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EXISTENCE AND MULTIPLICITY RESULTS FOR SEMILINEAR EQUATIONS WITH MEASURE DATA

Alberto Ferrero — Claudio Saccon

Abstract. In this paper, we study existence and nonexistence of solutions for the Dirichlet problem associated with the equation $-\Delta u = g(x,u) + \mu$ where μ is a Radon measure. Existence and nonexistence of solutions strictly depend on the nonlinearity $g(x,u)$ and suitable growth restrictions are assumed on it. Our proofs are obtained by standard arguments from critical theory and in order to find solutions of the equation, suitable functionals are introduced by mean of approximation arguments and iterative schemes.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a connected open bounded domain with smooth boundary and let $n > 2$. Denote by $\mathcal{M}(\Omega)$ the space of Radon measures, i.e. the dual space of the Banach space $C_0(\overline{\Omega})$ of continuous functions in $\overline{\Omega}$ which vanish on the boundary, endowed with the usual L^∞-norm. We study elliptic problems of the type

$$\begin{cases} -\Delta u = g(x,u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu \in \mathcal{M}(\Omega)$ and $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

Previous results on semilinear and quasilinear equations with measure data have been obtained, see [4]–[6], [8]. In some of these papers the model problem

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(1.1) is studied and existence and nonexistence results are obtained under the fundamental assumption $g(x,s)s \leq 0$. Some difficulties arise when existence of solutions of (1.1) is studied due to the fact that the associated action functional J is not defined in the whole Sobolev space $H_0^1(\Omega)$, where J is given by

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega G(x,v)\,dx - \int_\Omega v\,d\mu \tag{1.2}$$

for all $v \in H_0^1(\Omega)\cap C_0(\overline{\Omega})$. Here and in the rest of the paper we denote by $G(x,s)$ the function $\int_0^s g(x,t)\,dt$ for any $s \in \mathbb{R}$.

In some of the above mentioned papers (see [5], [8]), this difficulty is overcome by replacing the measure μ with a sequence of regular functions $\{\mu_m\}$ which converges weakly to μ in the sense of measures and $g(x,s)$ with a suitable sequence of truncated functions $g_m(x,s)$ (see (38) in [5]) so that, according to (1.2) the corresponding functionals $J^{(m)}$ are defined in the whole $H_0^1(\Omega)$. Then a sequence $\{u_m\}$ of solutions of the "regularized" problems

$$\begin{cases} -\Delta u_m = g_m(x,u_m) + \mu_m & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

is obtained via minimization and boundedness in $L^1(\Omega)$ of $g_m(x,u_m)$ is proved, provided that the assumption $g(x,s)s \leq 0$ holds. In order to pass to the limit in (1.3), a growth restriction at infinity on g is needed. This restriction involves the critical exponent $2_* = 2(n-1)/(n-2)$, see Theorem 3 in [5] and Theorem A.1 in [4]. Note that $2_* < 2^* = 2n/(n-2)$ where 2^* denotes the critical Sobolev exponent. Without this growth restriction the existence of solutions of (1.1) is guaranteed only under suitable assumptions on the measure μ, see [8] and [9].

Our purpose is to understand what happens when the condition $g(x,s)s \leq 0$ is dropped. A completely different behavior of problem (1.1) is expected without this assumption. In our first result we consider the equation (1.1) and assuming that $g(x,s)$ has a nonresonant linear behavior as $|s| \to \infty$, in Theorem 2.1 we prove that (1.1) admits a solution. Here the above mentioned approximation argument is employed: in the proof of Theorem 2.1 the measure μ is replaced by a sequence $\{\mu_m\}$ and the existence of a solution u_m for the corresponding problem (1.1) is obtained by [10]. Thanks to the linear asymptotic behavior at infinity of $g(x,s)$ we prove boundedness in $L^1(\Omega)$ of the sequence $\{u_m\}$ and using Theorem 8.1 in [14] we pass to the limit in the "regularized" problem thus obtaining the existence of a solution.

Then we investigate what happens when $g = g(s)$ has a superlinear behavior at infinity. To this purpose we study the problem

$$\begin{cases} -\Delta u = g(u) + \varepsilon\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where μ is a nonnegative nontrivial Radon measure and $\varepsilon > 0$. In Theorem 2.2, we assume that $g = g(s)$ satisfies the growth restriction $|g(s)| \le C(1 + |s|^{p-1})$ for any $s \in \mathbb{R}$ with $p < 2_* = 2(n-1)/(n-2)$. Here we prove the existence of a nonnegative solution u_ε of (1.4) for any $\varepsilon \in (0, \varepsilon^*)$ for a suitable $\varepsilon^* > 0$. Moreover we prove that the restriction $\varepsilon \in (0, \varepsilon^*)$ is also necessary for existence of nonnegative solutions of (1.4).

Our next purpose is to understand if (1.4) admits an extremal solution, i.e. a solution corresponding to $\varepsilon = \varepsilon^*$. In Theorem 2.5, we give a partial answer to this problem: we prove existence for $\varepsilon = \varepsilon^*$ at least when g is a polynomial-type function. The proof of Theorem 2.5 is obtained from a convergence result on the sequence of solutions u_ε as $\varepsilon \to \varepsilon^*$. In order to prove boundedness of $\{u_\varepsilon\}$, some precise asymptotic estimates are needed and therefore we restrict our result to the case of a polynomial-type nonlinearity g.

Then in Corollary 2.4, by a super-subsolution argument, we prove existence of solutions of (1.4) for $\mu \in \mathcal{M}(\Omega)$ without any restriction on the sign of μ.

Next, we investigate whether the model problem (1.4) admits at least two positive solutions. It was proved in [16] that for $g(s) = |s|^{p-2}s$, $p \le 2^*$ and $\mu \in H^{-1}(\Omega)$, $\mu \ge 0$, then (1.4) admits two positive solutions for any ε small enough. In Theorem 2.6 we extend this multiplicity result to the case of a nonnegative nontrivial Radon measure under the more restrictive growth condition $p < 2_*$. Actually, our multiplicity result is proved for a more general polynomial-type nonlinearity g. In the proof of Theorem 2.6, precise asymptotic estimates are needed in order to prove boundedness of Palais–Smale sequences (see Lemma 9.2) and hence the extension of this result to the more general nonlinearity g introduced in Theorem 2.2 is far from being straightforward.

In the proofs of Theorems 2.2 and 2.6 an iterative scheme is introduced. This approach enables us to define a new equation and a corresponding action functional denoted by $I_{N,\varepsilon}$ (see (4.11), (4.12) for the definition of the equation and the functional $I_{N,\varepsilon}$). This new problem is equivalent to (1.4) in the sense that admits a solution if and only if (1.4) admits a solution. The functional $I_{N,\varepsilon}$ is obtained after suitable "horizontal and vertical translations" from the functional
$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega G(v)\,dx - \varepsilon\int_\Omega v\,d\mu,$$
for all $v \in H_0^1(\Omega) \cap C_0(\overline{\Omega})$, where $G(s) = \int_0^s g(t)\,dt$ (see Theorem 2.7). With this procedure, we obtain a functional $I_{N,\varepsilon}$ defined in the whole $H_0^1(\Omega)$ from a functional which is defined only on a subspace of $H_0^1(\Omega)$. The multiplicity result in Theorem 2.6 then follows from an application to the functional $I_{N,\varepsilon}$ of the mountain-pass theorem [3].

Finally, we want to understand whether the growth restriction $p < 2_*$ on $g(s) = |s|^{p-2}s$ is also necessary for the existence of nonnegative solutions of (1.4).

We clarify this question proving in Proposition 2.8 that for $p \geq 2_*$, (1.4) does not admit a nonnegative solution when μ is a Dirac measure concentrated at an interior point of Ω. Since the Dirac mass is a concentrated measure, one may ask whether existence of nonnegative solutions of (1.4) can be obtained also for $p \geq 2_*$ when μ is a diffuse measure (see [8] for the definition of diffuse and concentrated measure). The answer in this sense is negative, since there exist diffuse measures for which (1.4) does not admit a nonnegative solution. More precisely, in Proposition 2.9 we prove that for any $p \geq 2_*$ there exists a $\mu \in L^1(\Omega)$ such that (1.4) does not admit a nonnegative solution.

This paper is organized as follows. The next section is devoted to the statements of the main results and Sections 3–12 are devoted to their proofs.

Acknowledgement. While this paper was being published the authors discovered that the superlinear case had been treated, in a quite comprehensive way, by Amann and Quittner in a previous paper (see [2]), also in the case of a more general linear elliptic operator $Lu = -\mathrm{div}(a_{ij}(x)D_j u)$ (plus lower order terms).

In [2], however the approach is quite different from the previously described one and is basically non variational, the main tool being the *fixed point index* for order-preserving maps. The proof of the main result relies on continuation properties of the index with respect to the parameter and on an a priori bound for all the solutions. Getting such properties requires some amount of regularity for the coefficients of the linear part which are supposed to be $C^2(\overline{\Omega})$.

On the contrary in the present paper the variational aspect of the problem is recovered, which allows to impose only minimal assumptions. Indeed, although for the sake of simplicity only the Laplace operator is considered herein, the results of Theorems 2.2, 2.5 and 2.6 can be easily extended to the case of the linear part being $Lu = -\mathrm{div}(a_{ij}(x)D_j u)$, with a_{ij} just measurable and bounded (and uniformly elliptic of course). To see this it suffices to observe that what is actually needed to perform the iterative scheme is the validity of (c) of Lemma 4.3, which follows from the following regularity argument: if $u \in W_0^{1,q}$ $q < N/(N-1)$, $Lu = h$ with $h \in L^r$ $r \geq 1$, then $u \in L^{(r^*)^*}$ when $r < N/2$ and $u \in C^0$ when $r > N/2$. This can be easily deduced by the regularity results of [14] (it is worth noticing that no properties on the second derivative of u are neither proved nor used). In [14] it is also shown that the operator Lu verifies the strong maximum principle used to prove positivity of the solutions.

As a final comment it should be noted that the iterative scheme used to recover the variational nature of the problem is also used in a forthcoming paper (see [11]), where a problem with jumping nonlinearity is faced, and where degree arguments seem not suffice to get (at least) the three solutions result proved therein.

2. Main results

In this section, we always assume that $n > 2$ and that $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary. In the rest of the paper we endow the Hilbert space $H_0^1(\Omega)$ with the scalar product

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \nabla v \, dx \quad \text{for all } u, v \in H_0^1(\Omega).$$

We study the Dirichlet problem

$$\begin{cases} -\Delta u = g(x,u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where by a solution of (2.1) we mean a function $u \in L^1(\Omega)$ with $g(x,u) \in L^1(\Omega)$ which satisfies

$$\int_\Omega -u\Delta\varphi \, dx = \int_\Omega g(x,u)\varphi \, dx + \int_\Omega \varphi \, d\mu \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}).$$

Here $C_0^2(\overline{\Omega})$ denotes the space of functions $C^2(\overline{\Omega})$ which vanish on $\partial\Omega$.

In our first result, we prove existence of solutions of (2.1) assuming that $g(x,s)$ has a linear asymptotic behavior as $|s| \to \infty$. Denote by $\sigma(-\Delta)$ the spectrum of $-\Delta$ with homogeneous Dirichlet boundary conditions and by $\lambda_1 < \lambda_2 \le \ldots$. the corresponding eigenvalues.

Let $g(x,s)$ be such that

$$h_\sigma(x) = \max_{|s| \le \sigma} |g(x,s)| \in L^2(\Omega) \quad \text{for all } \sigma > 0. \tag{2.2}$$

Assume that

$$\lim_{s \to \pm\infty} \frac{g(x,s)}{s} = \lambda(x) \quad \text{uniformly for a.e. } x \in \Omega \tag{2.3}$$

with $\lambda_k \le \lambda(x) \le \lambda_{k+1}$ and $\lambda_k < \lambda_{k+1}$ for a suitable $k \ge 0$ (here $\lambda_0 = 0$) and that

(2.4) there exists $\overline{s} > 0$ such that $g(x,s)s \ge 0$ for all $|s| > \overline{s}$ and for a.e. $x \in \Omega$.

Moreover, we assume that

$$\lambda_k < \lambda(x) \quad \text{on a subset of } \Omega \text{ of positive measure}, \tag{2.5}$$

$$\lambda(x) < \lambda_{k+1} \quad \text{on a subset of } \Omega \text{ of positive measure}. \tag{2.6}$$

Then we establish

Theorem 2.1. *Let $n > 2$ and assume that g satisfies* (2.2)–(2.6). *Then* (2.1) *admits a solution u. Moreover, $u \in W_0^{1,q}(\Omega)$ for any $q < n/(n-1)$.*

In our second result, we study the following problem

$$
\begin{cases}
-\Delta u = g(u) + \varepsilon\mu & \text{in } \Omega,\\
u \geq 0 & \text{in } \Omega,\\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{2.7}
$$

where g is superlinear at infinity, $\mu \in \mathcal{M}(\Omega)$ is a nonnegative nontrivial Radon measure and $\varepsilon > 0$. We assume that

$$g \in C^1(\mathbb{R}) \text{ is convex in } [0,\infty), \tag{2.8}$$

$$g'(s) > 0 \quad \text{for all } s > 0. \tag{2.9}$$

Moreover, suppose that there exists $C > 0$ such that

$$|g(s)| \leq \lambda|s| + C|s|^{p-1} \quad \text{for all } s \in \mathbb{R} \tag{2.10}$$

with $2 < p < 2_*$ and $\lambda \in (0, \lambda_1)$. Finally, we suppose that

$$\lim_{s\to\infty} \frac{g(s)}{s} = \infty. \tag{2.11}$$

In the sequel by minimal solution we mean the pointwise smallest nonnegative solution of (2.7).

