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ON THE STRUCTURE OF FIXED-POINT SETS OF UNIFORMLY LIPSCHITZIAN MAPPINGS

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Abstract. It is shown that the set of fixed points of any k-uniformly lipschitzian mapping in a uniformly convex space is a retract of a domain if k is close to 1.

1. Introduction

Let C be a nonempty, bounded, closed and convex subset of a Banach space X. We say that a mapping $T\colon C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for $x, y \in C$. The celebrated result of R. Bruck [1] asserts that if a nonexpansive mapping $T\colon C \to C$ has a fixed point in every nonempty closed convex subset of C which is invariant under T and if C is convex and weakly compact, then $\operatorname{Fix} T$, the set of fixed points, is a nonexpansive retract of C, (that is, there exists a nonexpansive mapping $R\colon C \to \operatorname{Fix} T$ such that $R_{|\operatorname{Fix} T} = I$). A few years ago, the Bruck result was extended by Domínguez Benavides and Lorenzo Ramirez [5] to the case of asymptotically nonexpansive mappings if the space X was sufficiently regular.

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On the other hand, the set of fixed points of a k-lipschitzian mapping can be very irregular for any $k > 1$. The following example has been communicated to us by K. Goebel:

Example 1.1. Let F be a nonempty closed subset of C. Fix $z \in F$, $0 < \varepsilon < 1$ and put

$$Tx = x + \varepsilon \operatorname{dist}(x, F)(z - x), \quad x \in C.$$

It is not difficult to see that $\operatorname{Fix} T = F$. Moreover, the Lipschitz constant of T tends to 1 if $\varepsilon \to 0$.

In 1973, Goebel and Kirk [7] introduced the class of uniformly lipschitzian mappings. Recall that a mapping $T: C \to C$ is k-uniformly lipschitizian if

$$\|T^n x - T^n y\| \leq k\|x - y\|$$

for every $x, y \in C$ and $n \in \mathbb{N}$.

Theorem 1.2 ([7]). *Let X be a uniformly convex Banach space with modulus of convexity δ_X and let C be a bounded, closed and convex subset of X. Suppose $T: C \to C$ is k-uniformly lipschitzian and*

$$k\left(1 - \delta_X\left(\frac{1}{k}\right)\right) < 1.$$

Then T has a fixed point in C. (Note that in a Hilbert space, $k < \sqrt{5}/2$).

It is known among specialists (folklore) that for k close to 1, the set of fixed points of T is connected. According to our knowledge, this fact has never been published, but it was mentioned several times at the conferences (R. Bruck). We would like to fill this gap by showing a little more: under the assumptions of Theorem 1.2, $\operatorname{Fix} T$ is not only connected but even a retract of C.

We note that Theorem 1.2 was significantly generalized by Lifschitz [10], Casini, Maluta [2] and Domínguez Benavides [4] but it is not very clear whether our statement is also valid in these cases. For recent results concerning uniformly lipschitzian mappings, see [3], [6], [9] and the references therein.

2. Main result

Let X be a uniformly convex Banach space. Recall that the modulus of convexity δ_X is the function $\delta_X: [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon\right\}$$

and, uniform convexity means $\delta_X(\varepsilon) > 0$ for $\varepsilon > 0$.

For $x, y \in C$ we use

$$r(y, \{T^i x\}) = \limsup_{i\to\infty} \|y - T^i x\| \quad \text{and} \quad r(C, \{T^i x\}) = \inf_{y\in C} r(y, \{T^i x\})$$

to denote the asymptotic radius of $\{T^i x\}$ at y and the asymptotic radius of $\{T^i x\}$ in C, respectively. It is well known that under the assumption of uniform convexity of X, the asymptotic center of $\{T^i x\}$ in C:

$$A(C, \{T^i x\}) := \{y \in C : r(y, \{T^i x\}) = r(C, \{T^i x\})\}$$

is a singleton.

Let $A\colon C \to C$ denote a mapping which associates with a given $x \in C$ a unique $z \in A(C, \{T^i x\})$, that is, $z = Ax$.

LEMMA 2.1. *Let X be a uniformly convex Banach space and C be a bounded, closed and convex subset of X. Then the mapping $A\colon C \to C$ is continuous.*

PROOF. On the contrary, suppose that there exist $x_0 \in C$ and $\varepsilon_0 > 0$ such that:

for all $\eta > 0$ there exists $x_1 \in C$ such that $\|x_1 - x_0\| < \eta$ and $\|z_1 - z_0\| \geq \varepsilon_0$,

where $\{z_0\} = A(C, \{T^i x_0\}), \{z_1\} = A(C, \{T^i x_1\})$.

Fix $\eta > 0$ and take $x_1 \in C$ such that

$$\|x_1 - x_0\| < \eta \quad \text{and} \quad \|z_1 - z_0\| \geq \varepsilon_0.$$

Let $R_0 = r(C, \{T^i x_0\}), R_1 = r(C, \{T^i x_1\})$ and $R = \limsup_{i\to\infty} \|z_1 - T^i x_0\|$. Notice that $R_0 < R$.

