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SOME RESULTS ON THE EXTENSION OF SINGLE- AND MULTIVALUED MAPS

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(*Submitted by L. Górniewicz*)

Abstract. Necessary and sufficient conditions for single-valued extensions of multivalued maps are discussed. Moreover, a quantitative version of a generalization of Dugundji's extension theorem for multivalued maps is obtained. Finally, the extension problem for compact maps is studied. Many of the results are new even for single-valued maps.

1. Introduction

The celebrated extension theorem of Dugundji [10] states that any continuous map $f\colon A \to Y$ defined on a closed subset A of a metric space X with values in a convex subset Y of a locally convex vector space possesses a continuous extension $F\colon X \to Y$. In the degree theory of multivalued maps, it is important to have such an extension result also for multivalued upper semicontinuous maps $f\colon A \multimap Y$. In particular, for the degree of noncompact maps, a standard proceeding is to restrict maps to so-called fundamental sets and then to extend them continuously (resp. upper semicontinuously).

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The only extension result which we know in this direction is a theorem of Ma [16] which closely follows Dugundji's proof and obtains a corresponding result for multivalued maps with compact convex values. In particular, it seems that no result is known which does not need convexity of the maps. However, in typical applications of degree theory, e.g. for the translation operator of differential equations in the absence of uniqueness, one not only has non-convex values but even maps with non-acyclic values which, however, typically are compositions of so-called R_δ-maps with single-valued maps.

We are therefore not only interested in the existence of upper semicontinuous extension for the above maps with non-convex values but also in preserving as much of the additional structure of these maps as possible. The first property in this connection which one might hope for is that the extension $F: X \multimap Y$ is even single-valued on $X \setminus A$. Another required property is that the extension of a compact map remains compact.

The existence of single-valued continuous extensions of a multivalued map is closely related to the approximation property as we will discuss in Section 2. The most important case of multivalued maps with the approximation property is that of upper semicontinuous maps with compact convex values. For such maps, we will obtain the existence of single-valued continuous extensions with very strong additional properties in Section 3. In particular, our result will contain the extension theorems of Dugundji and Ma as special cases, and in addition, we obtain a more quantitative version which yields the existence of retractions with very good approximation properties. In Section 4, we discuss the extension of continuous compact maps. We point out that many of our results are new even in case of single-valued maps.

2. Single-valued extensions and the approximation property

By a multivalued map $f: X \multimap Y$, we understand a map $f: X \to 2^Y$ with possibly empty values $f(x)$. As is usual practice, we put $f(A) := \bigcup_{x \in A} f(x)$ and

$$\mathrm{Gr}(f) := \{(x, y) \in X \times Y : y \in f(x)\}.$$

Recall that f is called *upper semicontinuous at* $x \subseteq X$, if for each open set $V \subseteq Y$ containing $f(x)$ there is some neighbourhood $U \subseteq X$ of x with $f(U) \subseteq V$. Upper semicontinuity on a set $M \subseteq X$ means upper semicontinuity at each $x \in M$.

We point out that the set X is implicitly involved in this definition, i.e. the upper semicontinuity of f on M is a stronger property than the upper semicontinuity of the restricted map $f|_M: M \multimap Y$. Also in the following definitions the spaces X and Y are implicitly involved.

Definition 2.1. Let X and Y be topological spaces, and $F: X \multimap Y$. If $W \subseteq X \times Y$ is a neighbourhood of $\mathrm{Gr}(F)$, we call a continuous map $\varphi: X \to Y$

a W-*approximation* (for F) if $\mathrm{Gr}(\varphi) \subseteq W$. Let $\mathcal{F}$ be a subset of the space $C(X,Y)$ of all continuous functions from X into Y.

(a) F has the *approximation property* (with respect to $\mathcal{F}$) if for each neighbourhood $W \subseteq X \times Y$ of $\mathrm{Gr}(F)$ there is a W-approximation in $\mathcal{F}$.

(b) F has the *homotopy approximation property* (with respect to $\mathcal{F}$) if it has the approximation property (with respect to $\mathcal{F}$) and if for each neighbourhood $W \subseteq X \times Y$ of $\mathrm{Gr}(F)$ there is some neighbourhood $W_0 \subseteq X \times Y$ of $\mathrm{Gr}(F)$ such that each two W_0-approximations (from $\mathcal{F}$) are joined by a homotopy $h\colon [0,1] \times X \to Y$ such that $h(t,\,\cdot\,)$ is a W-approximation for each $t \in [0,1]$.