Then the following holds

Theorem 2.2. *Let $n > 2$, $2 < p < 2_*$ and let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative nontrivial Radon measure. Assume that g satisfies* (2.8)–(2.11). *Then there exists an extremal value $\varepsilon^* > 0$ such that:*

(a) *If $\varepsilon \in (0, \varepsilon^*)$, then* (2.7) *admits a minimal solution u_ε.*
(b) *If $\varepsilon > \varepsilon^*$, then* (2.7) *admits no solutions.*

The assumption $\lambda \in (0, \lambda_1)$ in (2.10) is necessary for the existence of solutions of (2.7) as one can see from the following

Proposition 2.3. *Let $n > 2$, $2 < p < 2_*$, $\varepsilon > 0$ and let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative nontrivial Radon measure. Assume that g satisfies* (2.8)–(2.11) *and that $g(s) = \lambda s + o(s)$ as $s \to 0^+$ with $\lambda > 0$. If* (2.7) *admits a solution, then $\lambda < \lambda_1$.*

From Theorem 2.2, we derive an existence result for solutions of

$$
\begin{cases}
-\Delta u = g(u) + \varepsilon\mu & \text{in } \Omega,\\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{2.12}
$$

without any assumptions on the sign of $\mu \in \mathcal{M}(\Omega)$. We establish

Corollary 2.4. *Let $n > 2$, $2 < p < 2_*$ and $\mu \in \mathcal{M}(\Omega)$. Assume that $g(s)$ and $-g(-s)$ satisfy* (2.8)–(2.11) *for any $s > 0$. Then there exists $\overline{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \overline{\varepsilon})$ problem* (2.12) *admits a solution.*

Then we prove that if g is a polynomial-type function, then (2.7) admits a solution for $\varepsilon = \varepsilon^*$.

Theorem 2.5. *Let $n > 2$, $2 < p < 2_*$, $\varepsilon^* > 0$ as in Theorem* 2.2 *and let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative nontrivial Radon measure. Assume that $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \le q < p < 2_*$, $\lambda \ge 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$. Then* (2.7) *admits a solution for $\varepsilon = \varepsilon^*$.*

Next we prove the existence of a second solution for problem (2.7).

Theorem 2.6. *Let $n > 2$, $2 < p < 2_*$ and let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative nontrivial Radon measure. Assume that $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \le q < p < 2_*$, $\lambda \ge 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$. If $\varepsilon \in (0, \varepsilon^*)$ then* (2.7) *admits a second solution $U_\varepsilon > u_\varepsilon$ a.e. in Ω with ε^* and u_ε as in Theorem* 2.2.

In the next result we try to clarify the meaning of the procedure introduced in the proof of Theorem 2.2. Let μ be a nonnegative nontrivial Radon measure and let $\mu_m = \rho_m * \mu$ where $\{\rho_m\}$ is a sequence of mollifiers. Let $J^{(m)}$ be defined by

$$J^{(m)}(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega G(w)\,dx - \varepsilon\int_\Omega \mu_m w\,dx \quad \text{for all } w \in H_0^1(\Omega).$$

Let $v_{N,\varepsilon}^{(m)}$, $\gamma_{N,\varepsilon}^{(m)}$, $h_{N,\varepsilon}^{(m)}(x,s)$, $f_{N,\varepsilon}^{(m)}$ be the functions defined in (4.2)–(4.7) with μ_m in place of μ and N as in Lemma 4.3. Let $I^{(m)}$ be the translated functional

$$I^{(m)}(w) = J^{(m)}(w + \gamma_{N,\varepsilon}^{(m)}) - J^{(m)}(\gamma_{N,\varepsilon}^{(m)}) \quad \text{for all } w \in H_0^1(\Omega). \tag{2.13}$$

We have

Theorem 2.7. *Let $I^{(m)}$ be the functional defined in* (2.13) *with $\varepsilon \in (0, \varepsilon^*)$.*

(a) *Assume that g satisfies* (2.8)–(2.11). *Then there exists $\overline{m} > 0$ such that for any $m > \overline{m}$ the functional $I^{(m)}$ admits a local minimizer $u_{N,\varepsilon}^{(m)} \ge 0$ which satisfies $u_{N,\varepsilon}^{(m)} \to u_{N,\varepsilon}$ in $H_0^1(\Omega)$ as $m \to \infty$. Moreover, $u_{N,\varepsilon} + \gamma_{N,\varepsilon}$ coincides with the minimal solution u_ε of* (2.7) *found in Theorem* 2.2.

(b) *Assume that $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \le q < p < 2_*$, $\lambda \ge 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$. Then there exists $\overline{m} > 0$ such that for any $m > \overline{m}$ the functional $I^{(m)}$ admits a second critical point $U_{N,\varepsilon}^{(m)} \ge 0$ which satisfies $U_{N,\varepsilon}^{(m)} \to U^*$ in $H_0^1(\Omega)$ as $m \to \infty$ up to subsequences. Moreover, $U = U^* + \gamma_{N,\varepsilon}$ is a solution of* (2.7) *which satisfies $U > u_\varepsilon$ a.e. in Ω.*

Finally, we show that the growth restriction (2.10) on g is optimal. To this purpose consider the model problem

$$\begin{cases} -\Delta u = u^{p-1} + \varepsilon\mu & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Then we have

Proposition 2.8. *Let $n > 2$, $p \geq 2_*$ and let $\mu = \delta_a$ be a Dirac measure concentrated at $a \in \Omega$. Then* (2.14) *admits no solutions for any $\varepsilon > 0$.*

Assume again that $p \geq 2_*$. We prove that (2.14) admits no solutions also for some $\mu \in L^1(\Omega)$.

Proposition 2.9. *Let $n > 2$, $p \geq 2_*$. Then there exists $\mu \in L^1(\Omega)$ such that* (2.14) *admits no solutions for any $\varepsilon > 0$.*

3. Proof of Theorem 2.1

We start with some preliminary lemmas. In the following two subsections, we distinguish the cases $k \geq 1$ and $k = 0$ in (2.3)–(2.6).

3.1. The case $\lambda_k \leq \lambda(x) \leq \lambda_{k+1}$, $k \geq 1$. Let $G(x,s) = \int_0^s g(x,t)\,dt$ and for $\mu \in L^2(\Omega)$ define the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega G(x,v)\,dx - \int_\Omega \mu v\,dx \quad \text{for all } v \in H_0^1(\Omega). \tag{3.1}$$

By (2.2)–(2.6) we deduce that the functional $J \in C^1(H_0^1(\Omega))$ so that the critical points of J solve (2.1). We state the following

Lemma 3.1. *Assume that g satisfies* (2.2)–(2.6). *If $\mu \in L^2(\Omega)$ then* (2.1) *admits a solution $u \in H_0^1(\Omega)$.*

Proof. See [10]. □

Let $f \in L^2(\Omega)$ and consider the following linear problem

$$\begin{cases} -\Delta u = \lambda(x)u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

with $\lambda(x)$ as in (2.3). By Lemma 3.1 we infer that (3.2) admits a solution $u \in H_0^1(\Omega)$. Then, we establish

Lemma 3.2. *Let $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be the corresponding solution of* (3.2) *with $\lambda(x)$ as in* (2.3). *Then there exists a constant $C > 0$ such that*

$$\|u\|_{H_0^1} \leq C\|f\|_{L^2}.$$

PROOF. Let $L = -\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ and let $K\colon H_0^1(\Omega) \to H_0^1(\Omega)$ be the linear operator defined by

$$Kv = L^{-1}(\lambda(x)v) \quad \text{for all } v \in H_0^1(\Omega).$$

By (2.5)–(2.6) and Lemma 3 in [10], we deduce that the linear problem

$$\begin{cases} -\Delta u = \lambda(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits only the trivial solution and hence, since K is a compact linear operator, we have $1 \in \rho(K)$ where $\rho(K)$ denotes the resolvent set of K. In particular this implies that $(K - I)^{-1} \in \mathcal{L}(H_0^1(\Omega); H_0^1(\Omega))$ where I denotes the identity in $H_0^1(\Omega)$. If $u \in H_0^1(\Omega)$ is a solution of (3.2), then applying the operator L^{-1} to both sides of (3.2), we obtain $(K - I)u = -L^{-1}f$ and in turn $u = -(K - I)^{-1}L^{-1}f$. Then, the last identity yields

$$\|u\|_{H_0^1} \le \|(K - I)^{-1}\|_{\mathcal{L}(H_0^1;H_0^1)} \|L^{-1}\|_{\mathcal{L}(H^{-1};H_0^1)} \|f\|_{L^2}$$

which concludes the proof of the lemma. □

3.2. The case $0 \le \lambda(x) \le \lambda_1$. For $\mu \in L^2(\Omega)$ consider the functional J defined in (3.1). The existence of a critical point for J may be obtained by minimization.

LEMMA 3.3. *Assume that g satisfies* (2.2)–(2.6) *with $k = 0$. If $\mu \in L^2(\Omega)$, then* (2.1) *admits a solution $u \in H_0^1(\Omega)$.*

PROOF. We first prove that J is bounded from below. To this purpose, let $\{u_m\}$ be a sequence such that $\|u_m\|_{H_0^1} \to \infty$. Consider now

$$\frac{J(u_m)}{\|u_m\|_{H_0^1}^2} = \frac{1}{2} - \int_\Omega \frac{G(x, u_m)}{\|u_m\|_{H_0^1}^2}\, dx - \frac{1}{\|u_m\|_{H_0^1}^2} \int_\Omega \mu u_m\, dx. \tag{3.3}$$

By (2.3)–(2.4), we deduce that for any $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that

$$\left| G(x, s) - \frac{1}{2}\lambda(x)s^2 \right| < \varepsilon \frac{s^2}{2} \quad \text{for all } x \in \Omega \text{ and all } |s| > \sigma_\varepsilon.$$

Hence, by (2.2), we obtain

$$\left| \int_\Omega G(x, u_m)\, dx - \int_\Omega \frac{1}{2}\lambda(x)u_m^2\, dx \right| \le \sigma_\varepsilon \|h_{\sigma_\varepsilon}\|_{L^1} + \frac{1}{2}\lambda_1 \sigma_\varepsilon^2 |\Omega| + \frac{\varepsilon}{2} \int_\Omega u_m^2\, dx.$$

Letting $m \to \infty$ and $\varepsilon \to 0$, this yields

$$\left| \int_\Omega \frac{G(x, u_m)}{\|u_m\|_{H_0^1}^2}\, dx - \int_\Omega \frac{1}{2}\lambda(x) \frac{u_m^2}{\|u_m\|_{H_0^1}^2}\, dx \right| \to 0 \quad \text{as } m \to \infty. \tag{3.4}$$

Define $v_m = u_m / \|u_m\|_{H_0^1}$. Up to subsequences we may assume that $v_m \rightharpoonup v$ in $H_0^1(\Omega)$ and hence by (2.6), (3.4), compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and weak lower semicontinuity of the H_0^1-norm, we obtain

$$\begin{aligned}(3.5)\qquad \lim_{m\to\infty} \int_\Omega \frac{G(x,u_m)}{\|u_m\|_{H_0^1}^2}\,dx &= \lim_{m\to\infty} \frac{1}{2}\int_\Omega \lambda(x) v_m^2\,dx = \frac{1}{2}\int_\Omega \lambda(x) v^2\,dx \\ &\le \frac{1}{2}\lambda_1 \|v\|_{L^2}^2 \le \frac{1}{2}\|v\|_{H_0^1}^2 \le \liminf_{m\to\infty} \frac{1}{2}\|v_m\|_{H_0^1}^2 = \frac{1}{2}.\end{aligned}$$

We claim that the limit in the first line of (3.5) is strictly less than 1/2. Indeed, if we suppose that the equality holds, then by (3.5) we have $\lambda_1 \|v\|_{L^2}^2 = \|v\|_{H_0^1}^2$ which proves that v is an eigenfunction relative to λ_1; in particular $v^2 > 0$ in Ω. Therefore by (2.6) we have

$$\frac{1}{2}\int_\Omega \lambda(x) v^2\,dx < \frac{1}{2}\lambda_1 \|v\|_{L^2}^2$$

and hence the strict inequality in (3.5) holds.

Finally by (3.3) we infer that there exist $C > 0$ and $\overline{m} > 0$ such that

$$(3.6)\qquad J(u_m) > C\|u_m\|_{H_0^1}^2 \quad \text{for all } m > \overline{m}.$$

This proves that J is bounded from below. Next suppose that $\{w_m\}$ is a minimizing sequence for J. Then applying inequality (3.6) to $\{w_m\}$ we deduce that $\{w_m\}$ is bounded in $H_0^1(\Omega)$ and hence by (2.3), compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and weak lower semicontinuity of the H_0^1-norm we conclude that the weak limit of $\{w_m\}$ in $H_0^1(\Omega)$ is a global minimizer for J. □

3.3. End of the proof of Theorem 2.1. Let $\{f_m\} \subset L^1(\Omega) \cap L^2(\Omega)$ be a bounded sequence in $L^1(\Omega)$ and consider the following nonlinear problem

$$(3.7)\qquad \begin{cases} -\Delta u = g(x,u) + f_m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 3.1 for the case $\lambda_k \le \lambda(x) \le \lambda_{k+1}$ and by Lemma 3.3 for the case $0 \le \lambda(x) \le \lambda_1$, we deduce that (3.7) admits a solution $u_m \in H_0^1(\Omega)$ for any $m \in \mathbb{N}$. We recall that u_m solves (3.7) in $H^{-1}(\Omega)$, i.e.

$$(3.8)\qquad \int_\Omega \nabla u_m \nabla v\,dx = \int_\Omega g(x,u_m) v\,dx + \int_\Omega f_m v\,dx \quad \text{for all } v \in H_0^1(\Omega).$$

Then, we establish

Lemma 3.4. *Let $\{f_m\} \subset L^1(\Omega) \cap L^2(\Omega)$ and suppose that $\|f_m\|_{L^1} < C$ for any $m \in \mathbb{N}$ and let $\{u_m\}$ be the corresponding sequence of solutions of* (3.7). *Then $\{u_m\}$ is bounded in $L^1(\Omega)$.*

PROOF. If we put $v_m = \operatorname{sign}(u_m)$, then $\|v_m\|_{L^\infty} = 1$ and

$$\|u_m\|_{L^1} = \int_\Omega u_m v_m \, dx. \tag{3.9}$$

Introduce the following linear problem

$$\begin{cases} -\Delta\varphi = \lambda(x)\varphi + v_m & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

By Lemmas 3.1, 3.3 and Lemma 3 in [10], we infer that for any $m \in \mathbb{N}$ problem (3.10) admits a unique solution $\varphi_m \in H_0^1(\Omega)$. Since $\|v_m\|_{L^\infty} = 1$ then $\{v_m\}$ is bounded $L^2(\Omega)$ and by Lemma 3.2 it follows that $\{\varphi_m\}$ is bounded in $H_0^1(\Omega)$. By Sobolev inequality and L^q-estimates for strictly elliptic linear operators (see [1]), we infer that $\{\varphi_m\}$ is bounded in $L^\infty(\Omega)$. We use φ_m as test function in (3.8) to obtain

$$\int_\Omega \nabla u_m \nabla \varphi_m \, dx = \int_\Omega g(x, u_m)\varphi_m \, dx + \int_\Omega f_m \varphi_m \, dx \quad \text{for all } m \in \mathbb{N}. \tag{3.11}$$

