let us take $F(t;\omega) = e^{-t} + \varepsilon n(t;\omega)$. In this case, the following results are obtained (see Figure 7):

6. The direct perturbation method

The direct expansion of the solution process is the most conventional and direct one among all the approximation techniques. The basic assumption is

$$x(t;\omega) = x^{(0)}(t;\omega) + \varepsilon x^{(1)}(t;\omega) + \varepsilon^2 x^{(2)}(t;\omega) + \varepsilon^3 x^{(3)}(t;\omega) + \dots$$

Substituting in the original equation (2.1) and equating the equal powers in both sides of the resulting equation one can get a set of linear differential equations to be solved with their corresponding deterministic initial conditions.

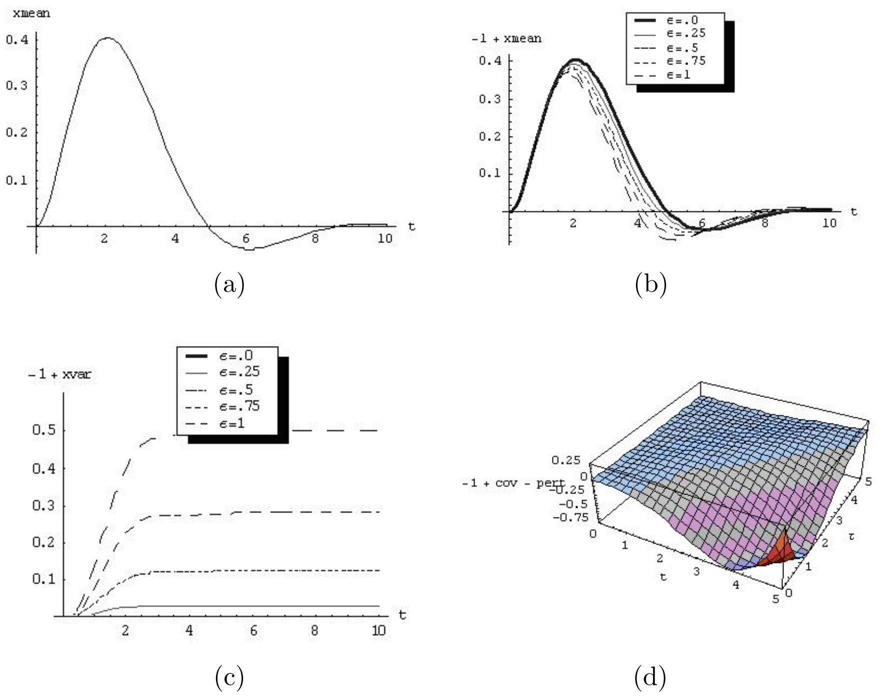


Figure 7. (a) The zero order approximation x_0; (b) the mean of the first order approximation x_1 at different ε; (c) the variance of the first order approximation at different ε; (d) the covariance of first order approximation at $\varepsilon = 0.5$

Example 6.1. While working on the prototype example of this paper, the following results are obtained (see Figure 8):

7. The Adomian decomposition method

In this method, the differential operator is so decomposed that equation (2.1) is rewritten in the following form:

$$Lx(t;\omega) = F(t;\omega) - R(x) - \varepsilon w^2 x^2(t;\omega),$$

where

$$Lx(t;\omega) = \frac{d^2x}{dt^2}, \qquad R(x) = \left(2w\xi\frac{d}{dt} + w^2\right)(x).$$

These decompositions transform the problem into an easier one. The general solution procedure is obtained when using the following:

$$x = x(0) + \dot{x}(0)t + \int_0^t \int_0^t F(t)\,dt\,dt - \int_0^t \int_0^t R(x)\,dt\,dt - \varepsilon w^2 \int_0^t \int_0^t x^2(t)\,dt\,dt \tag{7.1}$$