Most results which imply that a function has the approximation property also show that the function has the homotopy approximation property. Let us start with some examples.

Recall that a nonempty closed set K of a Hausdorff space Y is

(a) ∞-*proximally connected in* Y if each neighbourhood $U \subseteq Y$ of K contains a neighbourhood $V \subseteq U$ of K such that each continuous map from some finite-dimensional sphere into V has a continuous extension to the corresponding ball with values in U.

(b) UV^∞ *in* Y if each neighbourhood $U \subseteq Y$ of K contains a contractible neighbourhood $V \subseteq U$ of K.

If K is UV^∞ in Y then K is ∞-proximally connected in Y, but the converse is not true, in general. In view of [14, Proposition 1.12] (see also [15]), each R_δ-space K (i.e. each intersection of a decreasing sequence of nonempty compact contractible metric spaces) has the above two properties whenever Y is an ANR (see Section 4).

Proposition 2.2.

(a) (AC) *If X is a metric space, Y is a locally convex space, and $F\colon X \multimap Y$ is upper semicontinuous with nonempty compact convex values, then F has the approximation property with respect to $C(X,Y)$.*

(b) (AC) *If X and Y are metrizable, X has finite or countable covering dimension, and $F\colon X \multimap Y$ is upper semicontinuous and $F(x)$ is compact and UV^∞ in Y for each $x \in X$, then F has the homotopy approximation property with respect to $C(X,Y)$.*

(c) *Let Y be Hausdorff and $F\colon X \multimap Y$ be upper semicontinuous and $F(x)$ be ∞-proximally connected in Y for each $x \in X$. Assume in addition*

  (i) *X is a finite polyhedron or*

  (ii) *X is a compact ANR, Y is metrizable and $F(x)$ is compact for each $x \in X$.*

Then F has the homotopy approximation property with respect to the space $C(X,Y)$.

(d) *If $F\colon X \multimap Y$ has the approximation property with respect to some family $\mathcal{F}$ and $g\colon Y \to Z$ is continuous, then $g\circ F$ has the approximation property with respect to the family $\mathcal{F}_g := \{g\circ\varphi : \varphi \in \mathcal{F}\}$. If g is one-to-one and $g\colon Y \to g(Y)$ is an open map, then an analogous result holds for the homotopy approximation property.*

We use the symbol (AC) to tag that the axiom of choice is essentially involved in the proof (cf. also Remark 3.4).

Proof. (a) This is a special case of [3, Proposition 4.2]; for a particular case see also [7].

(b) By [14, Theorem 1.21] (see also [1]), F is weakly relatively approximable over $\emptyset$ in the sense of [14]. In particular, F has the approximation property. The same result shows that also the map $\widetilde{F}(t,x) := F(x)$ ($t \in [0,1]$) is weakly relatively approximable over $A := \{0,1\} \times X$. Hence, if $W \subseteq X \times Y$ is a neighbourhood of $\mathrm{Gr}(F)$, then $\widetilde{W} := [0,1] \times W$ is a neighbourhood of $\mathrm{Gr}(\widetilde{F})$, and so we find some open neighbourhood $\widetilde{W}_0$ of $\mathrm{Gr}(\widetilde{F})$ such that any continuous map $\varphi\colon A \to Y$ with $\mathrm{Gr}(\varphi) \subseteq \widetilde{W}_0$ has an extension to a $\widetilde{W}$-approximation. Then

$$W_0 := \{(x,y) \in X \times Y : (0,x,y) \in \widetilde{W}_0 \text{ and } (1,x,y) \in \widetilde{W}_0\}$$

is open and thus a neighbourhood of $\mathrm{Gr}(F)$. If φ_0, φ_1 are two W_0-approximations, the map $\varphi\colon A \to Y$, defined by $\varphi(t,x) := \varphi_t(x)$ ($t \in \{0,1\}$) is continuous with $\mathrm{Gr}(\varphi) \subseteq \widetilde{W}_0$ and thus has an extension to a $\widetilde{W}$-approximation $h\colon [0,1]\times X \to Y$. Then h is a homotopy connecting φ_0 with φ_1 and such that $h(t,\,\cdot\,)$ is a W-approximation for each $t \in [0,1]$.

(c) For the case that X is a polyhedron, the claim is a reformulation of [2, Theorem 8.11]. In the other case the claim is a reformulation of [12, Theorems 23.8 and 23.9] (see also [13, Theorems 5.12 and 5.13]) in view of Remark 2.8.