We need an estimate from above on $\int_\Omega g(x, u_m)\varphi_m \, dx$. By (2.3) we infer that for any $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that

$$\begin{cases} (\lambda(x) - \varepsilon)s \le g(x, s) \le (\lambda(x) + \varepsilon)s & \text{for all } x \in \Omega \text{ and all } s > \sigma_\varepsilon, \\ (\lambda(x) + \varepsilon)s \le g(x, s) \le (\lambda(x) - \varepsilon)s & \text{for all } x \in \Omega \text{ and all } s < -\sigma_\varepsilon, \end{cases}$$

and hence by (2.2), Sobolev embedding and Hölder inequality, we obtain

$$\begin{aligned} \int_\Omega g(x, u_m)\varphi_m \, dx \le{}& \int_{\{|u_m| \le \sigma_\varepsilon\}} g(x, u_m)\varphi_m \, dx \\ &+ \int_{\{\varphi_m \ge 0\} \cap \{u_m > \sigma_\varepsilon\}} (\lambda(x) + \varepsilon) u_m \varphi_m \, dx \\ &+ \int_{\{\varphi_m < 0\} \cap \{u_m < -\sigma_\varepsilon\}} (\lambda(x) + \varepsilon) u_m \varphi_m \, dx \\ &+ \int_{\{\varphi_m \ge 0\} \cap \{u_m < -\sigma_\varepsilon\}} (\lambda(x) - \varepsilon) u_m \varphi_m \, dx \\ &+ \int_{\{\varphi_m < 0\} \cap \{u_m > \sigma_\varepsilon\}} (\lambda(x) - \varepsilon) u_m \varphi_m \, dx \\ \le{}& C(\varepsilon)\|\varphi_m\|_{L^\infty} + \varepsilon \|u_m\|_{L^1} \|\varphi_m\|_{L^\infty} + \int_\Omega \lambda(x) u_m \varphi_m \, dx. \end{aligned} \tag{3.12}$$

Then, by (3.9)–(3.12) we have

$$\begin{aligned} 0 &= -\int_\Omega \nabla u_m \nabla \varphi_m \, dx + \int_\Omega g(x, u_m)\varphi_m \, dx + \int_\Omega f_m \varphi_m \, dx \\ &\le -\int_\Omega \nabla u_m \nabla \varphi_m \, dx + \int_\Omega \lambda(x) u_m \varphi_m \, dx \\ &\quad + C(\varepsilon)\|\varphi_m\|_{L^\infty} + \varepsilon \|u_m\|_{L^1} \|\varphi_m\|_{L^\infty} + \int_\Omega f_m \varphi_m \, dx \end{aligned}$$

$$= -\int_\Omega u_m v_m\,dx + C(\varepsilon)\|\varphi_m\|_{L^\infty} + \varepsilon\|u_m\|_{L^1}\|\varphi_m\|_{L^\infty} + \int_\Omega f_m\varphi_m\,dx$$
$$= -\|u_m\|_{L^1} + C(\varepsilon)\|\varphi_m\|_{L^\infty} + \varepsilon\|u_m\|_{L^1}\|\varphi_m\|_{L^\infty} + \int_\Omega f_m\varphi_m\,dx$$

and from this we obtain

$$(1-\varepsilon\|\varphi_m\|_{L^\infty})\|u_m\|_{L^1} \le C(\varepsilon)\|\varphi_m\|_{L^\infty} + \|f_m\|_{L^1}\|\varphi_m\|_{L^\infty}.$$

Since $\{f_m\}$ and $\{\varphi_m\}$ are bounded respectively in $L^1(\Omega)$ and $L^\infty(\Omega)$, choosing ε small enough, it follows immediately that $\{u_m\}$ is bounded in $L^1(\Omega)$. □

Let $\mu \in \mathcal{M}(\Omega)$. Then there exists a sequence $\{f_m\} \subset L^1(\Omega)\cap L^2(\Omega)$ such that $\|f_m\|_{L^1} < C$ and $f_m \rightharpoonup \mu$ in the sense of measures. Let $\{u_m\}$ be the corresponding sequence of solutions of (3.7). By Lemma 3.4 we deduce that the sequence $\{u_m\}$ is bounded in $L^1(\Omega)$ and by (2.2)–(2.3) and Theorem B.1 in [8] (see also Theorem 8.1 in [14]), it follows that the sequence $\{u_m\}$ is bounded in $W_0^{1,q}(\Omega)$ for any $1 \le q < n/(n-1)$. Up to subsequences we may assume that there exists u such that $u_m \rightharpoonup u$ in $W_0^{1,q}(\Omega)$ for any $1 \le q < n/(n-1)$. Moreover, by (2.2)–(2.3), Sobolev embedding and dominated convergence, it follows that $g(x,u_m) \to g(x,u)$ in $L^1(\Omega)$.

Passing to the limit in (3.8) with $v = \varphi \in \mathcal{D}(\Omega)$, we obtain

$$\int_\Omega \nabla u\nabla\varphi\,dx = \int_\Omega g(x,u)\varphi\,dx + \int_\Omega \varphi\,d\mu \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{3.13}$$

Note that the variational identity (3.13) holds for any $\varphi \in W_0^{1,q/(q-1)}(\Omega) \subset C_0(\overline{\Omega})$ for $q < n/(n-1)$. If we choose $\varphi \in C_0^2(\overline{\Omega})$, then integrating by parts it follows that

$$\int_\Omega -u\Delta\varphi\,dx = \int_\Omega g(x,u)\varphi\,dx + \int_\Omega \varphi\,d\mu \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}).$$

This completes the proof of the theorem. □

4. Proof of Theorem 2.2

We start with the following preliminary lemma.

Lemma 4.1. *Let $n > 2$, $p \in (2,2^*)$, $\lambda \in (0,\lambda_1)$ and $\varepsilon_1,\varepsilon_2,\kappa > 0$. If $a \in L^{n/2}(\Omega)$ and $f \in L^{2n/(n+2)}(\Omega)$ are nonnegative functions, then the problem*

$$\begin{cases} -\Delta u = \lambda u + \kappa u^{p-1} + \varepsilon_1 a(x)u + \varepsilon_2 f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

admits a nonnegative nontrivial solution for ε_1 and ε_2 small enough.

PROOF. The proof of this lemma follows from a standard argument from critical point theory. Define the functional

$$K(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{\lambda}{2}\int_\Omega (v^+)^2\,dx - \frac{\kappa}{p}\int_\Omega (v^+)^p\,dx \\ - \frac{\varepsilon_1}{2}\int_\Omega a(x)(v^+)^2\,dx - \varepsilon_2\int_\Omega fv\,dx$$

for all $v \in H_0^1(\Omega)$. Then $K \in C^1(H_0^1(\Omega))$ and by the weak comparison principle, any critical point of K is a nonnegative nontrivial solution of (4.1). By Sobolev embedding and Poincarè inequality we have

$$K(v) \ge \frac{1}{2}\frac{\lambda_1 - \lambda}{\lambda_1}\|v\|_{H_0^1}^2 - \frac{\kappa C_1}{p}\|v\|_{H_0^1}^p \\ - \varepsilon_1 C_2\|a\|_{L^{n/2}}\|v\|_{H_0^1}^2 - \varepsilon_2 C_3\|f\|_{L^{2n/(n+2)}}\|v\|_{H_0^1}$$

and hence for ε_1 and ε_2 small enough we have

there exists $\rho, R > 0$ such that $K(v) \ge R$ for all v such that $\|v\|_{H_0^1} = \rho$.

Moreover, there exists a nonnegative function w such that

$$\|w\|_{H_0^1} > \rho \quad \text{and} \quad K(w) < 0.$$

On the other hand, since $p \in (2, 2^*)$ then K satisfies the Palais–Smale condition. Therefore, the mountain-pass theorem [3] applies and (4.1) admits a nonnegative nontrivial solution. □

Let $v_{1,\varepsilon}$ be the unique solution of

$$\begin{cases} -\Delta v_{1,\varepsilon} = \varepsilon\mu & \text{in } \Omega, \\ v_{1,\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and by iteration define for $k = 1, 2, \ldots$

$$\begin{cases} -\Delta v_{k+1,\varepsilon} = g\Big(\sum_{i=1}^{k} v_{i,\varepsilon}\Big) - g\Big(\sum_{i=1}^{k-1} v_{i,\varepsilon}\Big) & \text{in } \Omega, \\ v_{k+1,\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

The functions $v_{k,\varepsilon}$ are well defined in view of Theorem 8.1 in [14] and they are nonnegative in view of the weak comparison principle (see Lemma 3 in [7]). Moreover, they satisfy

$$v_{i,\varepsilon} \in L^q(\Omega) \quad \text{for all } q \in \Big[1, \frac{n}{n-2}\Big) \text{ and all } i \ge 1. \tag{4.4}$$

Suppose that u is a solution of (2.7). We introduce the functions $u_{k+1,\varepsilon} = u_{k,\varepsilon} - v_{k+1,\varepsilon}$ for $k = 0, 1, \ldots$ where $u_{0,\varepsilon} = u$. Then, the functions $u_{k+1,\varepsilon}$ solve

$$\begin{cases} -\Delta u_{k+1,\varepsilon} = g\bigg(u_{k+1,\varepsilon} + \displaystyle\sum_{i=1}^{k+1} v_{i,\varepsilon}\bigg) - g\bigg(\displaystyle\sum_{i=1}^{k} v_{i,\varepsilon}\bigg) & \text{in } \Omega, \\ u_{k+1,\varepsilon} \geq 0 & \text{in } \Omega, \\ u_{k+1,\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

For $k \geq 1$ introduce the odd function $h_{k,\varepsilon}(x, s)$, defined by

$$h_{k,\varepsilon}(x, s) = \begin{cases} g(s + \gamma_{k,\varepsilon}) - g(\gamma_{k,\varepsilon}) & \text{if } s \geq 0, \\ h_{k,\varepsilon}(x, s) = -h_{k,\varepsilon}(x, -s) & \text{if } s < 0, \end{cases} \tag{4.6}$$

and the function $f_{k,\varepsilon}$ defined by

$$f_{k,\varepsilon} = g(\gamma_{k,\varepsilon}) - g(\gamma_{k-1,\varepsilon}). \tag{4.7}$$

where $\gamma_{k,\varepsilon} = \sum_{i=1}^{k} v_{i,\varepsilon}$ for $k \geq 1$ and $\gamma_{0,\varepsilon} = 0$. Here and in the sequel we denote by $h'_{k,\varepsilon}(x, s)$ the derivative of $h_{k,\varepsilon}(x, s)$ with respect to s. Then, by adding and subtracting $g(\gamma_{k+1,\varepsilon})$ in (4.5), we see that any solution u of problem (4.5) is a nonnegative solution of

$$\begin{cases} -\Delta u = h_{k+1,\varepsilon}(x, u) + f_{k+1,\varepsilon} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

that is a function $u \in L^1(\Omega)$ such that $h_{k+1,\varepsilon}(x, u) \in L^1(\Omega)$ and

$$\int_\Omega -u\Delta\varphi \, dx = \int_\Omega h_{k+1,\varepsilon}(x, u)\varphi \, dx + \int_\Omega f_{k+1,\varepsilon}\varphi \, dx \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}).$$

The existence of a solution of (2.7) then follows once we prove the existence of a solution of (4.5) or equivalently of (4.8) for a suitable k.

We start with the following technical lemma

Lemma 4.2. *Let $u \in L^\alpha(\Omega)$ and $v \in L^\beta(\Omega)$ then $uv \in L^{\alpha\beta/(\alpha+\beta)}(\Omega)$.*

Proof. This is a straightforward application of the Hölder inequality with $p = (\alpha + \beta)/\beta$ and $q = (\alpha + \beta)/\alpha$. $\square$

Next we prove

Lemma 4.3. *Let $n > 2$ and assume that g satisfies (2.8)–(2.11). Then:*

(a) *there exists a constant $C > 0$ such that*

$$h_{k,\varepsilon}(x, s) \leq (\lambda_1 + C(\gamma_{k,\varepsilon} + s)^{p-2})s \quad \text{for all } x \in \Omega, \text{ all } s \geq 0 \text{ and all } k \geq 1;$$

(b) *there exists a constant $C > 0$ such that*

$$h'_{k,\varepsilon}(x, s) \leq \lambda_1 + C(\gamma_{k,\varepsilon} + s)^{p-2} \quad \text{for all } x \in \Omega, \text{ all } s \geq 0 \text{ and all } k \geq 1;$$

(c) *there exists $N \in \mathbb{N}$ such that $v_{N,\varepsilon} \in L^\infty(\Omega)$ and $f_{N,\varepsilon} \in L^{2n/(n+2)}(\Omega)$.*

Proof. By (2.8)–(2.10) for $k = 1, 2, \ldots$ we have

$$0 \le h_{k,\varepsilon}(x,s) = g(\gamma_{k,\varepsilon} + s) - g(\gamma_{k,\varepsilon}) \le g'(\gamma_{k,\varepsilon} + s)s \le (\lambda_1 + C(\gamma_{k,\varepsilon} + s)^{p-2})s$$

for any $x \in \Omega$ and $s \ge 0$. This proves (a) and (b).

Using again (2.8)–(2.10) for $k = 1, 2, \ldots$ we have

$$\begin{aligned}0 \le g\bigg(\sum_{i=1}^{k} v_{i,\varepsilon}\bigg) - g\bigg(\sum_{i=1}^{k-1} v_{i,\varepsilon}\bigg) \\ \le g'\bigg(\sum_{i=1}^{k} v_{i,\varepsilon}\bigg)v_{k,\varepsilon} \le \bigg(\lambda_1 + C\bigg(\sum_{i=1}^{k} v_{i,\varepsilon}\bigg)^{p-2}\bigg)v_{k,\varepsilon}\end{aligned} \tag{4.9}$$

where $C > 0$ is the constant introduced in (a). Since $p < 2_*$, by (4.4) we deduce that there exists $\alpha_0 > n/2$ such that $v_{i,\varepsilon}^{p-2} \in L^{\alpha_0}(\Omega)$ for any $i \ge 1$. With this choice of α_0 define $c = (2\alpha_0 - n)/n\alpha_0$. For a fixed $k \ge 1$ assume that $v_{k,\varepsilon} \in L^{\beta_k}(\Omega)$. Then by Lemma 4.2 and (4.4) it follows that

$$\bigg(\sum_{i=1}^{k} v_{i,\varepsilon}\bigg)^{p-2} v_{k,\varepsilon} \in L^{\alpha_0\beta_k/(\alpha_0+\beta_k)}(\Omega). \tag{4.10}$$

We distinguish three cases.