Choose $\varepsilon > 0$. Then

$$\begin{cases} \|z_1 - T^i x_0\| < R + \varepsilon, \\ \|z_0 - T^i x_0\| < R_0 + \varepsilon < R + \varepsilon, \\ \|z_0 - z_1\| \geq \varepsilon_0, \end{cases}$$

for all but finitely many i. It follows from the properties of δ_X that

$$\left\|T^i x_0 - \frac{z_1 + z_0}{2}\right\| \leq \left(1 - \delta_X\left(\frac{\varepsilon_0}{R + \varepsilon}\right)\right)(R + \varepsilon)$$

and hence

$$R_0 < \limsup_{i\to\infty} \left\|T^i x_0 - \frac{z_1 + z_0}{2}\right\| \leq \left(1 - \delta_X\left(\frac{\varepsilon_0}{R + \varepsilon}\right)\right)(R + \varepsilon). \tag{2.1}$$

Moreover, for all but finitely many i,

$$\|T^i x_0 - z_1\| \leq \|T^i x_0 - T^i x_1\| + \|T^i x_1 - z_1\| \leq k\|x_0 - x_1\| + R_1 + \varepsilon$$

and hence

$$\limsup_{i\to\infty} \|T^i x_0 - z_1\| = R \leq k\eta + R_1 + \varepsilon. \tag{2.2}$$

Similarly,

$$R_1 < \limsup_{i\to\infty} \|T^i x_1 - z_0\| \leq k\eta + R_0 + \varepsilon. \tag{2.3}$$

From (2.2) and (2.3), we have

$$R \leq k\eta + R_1 + \varepsilon < 2k\eta + 2\varepsilon + R_0. \tag{2.4}$$

Combining (2.4) with (2.1) and applying the monotonicity of δ_X, we obtain

$$R_0 < \left(1 - \delta_X\left(\frac{\varepsilon_0}{2k\eta + 3\varepsilon + R_0}\right)\right)(2k\eta + 3\varepsilon + R_0).$$

Letting $\eta, \varepsilon \to 0$ and using the continuity of δ_X, we conclude that

$$1 \leq 1 - \delta_X\left(\frac{\varepsilon_0}{R_0}\right) < 1.$$

This contradiction proves the continuity of the mapping A. $\square$

We are now in a position to prove our main result.

Theorem 2.2. *Let X be a uniformly convex Banach space with modulus of convexity δ_X and let C be a bounded, closed and convex subset of X. Suppose $T\colon C \to C$ is k-uniformly lipschitzian and*

$$k\left(1 - \delta_X\left(\frac{1}{k}\right)\right) < 1. \tag{2.5}$$

Then $\operatorname{Fix} T$ *is a retract of C.*

Proof. Fix $x \in C$ and let $z = Ax$. If $r(C, \{T^i x\}) = 0$ or $r(z, \{T^i z\}) = 0$, then $z = Tz$ and consequently $A^n x = z$ for $n > 0$.

Assume that $r(C, \{T^i x\}) > 0$ and $r(z, \{T^i z\}) > 0$. We follow the arguments from [7, Theorem 1]. Fix $\varepsilon > 0$, $\varepsilon \leq r(z, \{T^i z\})$ and choose j such that $\|z - T^j z\| \geq r(z, \{T^i z\}) - \varepsilon$. There exists N such that

$$\|z - T^i x\| \leq r(C, \{T^i x\}) + \varepsilon \leq k(r(C, \{T^i x\}) + \varepsilon)$$

for each $i > N$ (we assume that $k \geq 1$). Hence, for $i - j \geq N$,

$$\|T^j z - T^i x\| \leq k\|z - T^{i-j} x\| \leq k(r(C, \{T^i x\}) + \varepsilon).$$

Put $r_0 := k(r(C, \{T^i x\}) + \varepsilon)$. It follows from the properties of δ_X that

$$\left\|\frac{z + T^j z}{2} - T^i x\right\| \leq \left(1 - \delta_X\left(\frac{\|z - T^j z\|}{r_0}\right)\right) r_0 \leq \left(1 - \delta_X\left(\frac{r(z, \{T^i z\}) - \varepsilon}{r_0}\right)\right) r_0$$

for $i \geq N + j$ and hence

$$r(C, \{T^i x\}) \leq \left(1 - \delta_X\left(\frac{r(z, \{T^i z\}) - \varepsilon}{r_0}\right)\right) r_0.$$

Letting $\varepsilon \to 0$ and using the continuity of δ_X, we obtain

$$r(C, \{T^i x\}) \leq \left(1 - \delta_X\left(\frac{r(z, \{T^i z\})}{kr(C, \{T^i x\})}\right)\right) kr(C, \{T^i x\})$$

and consequently

$$r(z, \{T^i z\}) \leq k\delta_X^{-1}\left(1 - \frac{1}{k}\right) r(C, \{T^i x\}) \leq \alpha r(x, \{T^i x\}),$$

where $\alpha := k\delta_X^{-1}(1 - 1/k) < 1$ by (2.5). Moreover,

$$\|Ax - x\| = \|z - x\| \leq r(z, \{T^i x\}) + r(x, \{T^i x\}) \leq 2r(x, \{T^i x\}).$$

By iteration,

$$\|A^{n+1}x - A^n x\| \leq 2\alpha^n r(x, \{T^i x\}) \leq 2\alpha^n \operatorname{diam} C \tag{2.6}$$

for $x \in C$, $n = 0, 1, \ldots$ Thus

$$\sup_{x \in C} \|A^m x - A^n x\| \leq \frac{2\alpha^n}{1 - \alpha} \operatorname{diam} C \to 0 \quad \text{if } n, m \to \infty,$$

which implies that the sequence $\{A^n x\}$ converges uniformly to a function

$$Rx = \lim_{n \to \infty} A^n x.$$

It follows from Lemma 2.1 that $R: C \to C$ is continuous. Moreover, by standard arguments,

$$r(Rx, \{T^i Rx\}) \leq (1 + k)\|Rx - A^n x\| + r(A^n x, \{T^i A^n x\}_i) \to 0 \quad \text{if } n \to \infty.$$

Thus $Rx = TRx$ for every $x \in C$ and R is a retraction of C onto $\operatorname{Fix} T$. $\square$

REMARK 2.3. We have proved the continuity of R only, but it is expected that the resulting retraction enjoys some regularity properties. We leave this problem for future investigations.

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