(d) Let $W \subseteq X \times Z$ be an open neighbourhood of $\mathrm{Gr}(g\circ F)$. Then

$$W_F := \{(x,y) \in X \times Y : (x, g(y)) \subseteq W\}$$

is open and thus a neighbourhood of $\mathrm{Gr}(F)$. If φ is a W_F-approximation, then $g\circ\varphi$ is a W-approximation. Hence, $g \circ F$ has the approximation property with respect to $\mathcal{F}_g$. Moreover, if F has the homotopy approximation property, we find some neighbourhood $W_0 \subseteq X \times Y$ of $\mathrm{Gr}(F)$ such that each two W_0-approximations from $\mathcal{F}$ are joined by a homotopy $h\colon [0,1] \times X \to Y$ such that $h(t,\,\cdot\,)$ is a W_F-approximation for each $t \in [0,1]$. Each $(x_0,y_0) \in W_0$ has an open neighbourhood of the form $U \times V \subseteq X \times Y$ with $U \times V \subseteq W_0$. Then $g(V)$ is open in $g(Y)$, i.e. there is some open set $O \subseteq Z$ with $O \cap g(Y) = g(V)$. Since g is one-to-one, we have $g^{-1}(O) = V$. Let $W_{g\circ F} \subseteq X \times Z$ be the union of all sets $U \times O$ obtained in this way. Then $W_{g \circ F}$ is open and a neighbourhood of $\mathrm{Gr}(g \circ F)$. Moreover, if $(x, g(y)) \in W_{g \circ F}$ then $(x, y) \in W_0$.

Let now $\Phi_i \in \mathcal{F}_g$ $(i = 1, 2)$ be two $W_{g \circ F}$-approximations. Then $\Phi_i = g \circ \varphi_i$ for some $\varphi_i \in \mathcal{F}$, and by our choice of $W_{g \circ F}$, φ_i is a W_0-approximation for $i = 1, 2$. Hence, we find a homotopy $h\colon [0,1] \times X \to Y$ joining φ_1 with φ_2 and such that $h(t, \cdot\,)$ is a W_F-approximation for each $t \in [0,1]$. It follows as above that $H(t, \cdot\,) := g(h(t, \cdot\,))$ is a W-approximation for each $t \in [0,1]$ which joins Φ_1 with Φ_2. □

For further examples, see e.g. [2, Section II.8].

The following result shows that without the approximation property one has practically no chance to obtain a single-valued extension of a multivalued map.

PROPOSITION 2.3. *Let X be a metric space, Y be a topological space, and $A \subseteq X$. Let $B \subseteq Z \subseteq X$ be such that for each Lipschitz-continuous function $\delta\colon B \to (0,1)$ there is a continuous map $s\colon B \to Z \setminus A$ with $d(s(x), x) \le \delta(x)$ for all $x \in B$. If $f\colon A \multimap Y$ has an extension $F\colon Z \multimap Y$ which is upper semicontinuous on B and single-valued on $Z \setminus A$, then $F|_B\colon B \multimap Y$ has the approximation property with respect to $C(X,Y)$.*

Proposition 2.3 means not only that the extended function F has the approximation property but even the original (unextended) function f must have this property, provided that A is "not too bad".

COROLLARY 2.4. *Let X be a metric space, Y be a topological space, and suppose that the closed set $A \subseteq X$ has the property that for each Lipschitz-continuous function $\delta\colon \partial A \to (0,\infty)$ there is a continuous map $s\colon \partial A \to X \setminus A$ with $d(s(x), x) \le \delta(x)$ for all $x \in \partial A$. If $f\colon A \multimap Y$ has an extension $F\colon U \multimap Y$ to some neighbourhood $U \subseteq X$ of A which is upper semicontinuous on ∂A and single-valued on $X \setminus A$, then $f|_{\partial A}\colon \partial A \multimap Y$ has the approximation property with respect to $C(X,Y)$.*

PROOF. Apply Proposition 2.3 with $X := U$, $B := \partial A$, and $Z := \overline{X \setminus A}$. □