Case 1. If $\beta_k > 1/c$ then $\alpha_0\beta_k/(\alpha_0 + \beta_k) > n/2$ and hence by (4.3), (4.9), (4.10), elliptic regularity [1] and Sobolev embedding, we obtain $v_{k+1,\varepsilon} \in W^{2,\alpha_0\beta_k/(\alpha_0+\beta_k)}(\Omega) \subset L^\infty(\Omega)$.

Case 2. If $\beta_k = 1/c$ with the same procedure of Case 1, we obtain $v_{k+1,\varepsilon} \in W^{2,n/2}(\Omega) \subset L^q(\Omega)$ for any $q \ge 1$ and with another iteration this yields $v_{k+2,\varepsilon} \in L^\infty(\Omega)$.

Case 3. If $\beta_k < 1/c$, using again (4.3), (4.9), (4.10), elliptic regularity [1] and Sobolev embedding, we obtain $v_{k+1,\varepsilon} \in W^{2,\alpha_0\beta_k/(\alpha_0+\beta_k)}(\Omega) \subset L^{\beta_{k+1}}(\Omega)$ with $\beta_{k+1} = \beta_k/(1 - c\beta_k)$. After a finite number of iterations, we find $\overline{k} \ge 1$ such that $\beta_{\overline{k}} \ge 1/c$ and hence applying Cases 1 and 2 it follows that $v_{\overline{k}+2,\varepsilon} \in L^\infty(\Omega)$.

We just proved the existence of $N \in \mathbb{N}$ such that $v_{N,\varepsilon} \in L^\infty(\Omega)$. Finally by (4.9), Theorem B.1 in [8], $p < 2_*$ and $v_{N,\varepsilon} \in L^\infty(\Omega)$ it follows that $f_{N,\varepsilon} \in L^{2n/(n+2)}(\Omega)$. $\square$

From now on we look for solutions of

$$\begin{cases} -\Delta u = h_{N,\varepsilon}(x,u) + f_{N,\varepsilon} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

where $h_{N,\varepsilon}$ and $f_{N,\varepsilon}$ are defined in (4.6)–(4.7). Let $I_{N,\varepsilon}$ be the functional defined by

$$I_{N,\varepsilon}(w) = \frac{1}{2}\|\nabla w\|_{L^2}^2 - \int_\Omega H_{N,\varepsilon}(x,w)\,dx - \int_\Omega f_{N,\varepsilon} w\,dx \tag{4.12}$$

where

$$H_{N,\varepsilon}(x,s)=\int_0^s h_{N,\varepsilon}(x,t)\,dt.$$

Then we establish:

Lemma 4.4. *Let $n>2$ and assume that g satisfies* (2.8)–(2.11). *Let N be as in Lemma* 4.3. *Then it is well defined the functional $I_{N,\varepsilon}\colon H_0^1(\Omega)\to\mathbb{R}$. Moreover, $I_{N,\varepsilon}\in C^2(H_0^1(\Omega))$ and its critical points solve problem* (4.11) *or equivalently problem* (4.5) *with $k=N-1$. Finally the operator $\nabla I_{N,\varepsilon}\colon H_0^1(\Omega)\to H_0^1(\Omega)$, defined by*

$$(\nabla I_{N,\varepsilon}(u),v)_{H_0^1}=\langle I'_{N,\varepsilon}(u),v\rangle\quad\text{for all } u,v\in H_0^1(\Omega)$$

can be decomposed as

$$\nabla I_{N,\varepsilon}(u)=u+Ku=u+\Delta^{-1}(h_{N,\varepsilon}(x,u)+f_{N,\varepsilon})$$

where $K\colon H_0^1(\Omega)\to H_0^1(\Omega)$ is a compact operator.

Proof. The proof of this lemma is standard and it follows from Lemma 4.3(a) and the fact that by (4.4) there exists $\alpha_0>n/2$ such that $\gamma_{N,\varepsilon}^{p-2}\in L^{\alpha_0}(\Omega)$. □

Before proving existence of solutions for problem (4.11), we need a monotonicity result on the functions $h_{N,\varepsilon}$ and $f_{N,\varepsilon}$ with respect to ε. To this purpose we recall the following maximum principle.

Lemma 4.5. *Let $\mu\in\mathcal{M}(\Omega)$ be a nonnegative nontrivial Radon measure and let $u\in L^1(\Omega)$ be a solution of*

$$\int_\Omega -u\Delta\varphi\,dx=\int_\Omega\varphi\,d\mu\quad\text{for all } \varphi\in C_0^2(\overline{\Omega}).\tag{4.13}$$

Then $u>0$ in Ω in the sense that $\inf_K u>0$ *for any compact set $K\subset\Omega$.*

Proof. Fix a compact set $K_1\subset\Omega$ such that $\mu(K_1)>0$. For any open set Ω_1 such that $\overline{\Omega}_1\subset\Omega$ and any function $\phi\in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\phi\subset\Omega_1$, consider the following Dirichlet problem

$$\begin{cases}-\Delta\psi=\phi & \text{in } \Omega,\\ \psi=0 & \text{on } \partial\Omega.\end{cases}\tag{4.14}$$

Then denoting by $G(x,y)$ the Green function for $-\Delta$ in Ω, from (4.13) we obtain

$$\begin{aligned}\int_\Omega u\phi\,dx&=\int_\Omega -u\Delta\psi\,dx=\int_\Omega\psi\,d\mu=\int_\Omega\bigg(\int_\Omega G(x,y)\phi(y)\,dy\bigg)\,d\mu(x)\\ &\geq\int_{K_1}\bigg(\int_{\Omega_1}G(x,y)\phi(y)\,dy\bigg)\,d\mu(x)\geq C\mu(K_1)\int_\Omega\phi\,dy,\end{aligned}$$

for all $\phi \in C^\infty(\overline{\Omega})$, $\phi \geq 0$, $\operatorname{supp}\phi \subset \Omega_1$, where $C > 0$ is a constant which depends on $G(x,y)$, K_1, Ω_1. This proves that $u(x) \geq C\mu(K_1)$ for a.e. $x \in \Omega_1$ and hence the proof of the lemma is complete. □

Now we are ready to prove the following

Lemma 4.6. *Let $h_{N,\varepsilon}(x,s)$ and $f_{N,\varepsilon}$ be the functions introduced in* (4.6)–(4.7). *Then:*

(a) *for $s \geq 0$, $h_{N,\varepsilon}(x,s)$ is nondecreasing with respect to ε in the sense that for any $\varepsilon_1 < \varepsilon_2$ we have*

$$h_{N,\varepsilon_1}(x,s) \leq h_{N,\varepsilon_2}(x,s) \quad \textit{for a.e. } x \in \Omega \textit{ and } s \geq 0;$$

(b) *$f_{N,\varepsilon}$ is strictly increasing with respect to ε in the sense that for any $\varepsilon_1 < \varepsilon_2$ we have*

$$f_{N,\varepsilon_1}(x) < f_{N,\varepsilon_2}(x) \qquad \textit{for a.e. } x \in \Omega.$$

Proof. First of all note that in view of the Lemma 4.5, the functions $v_{i,\varepsilon}$ defined in (4.2)–(4.3) are strictly positive almost everywhere in Ω for any $i \in \{1,\dots,N\}$. Therefore, using (2.8)–(2.9), by induction on $i = 1,\dots,N$ we prove that $v_{i,\varepsilon}$ is strictly increasing with respect to ε for any $i \in \{1,\dots,N\}$. This is trivial for $i = 1$. Assuming our claim true for any $i \leq k$, we prove that if $\varepsilon_1 < \varepsilon_2$ then $v_{k+1,\varepsilon_1} < v_{k+1,\varepsilon_2}$ almost everywhere in Ω. By (2.8)–(2.9) we have

$$\frac{d}{d\varepsilon}[g(\gamma_{k,\varepsilon}) - g(\gamma_{k-1,\varepsilon})] \geq g'(\gamma_{k-1,\varepsilon})\frac{d}{d\varepsilon}v_{k,\varepsilon} \geq 0. \tag{4.15}$$

where the derivative $dv_{k,\varepsilon}/d\varepsilon$ is defined by

$$\left(\frac{d}{d\varepsilon}v_{k,\varepsilon}\right)\bigg|_{\varepsilon=\varepsilon_0} = \lim_{t\to 0}\frac{v_{k,\varepsilon_0+t} - v_{k,\varepsilon_0}}{t} \quad \text{in } L^q(\Omega),\ q \in \left[1, \frac{n}{n-2}\right).$$

The previous limit is well defined by induction on k since in view of Theorem 8.1 in [14], the linear operator $(-\Delta)^{-1}\colon \mathcal{M}(\Omega) \to L^q(\Omega)$ is continuous for any $q \in [1, n/(n-2))$. If we assume by contradiction that the map $\varepsilon \mapsto g(\gamma_{k,\varepsilon}) - g(\gamma_{k-1,\varepsilon})$ is not strictly increasing then by (4.15) we infer that there exist $\varepsilon_1 < \varepsilon_2$ such that $g'(\gamma_{k-1,\varepsilon})dv_{k,\varepsilon}/d\varepsilon = 0$ for any $\varepsilon \in (\varepsilon_1,\varepsilon_2)$. Since $g'(\gamma_{k-1,\varepsilon}) > 0$, this contradicts the fact that the map $\varepsilon \mapsto v_{k,\varepsilon}$ is strictly increasing. The strict monotonicity of $\varepsilon \mapsto v_{k+1,\varepsilon}$ then follows immediately from (4.3) and Lemma 4.5.

By (2.8)–(2.9) and (4.6) we have

$$\frac{d}{d\varepsilon}h_{N,\varepsilon}(x,s) = [g'(s+\gamma_{N,\varepsilon}) - g'(\gamma_{N,\varepsilon})]\frac{d}{d\varepsilon}\gamma_{N,\varepsilon} \geq 0 \quad \text{for a.e. } x \in \Omega, \text{ for all } s \geq 0.$$

This proves part (a). Finally, by (2.8)–(2.9) and (4.7), we have

$$\frac{d}{d\varepsilon}f_{N,\varepsilon} \geq g'(\gamma_{N-1,\varepsilon})\frac{d}{d\varepsilon}v_{N,\varepsilon} \quad \text{for a.e. } x \in \Omega.$$

Therefore, since $g'(\gamma_{N-1,\varepsilon}) > 0$ almost everywhere in Ω and $v_{N,\varepsilon}$ is strictly increasing with respect to ε, we conclude that $f_{N,\varepsilon}$ is also strictly increasing with respect to ε. $\square$

In the next lemma we prove an existence result for (4.11) for small ε.

Lemma 4.16. *Let $n > 2$ and assume that g satisfies* (2.8)–(2.11). *Then the set*

$$E = \{\varepsilon > 0 : (4.11) \textit{ has a nonnegative solution } u \in H_0^1(\Omega)\}$$

is an interval and $\inf E = 0$.

Proof. First we prove that E is nonempty. Denote for simplicity by v_k the functions $v_{k,1}$. Define by iteration the following sequence of functions

$$-\Delta\widetilde{v}_k = \bigg[\lambda_1 + C\bigg(\sum_{i=1}^{k-1}\widetilde{v}_i\bigg)^{p-2}\bigg]\widetilde{v}_{k-1} \tag{4.16}$$

if $k \geq 2$ and $\widetilde{v}_k = v_k$ if $k = 1$ where C denotes the positive constant introduced in (4.9). Assuming $\varepsilon < 1$, we claim that $v_{k,\varepsilon} \leq \varepsilon\widetilde{v}_k$. By (4.2) we deduce that $v_{1,\varepsilon} = \varepsilon v_1$. By induction on k, suppose that $v_{i,\varepsilon} \leq \varepsilon\widetilde{v}_i$ for any $i = 1,\dots,k$. Then, using again (4.9), we have

$$\begin{aligned}0 &\leq g\bigg(\sum_{i=1}^{k} v_{i,\varepsilon}\bigg) - g\bigg(\sum_{i=1}^{k-1} v_{i,\varepsilon}\bigg)\\ &\leq \bigg[\lambda_1 + C\bigg(\sum_{i=1}^{k} v_{i,\varepsilon}\bigg)^{p-2}\bigg]v_{k,\varepsilon} \leq \varepsilon\bigg[\lambda_1 + C\bigg(\sum_{i=1}^{k}\widetilde{v}_i\bigg)^{p-2}\bigg]\widetilde{v}_k\end{aligned}$$

and by (4.3), (4.16) and the weak comparison principle (see Lemma 3 in [7]), we deduce that $v_{k+1,\varepsilon} \leq \varepsilon\widetilde{v}_{k+1}$. This proves that $v_{k,\varepsilon} \leq \varepsilon\widetilde{v}_k$ for any $k \leq N$. This yields

$$0 \leq f_{N,\varepsilon} \leq g'(\gamma_{N,\varepsilon})v_{N,\varepsilon} \leq [\lambda_1 + C\gamma_{N,\varepsilon}^{p-2}]v_{N,\varepsilon} \leq \varepsilon\widetilde{f}_N \tag{4.17}$$

with

$$\widetilde{f}_N = \bigg[\lambda_1 + C\bigg(\sum_{i=1}^{N}\widetilde{v}_i\bigg)^{p-2}\bigg]\widetilde{v}_N.$$

On the other hand, by (2.8), (2.10), (4.6) we have for a fixed $\sigma \in (0, \lambda_1 - \lambda)$

$$\begin{aligned}|h_{N,\varepsilon}(x,s)| &\leq g'(|s| + \gamma_{N,\varepsilon})|s| \leq [(\lambda + \sigma) + C(|s| + \gamma_{N,\varepsilon})^{p-2}]|s|\\ &\leq (\lambda + \sigma)|s| + C_1|s|^{p-1} + \varepsilon^{p-2}C_2\bigg(\sum_{i=1}^{N}\widetilde{v}_i\bigg)^{p-2}|s|\end{aligned} \tag{4.18}$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$. By Lemma 4.3 and Theorem B.1 in [8], we infer that $\widetilde{f}_N \in L^{2n/(n+2)}(\Omega)$ and $a(x) = C_2(\sum_{i=1}^{N}\widetilde{v}_i)^{p-2} \in L^{n/2}(\Omega)$. Therefore, in view of Lemma 4.1, the homogeneous Dirichlet problem associated to the equation

$$-\Delta u = (\lambda+\sigma)u + C_1 u^{p-1} + \varepsilon^{p-2}a(x)u + \varepsilon \widetilde{f}_N$$

admits a nonnegative nontrivial solution $w \in H_0^1(\Omega)$ for ε small enough. Then, by (4.17), (4.18) we deduce that w is a supersolution of (4.11) and since the null function is a subsolution of (4.11), using the super-subsolution method, it follows that (4.11) admits a nonnegative minimal solution $u_{N,\varepsilon} \in H_0^1(\Omega)$. This proves that the set E is nonempty. Let us prove that E is an interval. By Lemma 4.6, we have that $h_{N,\varepsilon}(x,s)$ and $f_{N,\varepsilon}$ are nondecreasing with respect to ε. Therefore we deduce that if $\overline{\varepsilon} \in E$ then the corresponding solution of (4.11) is a supersolution for (4.11) corresponding to $\varepsilon \in (0,\overline{\varepsilon})$. Using again the super-subsolution method we infer that (4.11) admits a nonnegative solution $u_{N,\varepsilon} \in H_0^1(\Omega)$ for any $\varepsilon \in (0,\overline{\varepsilon})$. This completes the proof of the lemma. □

Next we prove a nonexistence result for (2.7) for ε large enough.