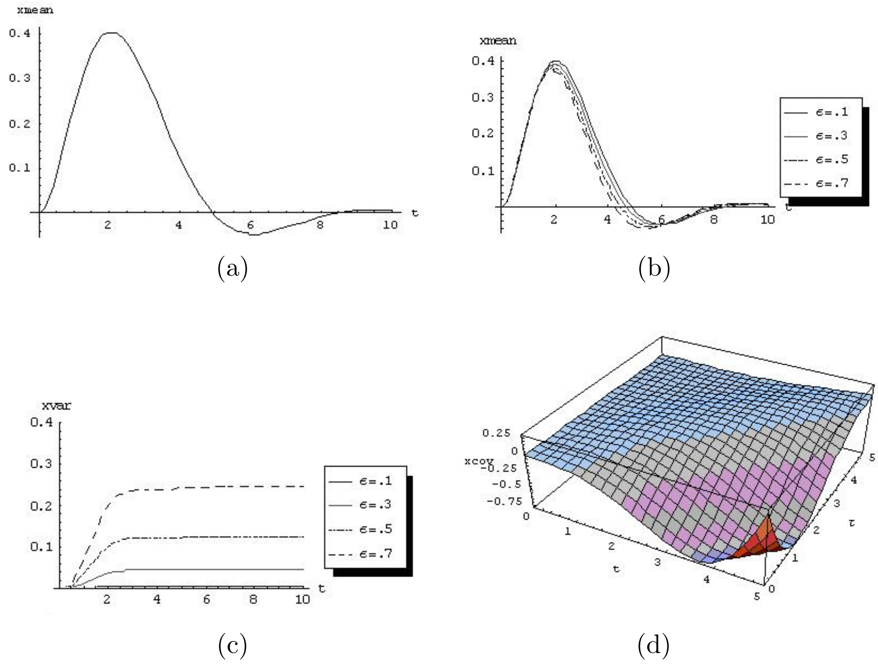


Figure 8. (a) The zero order approximation of the mean; (b) the first order approximation of the mean at different ε; (c) the first order approximation of the variance at different values of ε; (d) the covariance of the first approximation at $\varepsilon = 0.5$

The method also decomposes the solution process into

$$x^{(0)}(t;\omega) + x^{(1)}(t;\omega) + x^{(2)}(t;\omega) + \dots \tag{7.2}$$

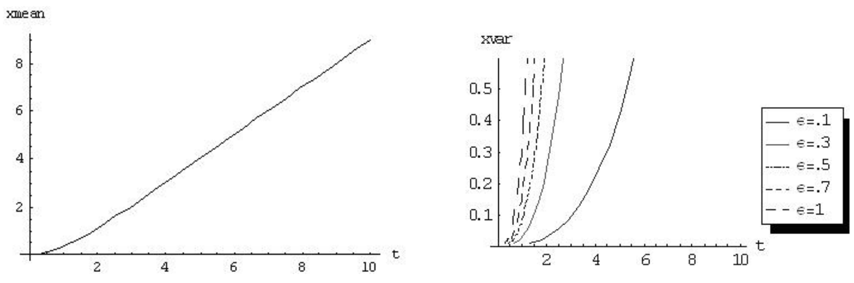
Substituting from equation (7.2) into (7.1), one can get the following iterative equations in the unknown kernels of equation (7.2):

$$\begin{aligned} x^{(0)}(t;\omega) &= x(0) + \dot{x}(0)t + \int_0^t \int_0^t F(t;\omega)\,dt\,dt, \\ x^{(1)}(t,\omega) &= -\int_0^t \int_0^t R(x^{(0)}\,dt\,dt - \varepsilon w^2 \int_0^t \int_0^t (x^{(0)})\,dt\,dt, \end{aligned}$$

Example 7.1. When solving the prototype example, we get the results in Figures 9 and 10. One can notice how the obtained results are distant from those of the previous techniques.

8. Conclusions

Concerning the quadratic nonlinearity problem and the prototype example used for illustrating the efficiency of the processed approximation techniques, one may suggest the use of the Pickard approximation which is very rapidly



(a) (b)

Figure 9. (a) The first order approximation of the mean; (b) the first order variance at different values of ε.

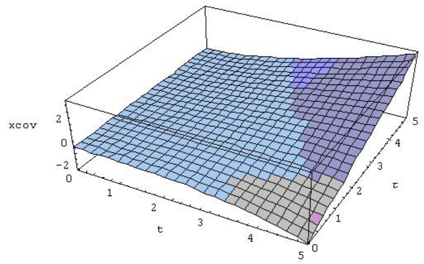


Figure 10. The first order covariance at $\varepsilon = 0.3$.

convergent to the solution, if convergent, and when using an efficient computer with an efficient symbolic program. The direct perturbation method produces good results. The WHEP technique seems to be an efficient one because of its corrections possibilities in spite of being analytically lengthy. The HPM is the easiest in computations, but expectedly depends highly on the initial guess. Concerning only first order approximation, the Adomian decompositions method is the worst among all the other executed techniques in this paper.

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