PROOF OF PROPOSITION 2.3. Let W be a neighbourhood of $\mathrm{Gr}(F|_B)$ in $B \times Y$. Let $\mathfrak{U}$ be the family of all open in Z sets $U \subseteq Z$ with $\operatorname{diam} U < \min\{1, \operatorname{diam} Z\}$ and the property that there is some open set $V \subseteq Y$ such that $F(U) \subseteq V$ and $(U \cap B) \times V \subseteq W$. Then $\mathfrak{U}$ covers B. Indeed, for each $x \in B$, the set W is a neighbourhood of the set $\{x\} \times F(x)$ in the space $B \times Y$, and so we find open sets $U \subseteq X$ and $V \subseteq Y$ with $x \in U$ and $F(x) \subseteq V$ such that $(U \cap B) \times V \subseteq W$. Since $F\colon Z \multimap Y$ is upper semicontinuous at $x \in B$ with $F(x) \subseteq V$, we conclude that x is contained in some set of $\mathfrak{U}$. Hence, the function $\delta\colon Z \to [0,1]$, defined by

$$\delta(x) := \sup\{\operatorname{dist}(x, Z \setminus U) : U \in \mathfrak{U}\},$$

has no zero on B. In view of the subsequent Lemma 2.5 δ is Lipschitz (with constant ≤ 1).

Let $s: B \to Z \setminus A$ be continuous with $d(s(x), x) \leq \delta(x)/2$ for all $x \in B$, and define $\varphi: B \to Y$ by $\{\varphi(x)\} = F(s(x))$. Then $\mathrm{Gr}(\varphi) \subseteq W$, because for each $x \in B$ there is some $U \in \mathfrak{U}$ with $d(s(x), x) < \mathrm{dist}(x, Z \setminus U)$, and so $s(x) \in Z \setminus (Z \setminus U) = U$; consequently $(x, \varphi(x)) \in (U \cap B) \times F(U) \subseteq (U \cap B) \times V \subseteq W$. □

Lemma 2.5. *Let X be a metric space and $f_i: X \to \mathbb{R}$ $(i \in I)$ a family of Lipschitz functions, all with a constant $\leq L < \infty$. Then either $f_+(x) := \sup_i f_i(x)$ is constantly ∞ or defines a Lipschitz function with constant $\leq L$. An analogous result holds for $f_-(x) := \inf_i f_i(x)$.*

Proof. Let $x, y \in X$ with $f_+(y) < \infty$. Choose sequences i_n, j_n with $f_{i_n}(x) \to f_+(x)$ and $f_{j_n}(y) \to f_+(y)$. Since

$$\begin{aligned} f_{i_n}(x) - f_{j_n}(y) &\leq f_{i_n}(x) - f_{i_n}(y) + f_+(y) - f_{j_n}(y) \\ &\leq Ld(x, y) + f_+(y) - f_{j_n}(y) \to Ld(x, y), \end{aligned}$$

we conclude that $f_+(x) - f_+(y) \leq Ld(x, y)$. In particular, $f_+(x) < \infty$, and interchanging x and y, we obtain $|f_+(x) - f_+(y)| \leq Ld(x, y)$. The claim for f_- follows from $f_-(x) = -\sup_i(-f_i)(x)$. □

Proposition 2.3 states, roughly speaking, that for finding a single-valued extension it is necessary to find an extension with the approximation property. Theorem 2.9 will show that finding an extension with the *homotopy* approximation property is even sufficient for finding a single-valued extension. Even slightly less is required.

In a metric space X and for $M \subseteq X$, we use the notation $B_r(M) := \{x \in X : \mathrm{dist}(x, M) < r\}$. For $M := \{x\}$, we write shorter $B_r(x) := B_r(\{x\})$.

The following definition is from [5]. In [6], the same property is called "ε-δ-upper semicontinuous".

Definition 2.6. Let Y be a metric space. A multivalued map $f: X \multimap Y$ is called *upper semicontinuous at $x \in X$ in the ε-sense*, if for each $\varepsilon > 0$ there is some neighbourhood $U \subseteq X$ of x with $f(U) \subseteq B_\varepsilon(f(x))$.

If f is upper semicontinuous at x, then f is upper semicontinuous at x in the ε-sense. The converse holds if $f(x)$ is compact. In a similar sense, we generalize the approximation properties.

Definition 2.7. Let X and Y be metric spaces, and $F: X \multimap Y$. We call a continuous map $\varphi: X \to Y$ an ε-approximation of F, if $\mathrm{Gr}(\varphi) \subseteq B_\varepsilon(\mathrm{Gr}(F))$, i.e. if φ is a $B_\varepsilon(\mathrm{Gr}(F))$-approximation; here we understand $X \times Y$ equipped with the max-distance, i.e. $\mathrm{Gr}(\varphi) \subseteq B_\varepsilon(\mathrm{Gr}(F))$ means that for each $x \in X$

there are $z \in X$ and $y \in F(z)$ satisfying $d(x,z) < \varepsilon$ and $d(\varphi(x),y) < \varepsilon$. Let $\mathcal{F} \subseteq C(X,Y)$.