Lemma 4.17. *Let $n > 2$ and assume that g satisfies* (2.8)–(2.11). *Then there exists $\widetilde{\varepsilon}$ such that* (2.7) *admits no solutions for $\varepsilon > \widetilde{\varepsilon}$.*

Proof. Fix $\varepsilon > 0$ and assume that (2.7) admits a solution u. Let e_1 be a positive eigenfunction of $-\Delta$ associated to the first eigenvalue λ_1. Then we have

$$\int_\Omega -u\Delta e_1\,dx = \int_\Omega g(u)e_1\,dx + \varepsilon\int_\Omega e_1\,d\mu.$$

By (2.11) we infer that there exists $M > 0$ such that $g(s) > \lambda_1 s$ for any $s > M$, from which we obtain

$$\varepsilon\int_\Omega e_1\,d\mu = \int_{\{0\le u\le M\}}[\lambda_1 u - g(u)]e_1\,dx + \int_{\{u>M\}}[\lambda_1 u - g(u)]e_1\,dx \le \lambda_1 M\|e_1\|_{L^1}$$

and hence

$$\varepsilon \le \widetilde{\varepsilon} = \frac{\lambda_1 M\|e_1\|_{L^1}}{\int_\Omega e_1\,d\mu}.$$

This completes the proof of the lemma. □

End of the proof of Theorem 2.2. By Lemmas 4.16 and 4.17 we deduce that E is a nonempty bounded interval. If we define $\varepsilon^* = \sup E$, then we conclude that (2.7) admits a solution u_ε given by $u_\varepsilon = u_{N,\varepsilon} + \gamma_{N,\varepsilon}$ for $\varepsilon < \varepsilon^*$ and no solutions for $\varepsilon > \varepsilon^*$. □

5. Proof of Proposition 2.3

Define for $s \ge 0$ the function $\widetilde{g}(s) = g(s) - \lambda s \ge 0$. Let e_1 be a positive eigenfunction of $-\Delta$ associated to the first eigenvalue λ_1. If u solves (2.7) then it satisfies

$$\int_\Omega -u\Delta e_1\,dx = \int_\Omega \lambda u e_1\,dx + \int_\Omega \widetilde{g}(u)e_1\,dx + \varepsilon\int_\Omega e_1\,d\mu$$

and in turn

$$(\lambda_1 - \lambda) \int_\Omega u e_1 \, dx = \int_\Omega \widetilde{g}(u) e_1 \, dx + \varepsilon \int_\Omega e_1 \, d\mu > 0.$$

This proves that $\lambda < \lambda_1$. □

6. Proof of Corollary 2.4

Consider the decomposition $\mu = \mu^+ - \mu^-$ with $\mu^+, \mu^- \geq 0$. Introduce the following problem

$$\begin{cases} -\Delta u = g(u) + \varepsilon \mu^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

By Theorem 2.2 we infer that if ε is small enough then (6.1) admits a nonnegative solution u_1. Moreover, since $\mu^+ \geq \mu$ then u_1 is a supersolution for (2.12), i.e.

$$\int_\Omega -u_1 \Delta\varphi \, dx \geq \int_\Omega g(u_1)\varphi \, dx + \varepsilon \int_\Omega \varphi \, d\mu \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}), \ \varphi \geq 0.$$

On the other hand, if we introduce the problem

$$\begin{cases} -\Delta u = g(u) - \varepsilon \mu^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.2}$$

then by Theorem 2.2 and (2.8), we deduce that if ε is small enough then (6.2) admits a nonpositive solution u_2. Moreover, since $-\mu^- \leq \mu$ then u_2 is a subsolution for (2.12), i.e.

$$\int_\Omega -u_2 \Delta\varphi \, dx \leq \int_\Omega g(u_2)\varphi \, dx + \varepsilon \int_\Omega \varphi \, d\mu \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}), \ \varphi \geq 0.$$

We just proved the existence of a supersolution u_1 and a subsolution u_2 with $u_1 \geq u_2$. Since g is an increasing function, the existence of a solution of (2.12) then follows from the super-subsolution method (see Lemma 3 in [7] for more details).

7. Further properties of minimal solutions of (4.11)

In this section we denote by $u_{N,\varepsilon}$ the minimal solution of (4.11) found in Lemma 4.16. We prove that the function $u_{N,\varepsilon}$ is a local minimizer for the functional $I_{N,\varepsilon}$ for any $\varepsilon \in (0, \varepsilon^*)$. To this purpose we define the weighted eigenvalue problem depending on ε

$$\begin{cases} -\Delta\psi = \lambda g'(u_{N,\varepsilon} + \gamma_{N,\varepsilon})\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

and the number

$$\lambda_1(\varepsilon) = \inf \left\{ \|\nabla\psi\|_{L^2}^2 : \psi \in H_0^1(\Omega), \ \int_\Omega g'(u_{N,\varepsilon} + \gamma_{N,\varepsilon})\psi^2 \, dx = 1 \right\}. \tag{7.2}$$

First of all note that the integral in (7.2) is well defined since $g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon}) \in L^{n/2}(\Omega)$. This follows from (4.4) and $g'(s) = O(s^{p-2})$ as $s \to \infty$ with $p < 2_*$. Using the compact embedding $H_0^1(\Omega) \subset L^2(\Omega; g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon})\,dx)$, by standard arguments, we deduce that $\lambda_1(\varepsilon)$ admits a minimizer which is a solution of (7.1) with $\lambda = \lambda_1(\varepsilon)$. Thanks to the strong maximum principle (see Theorem 8.19 in [10]), we get that problem (7.2) admits a positive minimizer ψ_1. We are ready to prove the following

Lemma 7.1. *Let $n > 2$ and assume that g satisfies* (2.8)–(2.11). *If $\varepsilon \in (0,\varepsilon^*)$ then $u_{N,\varepsilon}$ is a local minimizer for $I_{N,\varepsilon}$.*

Proof. For any fixed $\varepsilon \in (0,\varepsilon^*)$, let $\overline{\varepsilon} \in (\varepsilon,\varepsilon^*)$. We claim that $\lambda_1(\varepsilon) > 1$. Consider the corresponding minimal solutions of (4.11) $u_{N,\varepsilon}$ and $u_{N,\overline{\varepsilon}}$. By Lemma 4.6, (2.8) and the weak comparison principle we have $u_{N,\varepsilon} \le u_{N,\overline{\varepsilon}}$ and

$$
\begin{aligned}
(7.3)\quad & \int_\Omega \nabla(u_{N,\overline{\varepsilon}} - u_{N,\varepsilon})\nabla w\,dx \\
&= \int_\Omega [h_{N,\overline{\varepsilon}}(x,u_{N,\overline{\varepsilon}}) - h_{N,\varepsilon}(x,u_{N,\varepsilon})]w\,dx + \int_\Omega (f_{N,\overline{\varepsilon}} - f_{N,\varepsilon})w\,dx \\
&> \int_\Omega [h_{N,\varepsilon}(x,u_{N,\overline{\varepsilon}}) - h_{N,\varepsilon}(x,u_{N,\varepsilon})]w\,dx \\
&= \int_\Omega [g(u_{N,\overline{\varepsilon}}+\gamma_{N,\varepsilon}) - g(u_{N,\varepsilon}+\gamma_{N,\varepsilon})]\,w\,dx \\
&\ge \int_\Omega g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon})(u_{N,\overline{\varepsilon}} - u_{N,\varepsilon})w\,dx
\end{aligned}
$$

for all $w \in H_0^1(\Omega)$, $w > 0$ a.e. in Ω. Choosing $w = \psi_1 > 0$, by (7.1) and (7.3) we obtain

$$
\begin{aligned}
\lambda_1(\varepsilon)\int_\Omega g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon})(u_{N,\overline{\varepsilon}} - u_{N,\varepsilon})\psi_1\,dx &= \int_\Omega \nabla\psi_1\nabla(u_{N,\overline{\varepsilon}} - u_{N,\varepsilon})\,dx \\
&> \int_\Omega g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon})(u_{N,\overline{\varepsilon}} - u_{N,\varepsilon})\psi_1\,dx \ge 0
\end{aligned}
$$

which proves that $\lambda_1(\varepsilon) > 1$. Since

$$\|\nabla\psi\|_{L^2}^2 \ge \lambda_1(\varepsilon)\int_\Omega g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon})\psi^2\,dx \quad \text{for all } \psi \in H_0^1(\Omega)$$

then, for all $\psi \in H_0^1(\Omega)$, we obtain

$$(7.4)\qquad \|\nabla\psi\|_{L^2}^2 - \int_\Omega g'(u_{N,\varepsilon}+\gamma_{N,\varepsilon})\psi^2\,dx \ge \left(1 - \frac{1}{\lambda_1(\varepsilon)}\right)\|\nabla\psi\|_{L^2}^2.$$

By (4.6), Lemma 4.4 and (7.4), we deduce that there exists $C > 0$ such that

$$(7.5)\qquad \langle I''_{N,\varepsilon}(u_{N,\varepsilon})\psi,\psi\rangle = \|\nabla\psi\|_{L^2}^2 - \int_\Omega h'_{N,\varepsilon}(x,u_{N,\varepsilon})\psi^2\,dx \ge C\|\nabla\psi\|_{L^2}^2$$

for all $\psi \in H_0^1(\Omega)$. This proves that $u_{N,\varepsilon}$ is a local minimizer for $I_{N,\varepsilon}$. □

8. Proof of Theorem 2.5

Before proving the existence of a solution of (2.7) for $\varepsilon = \varepsilon^*$, we need the following technical inequality

Lemma 8.1. *For any $r > 2$ there exists $\sigma > 0$ such that*

$$(r-1)(t+s)^{r-2}s^2-(t+s)^{r-1}s+t^{r-1}s \geq \sigma[(t+s)^{r-1}s-t^{r-1}s-(r-1)t^{r-2}s^2] \tag{8.1}$$

for any $s, t \geq 0$.

Proof. The proof of (8.1) becomes trivial for $r = 2$ and hence we may assume $r > 2$. Divide both sides of (8.1) by $s^r > 0$ and put $y = t/s$. Then the proof of the lemma is complete once we prove that

$$(r-1)(1+y)^{r-2} - (1+y)^{r-1} + y^{r-1} \geq \sigma[(1+y)^{r-1} - y^{r-1} - (r-1)y^{r-2}] \tag{8.2}$$

for all $y \geq 0$. Define for $y \geq 0$ the functions

$$\Psi(y) = (r-1)(1+y)^{r-2} + y^{r-1} - (1+y)^{r-1} \tag{8.3}$$

and

$$\Phi(y) = (1+y)^{r-1} - y^{r-1} - (r-1)y^{r-2}. \tag{8.4}$$

By Lagrange Theorem we have

$$\Phi(y) = \frac{(r-1)(r-2)}{2}\xi_y^{r-3} > 0 \quad \text{for all } y \geq 0$$

for a suitable $\xi_y \in (y, y+1)$. On the other hand, by (8.3) we have

$$\widetilde{\Psi}(y) = y^{-(r-2)}\Psi(y) = (r-1)\left(1+\frac{1}{y}\right)^{r-2} + y - y\left(1+\frac{1}{y}\right)^{r-1} \quad \text{for all } y > 0.$$

By a second order Taylor expansion we obtain

$$\widetilde{\Psi}(y) = \frac{(r-1)(r-2)}{2}\frac{1}{y} + o\left(\frac{1}{y}\right) \to 0 \quad \text{as } y \to \infty. \tag{8.5}$$

Moreover,

$$\widetilde{\Psi}''(y) = \frac{(r-1)(r-2)(r-3)}{y^4}\left(1+\frac{1}{y}\right)^{r-4} + \frac{(r-1)(r-2)}{y^3}\left(1+\frac{1}{y}\right)^{r-3} > 0$$

for all $y > 0$ and this with (8.5) implies that $\widetilde{\Psi}(y) > 0$ for any $y > 0$ and in turn that $\Psi(y) > 0$ for any $y \geq 0$. Using again a second order Taylor expansion we obtain

$$\Psi(y) = \frac{(r-1)(r-2)}{2}y^{r-3} + o(y^{r-3}) \quad \text{and} \quad \Phi(y) = \frac{(r-1)(r-2)}{2}y^{r-3} + o(y^{r-3})$$

as $y \to \infty$. Therefore, there exists $\overline{y} > 0$ such that $\Psi(y)/\Phi(y) > 1/2$ for any $y > \overline{y}$ and hence

$$\Psi(y) > \sigma\Phi(y) \quad \text{for all } y > \overline{y} \text{ and all } \sigma \in (0, 1/2). \tag{8.6}$$

On the other hand, since Φ and Ψ are strictly positive in $[0, \infty)$ then

$$\Psi(y) > \sigma\Phi(y) \quad \text{for all } y \in [0, \overline{y}] \text{ and all } \sigma \in (0, M) \tag{8.7}$$

where $M = \min_{s\in[0,\overline{y}]} \Psi(s)/\Phi(s)$.

Finally, by (8.6)–(8.7) we obtain $\Psi(y) > \sigma\Phi(y)$ for all $y \geq 0$ with $\sigma < \min\{1/2, M\}$. This proves (8.1). □

Next we deal with the existence of a solution of (4.11) for $\varepsilon = \varepsilon^* = \sup E$.

LEMMA 8.2. *Let $n > 2$ and assume that $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \leq q < p < 2_*$, $\lambda \geq 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$. If $\varepsilon = \varepsilon^*$ then* (4.11) *admits a minimal solution u_{N,ε^*}.*

PROOF. Consider $\varepsilon \in (0, \varepsilon^*)$ and the corresponding minimal solution $u_{N,\varepsilon}$. Then $u_{N,\varepsilon}$ solves

$$\int_\Omega \nabla u_{N,\varepsilon} \nabla w \, dx = \int_\Omega h_{N,\varepsilon}(x, u_{N,\varepsilon}) w \, dx + \int_\Omega f_{N,\varepsilon} w \, dx \tag{8.8}$$

for all $w \in H_0^1(\Omega)$. By (4.6) and (7.4), we have

$$\int_\Omega |\nabla u_{N,\varepsilon}|^2 \, dx \geq \int_\Omega h'_{N,\varepsilon}(x, u_{N,\varepsilon}) u_{N,\varepsilon}^2 \, dx. \tag{8.9}$$

Choosing $w = u_{N,\varepsilon}$ in (8.8), this yields

$$\int_\Omega h'_{N,\varepsilon}(x, u_{N,\varepsilon}) u_{N,\varepsilon}^2 \, dx \leq \int_\Omega h_{N,\varepsilon}(x, u_{N,\varepsilon}) u_{N,\varepsilon} \, dx + \int_\Omega f_{N,\varepsilon} u_{N,\varepsilon} \, dx. \tag{8.10}$$

Replacing $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ in (4.6) and applying Lemma 8.1, we obtain for $\sigma > 0$ small enough

$$\begin{aligned} \int_\Omega & h'_{N,\varepsilon}(x, u_{N,\varepsilon}) u_{N,\varepsilon}^2 \, dx \\ & \geq (1 + \sigma) \int_\Omega h_{N,\varepsilon}(x, u_{N,\varepsilon}) u_{N,\varepsilon} \, dx - \sigma \int_\Omega g'(\gamma_{N,\varepsilon}) u_{N,\varepsilon}^2 \, dx. \end{aligned} \tag{8.11}$$

By (8.10) and (8.11) we have

$$\sigma \int_\Omega h_{N,\varepsilon}(x, u_{N,\varepsilon}) u_{N,\varepsilon} \, dx \leq \sigma \int_\Omega g'(\gamma_{N,\varepsilon}) u_{N,\varepsilon}^2 \, dx + \int_\Omega f_{N,\varepsilon} u_{N,\varepsilon} \, dx$$

and hence by (8.8) with $w = u_{N,\varepsilon}$ we obtain

$$\sigma \|\nabla u_{N,\varepsilon}\|_{L^2}^2 \leq \sigma \int_\Omega g'(\gamma_{N,\varepsilon}) u_{N,\varepsilon}^2 \, dx + (\sigma + 1) \int_\Omega f_{N,\varepsilon} u_{N,\varepsilon} \, dx. \tag{8.12}$$

If we define

$$\Lambda_1(\varepsilon) = \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla\psi\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon})\psi^2\,dx}$$

then by (8.12), Hölder inequality, Sobolev embedding and Lemma 4.6, we infer

$$\sigma\bigg(1 - \frac{1}{\Lambda_1(\varepsilon)}\bigg)\|\nabla u_{N,\varepsilon}\|_{L^2}^2 \le (\sigma+1)C\|f_{N,\varepsilon^*}\|_{L^{2n/(n+2)}}\|u_{N,\varepsilon}\|_{H_0^1}.$$

This implies that $\{u_{N,\varepsilon}\}$ is bounded in $H_0^1(\Omega)$ uniformly with respect to ε once we prove that

$$\lim_{\varepsilon\to\varepsilon^*} \Lambda_1(\varepsilon) > 1. \tag{8.13}$$

To this purpose consider a sequence $\{\psi_{1,\varepsilon}\}$ of minimizers of $\Lambda_1(\varepsilon)$ such that $\psi_{1,\varepsilon} > 0$ in Ω and $\int_\Omega g'(\gamma_{N,\varepsilon})\psi_{1,\varepsilon}^2\,dx = 1$. Then $\{\psi_{1,\varepsilon}\}$ is bounded in $H_0^1(\Omega)$ and hence up to subsequences we may assume that $\psi_{1,\varepsilon} \rightharpoonup \psi^*$ in $H_0^1(\Omega)$ as $\varepsilon \to \varepsilon^*$. By Theorem B.1 in [8] and monotone convergence (see Lemma 4.6) we infer that $g'(\gamma_{N,\varepsilon}) \to g'(\gamma_{N,\varepsilon^*})$ in $L^{n/2}(\Omega)$ as $\varepsilon \to \varepsilon^*$ and since by Sobolev embedding $\psi_{1,\varepsilon}^2 \rightharpoonup (\psi^*)^2$ in $L^{n/(n-2)}(\Omega)$, then we have $\int_\Omega g'(\gamma_{N,\varepsilon})\psi_{1,\varepsilon}^2\,dx \to \int_\Omega g'(\gamma_{N,\varepsilon^*})(\psi^*)^2\,dx$ up to subsequences. This proves that

$$\int_\Omega g'(\gamma_{N,\varepsilon^*})(\psi^*)^2\,dx = 1. \tag{8.14}$$

For any fixed $\overline{\varepsilon} \in (0,\varepsilon^*)$, we deduce by Lemmas 4.5 and 4.6 that $u_{N,\varepsilon} \ge u_{N,\overline{\varepsilon}} > 0$ almost everywhere in Ω for any $\varepsilon \in (\overline{\varepsilon},\varepsilon^*)$. Moreover, by (8.14), weak lower semicontinuity of the H_0^1-norm and monotone convergence, we have

$$\begin{aligned}
\lim_{\varepsilon\to\varepsilon^*} \Lambda_1(\varepsilon) &\ge \frac{\|\nabla\psi^*\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon^*})(\psi^*)^2\,dx} > \frac{\|\nabla\psi^*\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon^*}+u_{N,\overline{\varepsilon}})(\psi^*)^2\,dx} \\
&= \lim_{\varepsilon\to\varepsilon^*} \frac{\|\nabla\psi^*\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon}+u_{N,\overline{\varepsilon}})(\psi^*)^2\,dx} \\
&\ge \lim_{\varepsilon\to\varepsilon^*} \frac{\|\nabla\psi^*\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon}+u_{N,\varepsilon})(\psi^*)^2\,dx} \ge \lim_{\varepsilon\to\varepsilon^*} \lambda_1(\varepsilon) \ge 1
\end{aligned}$$

where the last inequality follows from the proof of Lemma 7.1. This proves (8.13).

Up to subsequences we may assume that there exists $u^* \in H_0^1(\Omega)$ such that

$$u_{N,\varepsilon} \rightharpoonup u^* \quad \text{in } H_0^1(\Omega) \text{ as } \varepsilon \to \varepsilon^* \tag{8.15}$$

and

$$u_{N,\varepsilon} \to u^* \quad \text{a.e. in } \Omega \text{ as } \varepsilon \to \varepsilon^*.$$

By Lemma 4.4 we deduce that $h_{N,\varepsilon}(x,u) \in L^{2n/(n+2)}(\Omega)$ for any $u \in H_0^1(\Omega)$ and $\varepsilon > 0$. Therefore by Lemma 4.6 and monotone convergence we have

$$h_{N,\varepsilon}(x,u_{N,\varepsilon}) \to h_{N,\varepsilon^*}(x,u^*) \quad \text{in } L^{2n/(n+2)}(\Omega) \text{ as } \varepsilon \to \varepsilon^*. \tag{8.16}$$

For the same reason, we deduce that

$$f_{N,\varepsilon} \to f_{N,\varepsilon^*} \quad \text{in } L^{2n/(n+2)}(\Omega) \text{ as } \varepsilon \to \varepsilon^*. \tag{8.17}$$

Therefore by (8.15)–(8.17), passing to the limit in (8.8) we obtain

$$\int_\Omega \nabla u^* \nabla w \, dx = \int_\Omega h_{N,\varepsilon^*}(x, u^*) w \, dx + \int_\Omega f_{N,\varepsilon^*} w \, dx$$

for all $w \in H_0^1(\Omega)$. This completes the proof of the lemma. □

END OF THE PROOF OF THEOREM 2.5. The existence of a solution u_{ε^*} of (2.7) follows immediately from Lemma 8.2 defining $u_{\varepsilon^*} = u_{N,\varepsilon^*} + \gamma_{N,\varepsilon^*}$. □

9. Proof of Theorem 2.6

The existence of a second solution of (2.7) is obtained as critical point of the functional $I_{N,\varepsilon}$ defined in (4.12). In this section we will use the same notations as in the proof of Theorem 2.2. First we prove that $I_{N,\varepsilon}$ satisfies the PS condition. In order to prove boundedness of PS sequences we need a technical inequality.

LEMMA 9.1. *For any $r > 2$ there exists $\sigma > 0$ such that, for any $s, t \geq 0$,*

$$\begin{aligned}(t+s)^{r-1}s + t^{r-1}s - \frac{2}{r}(t+s)^r + \frac{2}{r}t^r \\ \geq \sigma\left[\frac{1}{r}(t+s)^r - \frac{1}{r}t^r - t^{r-1}s - \frac{r-1}{2}t^{r-2}s^2\right].\end{aligned} \tag{9.1}$$

PROOF. The proof of (9.1) becomes trivial for $r = 2$ and hence we may assume $r > 2$. Divide both sides of (9.1) by $s^r > 0$ and put $y = t/s$. The proof of (9.1) is equivalent to $\Psi(y) \geq \sigma\Phi(y)$ for all $y \geq 0$ with

$$\Psi(y) = (1+y)^{r-1} + y^{r-1} - \frac{2}{r}[(1+y)^r - y^r], \tag{9.2}$$

$$\Phi(y) = \frac{1}{r}[(1+y)^r - y^r] - y^{r-1} - \frac{r-1}{2}y^{r-2}. \tag{9.3}$$

The inequality (9.1) may be obtained applying the procedure introduced in the proof of Lemma 8.1 to the functions Ψ and Φ defined in (9.2)–(9.3). □

We are ready to prove that the functional $I_{N,\varepsilon}$ satisfies the PS condition.

LEMMA 9.2. *Let $n > 2$ and assume that $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \leq q < p < 2_*$, $\lambda \geq 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$. Then $I_{N,\varepsilon}$ satisfies the* PS *condition.*

Proof. Let $\{u_m\}$ be a PS sequence for $I_{N,\varepsilon}$. Then we have

$$(9.4)\quad \begin{aligned} C + o(\|u_m\|_{H_0^1}) &= (2+\sigma)I_{N,\varepsilon}(u_m) - \langle I'_{N,\varepsilon}(u_m), u_m\rangle \\ &= \frac{\sigma}{2}\|u_m\|^2_{H_0^1} + \int_\Omega [h_{N,\varepsilon}(x,u_m)u_m - (2+\sigma)H_{N,\varepsilon}(x,u_m)]\,dx \\ &\qquad - (1+\sigma)\int_\Omega f_{N,\varepsilon}u_m\,dx. \end{aligned}$$

Replacing $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ in (4.6) and applying Lemma 9.1, we obtain for $\sigma > 0$ small enough

$$(9.5)\quad \int_\Omega [h_{N,\varepsilon}(x,u_m)u_m - (2+\sigma)H_{N,\varepsilon}(x,u_m)]\,dx \geq -\frac{\sigma}{2}\int_\Omega g'(\gamma_{N,\varepsilon})u_m^2\,dx.$$

Therefore, by (9.4) and (9.5) we infer

$$(9.6)\quad C + o(\|u_m\|_{H_0^1}) \geq \frac{\sigma}{2}\bigg(\|\nabla u_m\|^2_{L^2} - \int_\Omega g'(\gamma_{N,\varepsilon})u_m^2\,dx\bigg) - (1+\sigma)\int_\Omega f_{N,\varepsilon}u_m\,dx.$$

But from (7.2) and the proof of Lemma 7.1, we deduce that

$$\Lambda_1(\varepsilon) = \inf_{\psi\in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla\psi\|^2_{L^2}}{\int_\Omega g'(\gamma_{N,\varepsilon})\psi^2\,dx} \geq \lambda_1(\varepsilon) > 1$$

and hence by (9.6), Hölder inequality and Sobolev embedding, we obtain

$$C + o(\|u_m\|_{H_0^1}) \geq \frac{\sigma}{2}\bigg(1 - \frac{1}{\Lambda_1(\varepsilon)}\bigg)\|\nabla u_m\|^2_{L^2} - (1+\sigma)C\|f_{N,\varepsilon}\|_{L^{2n/(n+2)}}\|u_m\|_{H_0^1}$$

and in turn

$$\frac{\sigma}{2}\bigg(1 - \frac{1}{\Lambda_1(\varepsilon)}\bigg)\|u_m\|^2_{H_0^1} \leq C + o(\|u_m\|_{H_0^1}) + (1+\sigma)C\|f_{N,\varepsilon}\|_{L^{2n/(n+2)}}\|u_m\|_{H_0^1}$$

which proves that the sequence $\{u_m\}$ is bounded in $H_0^1(\Omega)$. Up to subsequences, we may assume that there exists $u \in H_0^1(\Omega)$ such that $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_m \to u$ in $L^r(\Omega)$ for any $r \in [1, 2^*)$.

Since $\{u_m\}$ is a PS sequence, by Lemma 4.4 it follows that $u_m = \nabla I_{N,\varepsilon}(u_m) - Ku_m$ and $u_m \to u$ strongly in $H_0^1(\Omega)$. □

Next we prove the existence of a second solution of (4.11).

Lemma 9.3. *Let $n > 2$ and assume that $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \leq q < p < 2_*$, $\lambda \geq 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$. Then for any $\varepsilon \in (0, \varepsilon^*)$, (4.11) admits a nonnegative solution $U_{N,\varepsilon}$ with $I_{N,\varepsilon}(U_{N,\varepsilon}) > I_{N,\varepsilon}(u_{N,\varepsilon})$. Moreover, $U_{N,\varepsilon} > u_{N,\varepsilon}$ almost everywhere in Ω.*

Proof. In Lemma 9.2 we proved that the functional $I_{N,\varepsilon}$ satisfies the PS condition. In view of (7.5) in Lemma 7.1, we deduce that there exist $\rho > 0$ and $R > I_{N,\varepsilon}(u_{N,\varepsilon})$ such that

$$(9.7)\quad I_{N,\varepsilon}(\psi) \geq R \quad \text{for all } \psi \text{ such that } \|\psi - u_{N,\varepsilon}\|_{H_0^1} = \rho.$$

Moreover, since $p > 2$ we deduce that there exists a nonnegative nontrivial function $w \in H_0^1(\Omega)$ such that $\|w - u_{N,\varepsilon}\|_{H_0^1} > \rho$ and $I_{N,\varepsilon}(w) < I_{N,\varepsilon}(u_{N,\varepsilon})$. Then, we define the functional

$$\widetilde{I}_{N,\varepsilon}(\psi) = \frac{1}{2}\|\nabla\psi\|_{L^2}^2 - \int_\Omega H_{N,\varepsilon}(x,\psi^+)\,dx - \int_\Omega f_{N,\varepsilon}\psi\,dx \tag{9.8}$$

for all $\psi \in H_0^1(\Omega)$. Replacing $I_{N,\varepsilon}$ with $\widetilde{I}_{N,\varepsilon}$ in the proofs of Lemmas 4.4 and 9.2 it follows that $\widetilde{I}_{N,\varepsilon} \in C^1(H_0^1(\Omega))$ and satisfies the PS condition.

Introduce the set of paths Γ defined by

$$\Gamma = \{\gamma \in C^0([0,1]; H_0^1(\Omega)) : \gamma(0) = u_{N,\varepsilon},\ \gamma(1) = w\}$$

and define the minimax level

$$c_\varepsilon = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widetilde{I}_{N,\varepsilon}(\gamma(t)).$$

Since $I_{N,\varepsilon}(\psi) \le \widetilde{I}_{N,\varepsilon}(\psi)$ for any $\psi \in H_0^1(\Omega)$, $I_{N,\varepsilon}(u_{N,\varepsilon}) = \widetilde{I}_{N,\varepsilon}(u_{N,\varepsilon})$ and $I_{N,\varepsilon}(w) = \widetilde{I}_{N,\varepsilon}(w)$, by (9.7) it follows that $c_\varepsilon > \widetilde{I}_{N,\varepsilon}(u_{N,\varepsilon}) > \widetilde{I}_{N,\varepsilon}(w)$. By the the mountain-pass theorem [3], we infer that $\widetilde{I}_{N,\varepsilon}$ admits a critical point $U_{N,\varepsilon}$ at level c_ε. In particular $U_{N,\varepsilon}$ solves the equation

$$-\Delta U_{N,\varepsilon} = h_{N,\varepsilon}(x, U_{N,\varepsilon}^+) + f_{N,\varepsilon}.$$

Moreover, by the weak comparison principle, it follows that $U_{N,\varepsilon} \ge 0$. This proves that $U_{N,\varepsilon}$ is a nonnegative solution of (4.11). Since $u_{N,\varepsilon}$ is the nonnegative pointwise smallest solution of (4.11) and $u_{N,\varepsilon}$ does not coincides with $U_{N,\varepsilon}$, by Lemma 4.5 we conclude that $U_{N,\varepsilon} > u_{N,\varepsilon}$ almost everywhere in Ω. □

End of the Proof of Theorem 2.6. The proof of the theorem follows immediately from Lemma 9.3 defining $U_\varepsilon = U_{N,\varepsilon} + \gamma_{N,\varepsilon}$ for any $\varepsilon \in (0,\varepsilon^*)$. □

10. Proof of Theorem 2.7

Let $\{\rho_m\}$ be a sequence of mollifiers and let $\mu_m = \rho_m * \mu$. For any $m \in \mathbb{N}$, let $v_{i,\varepsilon}^{(m)}$ be the functions defined in (4.2)–(4.3) with μ_m in place of μ and let $\gamma_{k,\varepsilon}^{(m)} = \sum_{i=1}^k v_{i,\varepsilon}^{(m)}$ for $k = 1,\dots,N$ where N is the integer introduced in the proof of Theorem 2.2. In this proof we fix $\varepsilon \in (0,\varepsilon^*)$ where ε^* is the extremal value for the existence of solutions of (2.7). We proved in Lemma 7.1 that for any $\varepsilon \in (0,\varepsilon^*)$, the functional $I_{N,\varepsilon}$ defined in (4.12) admits a local minimizer $u_{N,\varepsilon}$. We show that for any m large enough the functional $I_{N,\varepsilon}^{(m)}$ also admits a local minimum (see (4.12) for the definition of $I_{N,\varepsilon}^{(m)}$). First we prove the following

Lemma 10.1. *Let $\gamma_{N,\varepsilon}^{(m)}$, $h_{N,\varepsilon}^{(m)}$, $H_{N,\varepsilon}^{(m)}$ defined as in the proof of Theorem 2.2 with μ_m in place of μ. If $s_m \to s$ in $\mathbb{R}$ then:*

(a) *$H_{N,\varepsilon}^{(m)}(x,s_m) \to H_{N,\varepsilon}(x,s)$ as $m\to\infty$ for a.e. $x\in\Omega$;*

(b) $h_{N,\varepsilon}^{(m)}(x,s_m) \le a + b(x)s + cs^{p-1}$ *for all* m, *all* $s \in \mathbb{R}$ *and a.e.* $x \in \Omega$ *where* $a, c \in \mathbb{R}$, $b \in L^{\alpha_0}(\Omega)$, $\alpha_0 > n/2$ *and* $p < 2_*$;

(c) $h_{N,\varepsilon}^{(m)}(x,s_m) \to h_{N,\varepsilon}(x,s)$ *as* $m \to \infty$ *for a.e.* $x \in \Omega$;

(d) $(h_{N,\varepsilon}^{(m)})'(x,s_m) \to h'_{N,\varepsilon}(x,s)$ *as* $m \to \infty$ *for a.e.* $x \in \Omega$.

Proof. Since $\{\mu_m\}$ is bounded in $L^1(\Omega)$, by Theorem B.1 in [8], Sobolev embedding and (4.2)–(4.3) we have up to subsequences

$$v_{i,\varepsilon}^{(m)} \to v_{i,\varepsilon} \quad \text{in } L^r(\Omega) \text{ as } m \to \infty, \text{ for all } r \in [1, n/(n-2)). \tag{10.1}$$

Since $H_{N,\varepsilon}^{(m)}(x,s)$ and $H_{N,\varepsilon}(x,s)$ are even with respect to s, by (4.6), (10.1) we have up to subsequences

$$\begin{aligned}\lim_{m\to\infty} H_{N,\varepsilon}^{(m)}(x,s_m) &= \lim_{m\to\infty} \int_0^{|s_m|} [g(\gamma_{N,\varepsilon}^{(m)}+t) - g(\gamma_{N,\varepsilon}^{(m)})]\,dt \\ &= \lim_{m\to\infty} [G(\gamma_{N,\varepsilon}^{(m)} + |s_m|) - G(\gamma_{N,\varepsilon}^{(m)}) - g(\gamma_{N,\varepsilon}^{(m)})|s_m|] \\ &= G(\gamma_{N,\varepsilon} + |s|) - G(\gamma_{N,\varepsilon}) - g(\gamma_{N,\varepsilon})|s| \\ &= H_{N,\varepsilon}(x,|s|) = H_{N,\varepsilon}(x,s)\end{aligned}$$

for a.e. $x \in \Omega$ where $G(s) = \int_0^s g(t)\,dt$. This proves (a).

The proof of (b) follows from (10.1) and Lemma 4.3(a). The proofs of (c)–(d) are similar to the proof of (a) and they are based on the pointwise convergence $v_{i,\varepsilon}^{(m)} \to v_{i,\varepsilon}$ and the continuity of g and g'. □

Lemma 10.2. *Let* $\varepsilon \in (0,\varepsilon^*)$. *Then there exists* $\overline{m} > 0$ *such that for any* $m > \overline{m}$ *the functional* $I_{N,\varepsilon}^{(m)}$ *admits a local minimizer* $u_{N,\varepsilon}^{(m)} \ge 0$. *Moreover, we have* $u_{N,\varepsilon}^{(m)} \to u_{N,\varepsilon}$ *in* $H_0^1(\Omega)$ *as* $m \to \infty$.

Proof. We start by proving the existence of a local minimizer. In view of Lemma 9.3, define $\rho > 0$ and $R > I_{N,\varepsilon}(u_{N,\varepsilon})$ such that

$$I_{N,\varepsilon}(\psi) \ge R \quad \text{for all } \psi \in \partial B_\rho$$

where $B_\rho = \{\psi \in H_0^1(\Omega) : \|\psi - u_{N,\varepsilon}\|_{H_0^1} < \rho\}$. We claim that $I_{N,\varepsilon}^{(m)}$ converges uniformly to $I_{N,\varepsilon}$ on $\overline{B_\rho}$. It suffices to show that if $\{z_m\} \subset \overline{B_\rho}$ then up to subsequences we have

$$|I_{N,\varepsilon}^{(m)}(z_m) - I_{N,\varepsilon}(z_m)| \to 0 \quad \text{as } m \to \infty. \tag{10.2}$$

Indeed, since $\{z_m\}$ is bounded in $H_0^1(\Omega)$ up to subsequences there exists $z \in \overline{B_\rho}$ such that $z_m \to z$ in $L^r(\Omega)$ for any $r < 2^*$. By Lemma 10.1 we have

$$\begin{aligned}|I_{N,\varepsilon}^{(m)}(z_m) - I_{N,\varepsilon}(z_m)| \le \bigg| \int_\Omega H_{N,\varepsilon}^{(m)}(x,z_m)\,dx - \int_\Omega H_{N,\varepsilon}(x,z_m)\,dx \bigg| \\ + \bigg| \int_\Omega f_{N,\varepsilon}^{(m)} z_m\,dx - \int_\Omega f_{N,\varepsilon} z_m\,dx \bigg| \to 0\end{aligned}$$

as $m \to \infty$. This proves (10.2).

By (10.2) we infer that there exists $\overline{m} > 0$ such that

$$\inf_{\partial B_\rho} I_{N,\varepsilon}^{(m)} > I_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}) \quad \text{for all } m > \overline{m}. \tag{10.3}$$

On the other hand, since $\overline{B_\rho}$ is compact with respect to the weak topology of $H_0^1(\Omega)$, by weak lower semicontinuity, we infer that $I_{N,\varepsilon}^{(m)}$ admits a global minimum in $\overline{B_\rho}$ which, in view of (10.3), is achieved by a function $u_{N,\varepsilon}^{(m)} \in B_\rho$. Therefore $u_{N,\varepsilon}^{(m)}$ is a local minimizer for $I_{N,\varepsilon}^{(m)}$. Since $I_{N,\varepsilon}^{(m)}(|w|) \le I_{N,\varepsilon}^{(m)}(w)$ for any $w \in H_0^1(\Omega)$ we may assume that $u_{N,\varepsilon}^{(m)} \ge 0$ up to replace it with $|u_{N,\varepsilon}^{(m)}|$.

Since $I_{N,\varepsilon} \in C^2(H_0^1(\Omega))$, by Lemma 7.1 we may choose $\rho > 0$ small enough such that

$$\langle I''_{N,\varepsilon}(w)\psi, \psi\rangle \ge C\|\psi\|_{H_0^1}^2 \quad \text{for all } w \in \overline{B_\rho} \text{ and all } \psi \in H_0^1(\Omega) \tag{10.4}$$

for a suitable constant $C > 0$.

It remains to prove that $u_{N,\varepsilon}^{(m)} \to u_{N,\varepsilon}$ in $H_0^1(\Omega)$. We may assume up to subsequences that $u_{N,\varepsilon}^{(m)} \rightharpoonup \overline{u} \in \overline{B_\rho}$ in $H_0^1(\Omega)$. By Lemma 10.1(a), (b), (10.1) and weak lower semicontinuity of the $H_0^1(\Omega)$-norm we have

$$\begin{aligned} I_{N,\varepsilon}(\overline{u}) &\le \liminf_{m\to\infty} I_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}^{(m)}) \\ &\le \limsup_{m\to\infty} I_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}^{(m)}) \le \lim_{m\to\infty} I_{N,\varepsilon}^{(m)}(w) = I_{N,\varepsilon}(w) \end{aligned} \tag{10.5}$$

for all $w \in \overline{B_\rho}$. This proves that $\overline{u}$ is a minimizer for $I_{N,\varepsilon}$ in B_ρ. Moreover, choosing $w = \overline{u}$ in (10.5) we obtain $I_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}^{(m)}) \to I_{N,\varepsilon}(\overline{u})$. It follows immediately that $\|u_{N,\varepsilon}^{(m)}\|_{H_0^1} \to \|\overline{u}\|_{H_0^1}$ and hence by the weak convergence $u_{N,\varepsilon}^{(m)} \rightharpoonup \overline{u}$ we get $u_{N,\varepsilon}^{(m)} \to \overline{u}$ in $H_0^1(\Omega)$. The minimizer $\overline{u}$ necessarily coincides with $u_{N,\varepsilon}$ since by (10.4) $I_{N,\varepsilon}$ is strictly convex in $\overline{B_\rho}$. □

Next we prove that for m large enough, the functional $I_{N,\varepsilon}^{(m)}$ admits a second critical point.

Lemma 10.3. *Let $g(s) = \lambda|s|^{q-2}s + |s|^{p-2}s$ with $2 \le q < p < 2_*$, $\lambda \ge 0$ if $q > 2$ and $\lambda \in [0, \lambda_1)$ if $q = 2$ and let $\varepsilon \in (0, \varepsilon^*)$. Then there exists $\overline{m} > 0$ such that for any $m > \overline{m}$ the functional $I_{N,\varepsilon}^{(m)}$ admits a second critical point $U_{N,\varepsilon}^{(m)} \ge 0$ such that $U_{N,\varepsilon}^{(m)} \to U^*$ in $H_0^1(\Omega)$ as $m \to \infty$ up to subsequences. Moreover, U^* is a critical point for $I_{N,\varepsilon}$ with $I_{N,\varepsilon}(U_{N,\varepsilon}) > I_{N,\varepsilon}(u_{N,\varepsilon})$.*

Proof. Let $\overline{m}$ as in Lemma 10.2. According to (9.8), let $\widetilde{I}_{N,\varepsilon}^{(m)}$ be defined by

$$\widetilde{I}_{N,\varepsilon}^{(m)}(\psi) = \frac{1}{2}\|\nabla\psi\|_{L^2}^2 - \int_\Omega H_{N,\varepsilon}^{(m)}(x, \psi^+)\,dx - \int_\Omega f_{N,\varepsilon}^{(m)}\psi\,dx$$

for all $\psi \in H_0^1(\Omega)$. Then by (10.2), we deduce that

$$\widetilde{I}_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}) < \inf_{\partial B_\rho} \widetilde{I}_{N,\varepsilon}^{(m)} \quad \text{for all } m > \overline{m}$$

and moreover, by (10.2), there exists $w \geq 0$ such that $\|w - u_{N,\varepsilon}\|_{H_0^1} > \rho$ and $\widetilde{I}_{N,\varepsilon}^{(m)}(w) < \widetilde{I}_{N,\varepsilon}^{(m)}(u_{N,\varepsilon})$. Therefore by Lemma 9.2, it follows that the functional $\widetilde{I}_{N,\varepsilon}^{(m)}$ satisfies all the assumptions of the mountain-pass theorem [3] and hence admits a critical point $U_{N,\varepsilon}^{(m)}$ with $\widetilde{I}_{N,\varepsilon}^{(m)}(U_{N,\varepsilon}^{(m)}) > \widetilde{I}_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}) \geq \widetilde{I}_{N,\varepsilon}^{(m)}(u_{N,\varepsilon}^{(m)})$. Using the weak comparison principle it follows that $U_{N,\varepsilon}^{(m)} \geq 0$ and hence $U_{N,\varepsilon}^{(m)}$ is a critical point for $I_{N,\varepsilon}^{(m)}$. Then we prove that the sequence $\{U_{N,\varepsilon}^{(m)}\}$ is bounded in $H_0^1(\Omega)$. Using the uniform convergence (10.2) and the minimax characterization of $U_{N,\varepsilon}^{(m)}$, we deduce that there exists $C > 0$ such that

$$|I_{N,\varepsilon}^{(m)}(U_{N,\varepsilon}^{(m)})| < C \quad \text{for all } m \in \mathbb{N}.$$

Therefore, since $U_{N,\varepsilon}^{(m)}$ is a critical point for $I_{N,\varepsilon}^{(m)}$ then by (9.6) we have, for any $\sigma > 0$,

$$\begin{aligned}
C &\geq (2+\sigma) I_{N,\varepsilon}^{(m)}(U_{N,\varepsilon}^{(m)}) - \langle (I_{N,\varepsilon}^{(m)})'(U_{N,\varepsilon}^{(m)}), U_{N,\varepsilon}^{(m)} \rangle \\
&\geq \frac{\sigma}{2}\bigg[\|\nabla U_{N,\varepsilon}^{(m)}\|_{L^2}^2 - \int_\Omega g'(\gamma_{N,\varepsilon}^{(m)})(U_{N,\varepsilon}^{(m)})^2\,dx\bigg] - (1+\sigma)\int_\Omega f_{N,\varepsilon}^{(m)} U_{N,\varepsilon}^{(m)}\,dx \\
&\geq \frac{\sigma}{2}\bigg(1 - \frac{1}{\Lambda_1^{(m)}(\varepsilon)}\bigg)\|\nabla U_{N,\varepsilon}^{(m)}\|_{L^2}^2 - (1+\sigma)C\|f_{N,\varepsilon}^{(m)}\|_{L^{2n/(n+2)}}\|U_{N,\varepsilon}^{(m)}\|_{H_0^1}
\end{aligned}$$

where

$$\Lambda_1^{(m)}(\varepsilon) = \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla\psi\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon}^{(m)})\psi^2\,dx}.$$

Since the sequence $\{f_{N,\varepsilon}^{(m)}\}$ is bounded in $L^{2n/(n+2)}(\Omega)$ in order to prove boundedness of $\{U_{N,\varepsilon}^{(m)}\}$ we need to show that $\{\Lambda_1^{(m)}(\varepsilon)\}$ is bounded away from one for large m. Since $u_{N,\varepsilon}^{(m)}$ is a local minimizer for $I_{N,\varepsilon}^{(m)}$ for any $m > \overline{m}$, then we have

$$0 \leq \langle (I_{N,\varepsilon}^{(m)})''(u_{N,\varepsilon}^{(m)})\psi, \psi\rangle = \|\nabla\psi\|_{L^2}^2 - \int_\Omega g'(\gamma_{N,\varepsilon}^{(m)} + u_{N,\varepsilon}^{(m)})\psi^2\,dx$$

for all $\psi \in H_0^1(\Omega)$ and hence, by the proof of Lemma 7.1, we get

$$\Lambda_1^{(m)}(\varepsilon) \geq \lambda_1^{(m)}(\varepsilon) = \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla\psi\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon}^{(m)} + u_{N,\varepsilon}^{(m)})\psi^2\,dx} \geq 1$$

for all $m > \overline{m}$. Suppose by contradiction that there exists a subsequence still denoted by $\{\Lambda_1^{(m)}(\varepsilon)\}$ such that $\lim_{m\to\infty} \Lambda_1^{(m)}(\varepsilon) = 1$. Let $\{\psi_{1,\varepsilon}^{(m)}\}$ be a sequence of minimizer for $\Lambda_1^{(m)}(\varepsilon)$ such that

$$\int_\Omega g'(\gamma_{N,\varepsilon}^{(m)})(\psi_{1,\varepsilon}^{(m)})^2\,dx = 1.$$

Then $\{\psi_{1,\varepsilon}^{(m)}\}$ is bounded in $H_0^1(\Omega)$ so that we may assume that $\psi_{1,\varepsilon}^{(m)} \rightharpoonup \psi^*$ in $H_0^1(\Omega)$. Then by weak lower semicontinuity, Lemma 10.1(d), (10.1) and the fact that $g'(\gamma_{N,\varepsilon}^{(m)})$ is uniformly bounded in $L^{n/2}(\Omega)$ we have $\int_\Omega g'(\gamma_{N,\varepsilon})(\psi^*)^2 dx = 1$ and

$$\lim_{m\to\infty} \Lambda_1^{(m)}(\varepsilon) \geq \frac{\|\nabla\psi^*\|_{L^2}^2}{\int_\Omega g'(\gamma_{N,\varepsilon})(\psi^*)^2\,dx} \geq \lambda_1(\varepsilon) > 1$$

which is a contradiction. This proves that $\{U_{N,\varepsilon}^{(m)}\}$ is bounded in $H_0^1(\Omega)$ and hence up to subsequences there exists $U^* \in H_0^1(\Omega)$ such that $U_{N,\varepsilon}^{(m)} \rightharpoonup U^*$ in $H_0^1(\Omega)$ as $m \to \infty$. Since $U_{N,\varepsilon}^{(m)}$ is a critical point for $I_{N,\varepsilon}^{(m)}$, then from Lemma 4.4 and Lemma 10.1 it follows that U^* is a critical point for $I_{N,\varepsilon}$ and moreover $U_{N,\varepsilon}^{(m)} \to U^*$ in $H_0^1(\Omega)$. We have to prove that U^* does not coincide with $u_{N,\varepsilon}$. Since for any m large enough $U_{N,\varepsilon}^{(m)}$ is a mountain-pass critical point for $I_{N,\varepsilon}^{(m)}$ then by the proof of Lemma 10.2 it follows that for any $0 < \sigma < (\inf_{\partial B_\rho} \widetilde{I}_{N,\varepsilon} - \widetilde{I}_{N,\varepsilon}(u_{N,\varepsilon}))/2$ there exists $\overline{m} > 0$ such that

$$\widetilde{I}_{N,\varepsilon}^{(m)}(U_{N,\varepsilon}^{(m)}) \geq \inf_{\partial B_\rho} \widetilde{I}_{N,\varepsilon}^{(m)} \geq \inf_{\partial B_\rho} \widetilde{I}_{N,\varepsilon} - \sigma > \widetilde{I}_{N,\varepsilon}(u_{N,\varepsilon}) + \sigma.$$

Therefore, since $U_{N,\varepsilon}^{(m)} \to U^*$ in $H_0^1(\Omega)$ then

$$I_{N,\varepsilon}(U^*) = \widetilde{I}_{N,\varepsilon}(U^*) = \lim_{m\to\infty} \widetilde{I}_{N,\varepsilon}^{(m)}(U_{N,\varepsilon}^{(m)}) > \widetilde{I}_{N,\varepsilon}(u_{N,\varepsilon}) = I_{N,\varepsilon}(u_{N,\varepsilon}).$$

This completes the proof of the lemma. □

End of the Proof of Theorem 2.7. The proof of the proposition follows immediately from Lemmas 10.2 and 10.3 since

$$I^{(m)}(w) = J^{(m)}(w + \gamma_{N,\varepsilon}^{(m)}) - J^{(m)}(\gamma_{N,\varepsilon}^{(m)}) = I_{N,\varepsilon}^{(m)}(w)$$

for all $w \in H_0^1(\Omega)$, $w \geq 0$. □

11. Proof of Proposition 2.8

Suppose by contradiction that for any $\varepsilon > 0$ (2.14) admits a solution $u \in L^{p-1}(\Omega)$. Let $u_a(x) = \varepsilon G(x,a)$ where $G(x,a)$ denotes the Green function for $-\Delta$ in Ω. Then we have

$$\int_\Omega -(u - u_a)\Delta\varphi\,dx = \int_\Omega u^{p-1}\varphi\,dx \geq 0 \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}),\ \varphi \geq 0$$

and by the weak comparison principle (see Lemma 3 in [7]) we obtain

$$u \geq u_a \geq 0 \quad \text{a.e. in } \Omega. \tag{11.1}$$

On the other hand we have $u_a(x) \sim C|x-a|^{-n+2}$ as $x \to a$ and since $p \geq 2_*$, we infer $u_a \notin L^{p-1}(\Omega)$. And this with (11.1) contradicts $u \in L^{p-1}(\Omega)$.

12. Proof of Proposition 2.9

12.1. The case $p > 2_*$. Up to translation assume that $0 \in \Omega$. Choose $\mu = f \in L^1(\Omega)$ with $f(x) = |x|^\alpha$ with $\alpha \in (-n,-2)$ and consider the Poisson equation

$$\begin{cases} -\Delta v = \varepsilon f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we have $v(x) \sim C|x|^{\alpha+2}$ as $x \to 0$.

If we choose $\alpha \in (-n, -(n+2(p-1))/(p-1))$, then $v \notin L^{p-1}(\Omega)$. As in the proof of Proposition 2.8, using the weak comparison principle, we deduce that if $u \in L^{p-1}(\Omega)$ is a solution (2.14) with $\mu = f$, then $u \geq v \geq 0$ almost everywhere in Ω a contradiction.

12.2. The case $p = 2_*$. As in the previous case, assume $0 \in \Omega$ and choose $\mu = f \in L^1(\Omega)$ with $f(x) = |x|^{-n}/(\log|x/a|)^2$ and $B_{a/2}(0) \supset \overline{\Omega}$. Let $\overline{v}$ be a radial solution of $-\Delta\overline{v} = \varepsilon f$ in Ω. Then $\overline{v} = \overline{v}(r)$ solves the ordinary differential equation

$$-(r^{n-1}\overline{v}')' = \frac{\varepsilon}{r(\log(r/a))^2}, \quad \text{for all } r = |x| < \frac{a}{2}.$$

We may choose $\overline{v}$ in the form

$$\overline{v}(r) = -\int_r^{a/2} \frac{\varepsilon}{s^{n-1}\log(s/a)}\,ds, \quad \text{for all } r < \frac{a}{2}. \tag{12.1}$$

Let w be a harmonic function in Ω such that $v = w + \overline{v}$ is equal zero on $\partial\Omega$. Then v solves the Dirichlet problem

$$\begin{cases} -\Delta v = \varepsilon f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

If $u \in L^{p-1}(\Omega)$ is a solution of (2.14) with $\mu = f$, by the weak comparison principle we have $u \geq v \geq 0$ almost everywhere in Ω. If we show that $v \notin L^{p-1}(\Omega)$ we reach a contradiction. It is enough to prove that $\overline{v} \notin L^{p-1}(\Omega)$. By (12.1), we infer

$$\overline{v}(r) \geq \frac{C}{n-2+\beta}\left(r^{-(n-2+\beta)} - \left(\frac{a}{2}\right)^{-(n-2+\beta)}\right) \quad \text{for all } r < \frac{a}{2}$$

where $\beta \in (-n+2, 0)$ and C is a suitable positive constant. For $R > 0$ such that $B_R(0) \subset \Omega$ we have

$$\int_\Omega \overline{v}^{p-1}(x)\,dx \geq \left(\frac{C}{n-2+\beta}\right)^{p-1} \int_{B_R(0)} \left[|x|^{-(n-2+\beta)} - \left(\frac{a}{2}\right)^{-(n-2+\beta)}\right]^{p-1} dx$$

$$\geq \left(\frac{C}{n-2+\beta}\right)^{p-1} \int_{B_R(0)} \bigg[|x|^{-(p-1)(n-2+\beta)} - p|x|^{-(p-2)(n-2+\beta)} \left(\frac{a}{2}\right)^{-(n-2+\beta)}\bigg]\, dx$$

and since $p-1 = n/(n-2)$, denoting by ω_{n-1} the measure of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ we obtain

$$\begin{aligned}\int_\Omega \overline{v}^{p-1}(x)\, dx &\geq \left(\frac{C}{n-2+\beta}\right)^{p-1} \omega_{n-1} \\ &\quad\cdot \int_0^R \bigg[r^{-n\beta/(n-2)-1} - p\left(\frac{a}{2}\right)^{-(n-2+\beta)} r^{n-3-2\beta/(n-2)}\bigg]\, dr \\ &= \left(\frac{C}{n-2+\beta}\right)^{p-1} \omega_{n-1} \bigg[-\frac{n-2}{n\beta} R^{-n\beta/(n-2)} \\ &\quad - p\left(\frac{a}{2}\right)^{-(n-2+\beta)} \frac{n-2}{(n-2)^2 - 2\beta} R^{n-2-2\beta/(n-2)}\bigg] \to \infty\end{aligned}$$

as $\beta \to 0^-$. This proves that $\overline{v} \notin L^{p-1}(\Omega)$.

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Alberto Ferrero
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5
56127 Pisa, ITALY

E-mail address: ferrero@mail.dm.unipi.it

Claudio Saccon
Dipartimento di Matematica Applicata
Universitá di Pisa
Via Bonanno 25/B
56126 Pisa, ITALY

E-mail address: saccon@mail.dm.unipi.it