(a) F has the *weak approximation property* (with respect to $\mathcal{F}$) if for each $\varepsilon > 0$ there is an ε-approximation $\varphi \in \mathcal{F}$ of F.

(b) F has the *ε-homotopy approximation property* (with respect to $\mathcal{F}$) if it has the weak approximation property (with respect to $\mathcal{F}$) and if for each $\varepsilon > 0$ there is some $\varepsilon_0 > 0$ such that each two ε_0-approximations of F (from $\mathcal{F}$) are homotopic by a homotopy $h\colon [0,1] \times X \to Y$ such that $h(t,\,\cdot\,)$ is an ε-approximation of F for each $t \in [0,1]$.

(c) F has the *weak homotopy approximation property* (with respect to $\mathcal{F}$) if it has the approximation property (with respect to $\mathcal{F}$) and if for each $\varepsilon > 0$ there is some neighbourhood $W_0 \subseteq X \times Y$ of $\mathrm{Gr}(F)$ such that each two W_0-approximations (from $\mathcal{F}$) are homotopic by a homotopy $h\colon [0,1] \times X \to Y$ such that $h(t,\,\cdot\,)$ is an ε-approximation of F for each $t \in [0,1]$.

REMARK 2.8. The homotopy approximation property implies the weak homotopy approximation property, and the approximation property implies the weak approximation property; the converse holds if $\mathrm{Gr}(F)$ is compact (e.g. if X is compact and $F\colon X \multimap Y$ is upper semicontinuous with compact values). Moreover, in this case, the homotopy approximation property, the weak homotopy approximation property and the ε-homotopy approximation property are equivalent.

THEOREM 2.9. *Let X and Y be metric spaces, $A \subseteq X$ be closed, and let $f\colon A \multimap Y$ possess an extension $f\colon X \multimap Y$ which is upper semicontinuous on ∂A in the ε-sense and such that the restriction of f to a set $Z \supseteq X \setminus A$ has the weak homotopy approximation property or the ε-homotopy approximation property* (*with respect to some $\mathcal{F} \subseteq C(X,Y)$*)*. Then f possesses an extension $F\colon X \multimap Y$ which is single-valued on $X \setminus A$ and upper semicontinuous on $\overline{X \setminus A}$ in the ε-sense.*

At a first glance, Theorem 2.9 might appear somewhat disappointing, because it only states, roughly speaking, that if one finds an extension with the homotopy approximation property, then one also finds a single-valued extension. However, if the original map has the homotopy extension property and the set A is not "too bad", then such an extension exists, provided that the family $\mathcal{F}$ is appropriately chosen, as we will show now. Actually, this was the main reason why we introduced the family $\mathcal{F}$.

Recall that a set $A \subseteq X$ is called a *retract* of X, if there is a *retraction* $\rho\colon X \to A$ onto A, i.e. ρ is continuous with $\rho(x) = x$ for all $x \in A$.

Corollary 2.10. *Let X and Y be metric spaces, $U \subseteq X$, and A be a retract of U with a corresponding retraction ρ which is uniformly continuous. Assume also that ρ is "uniformly open" in the following sense: There is a function $\delta\colon (0,\infty) \to (0,\infty)$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and*

$$\rho(B_{\delta(\varepsilon)}(x) \cap U) \supseteq B_\varepsilon(\rho(x)) \cap Z_0 \quad (x \in Z), \tag{2.1}$$

where $U \setminus A \subseteq Z \subseteq U$ and $Z_0 := \rho(Z)$. Suppose that $f\colon A \multimap Y$ is upper semicontinuous in the ε-sense and that $f|_{Z_0}\colon Z_0 \multimap Y$ has the ε-homotopy approximation property with respect to some family $\mathcal{F}_0$. Then f has an extension to an upper semicontinuous map $F_0\colon U \multimap Y$ in the ε-sense which is single-valued on $U \setminus A$.

In Section 4, we will see that if U is a neighbourhood of A, then in many situations one even finds an extension $f\colon X \multimap Y$.

A simple example of a set A with the properties required in Corollary 2.10 is a closed ball in a normed space X when U is bounded. Then the canonical radial retraction ρ is Lipschitz and satisfies (2.1) with $\delta(\varepsilon) := c\varepsilon$ where $c > 0$ is a constant.

Proof of Corollary 2.10. Shrinking X if necessary, it is no loss of generality to assume $X = U$. Let $\mathcal{F}$ be the family of all functions of the form $\varphi \circ \rho$ where $\varphi \in \mathcal{F}_0$. Extend f to an upper semicontinuous in the ε-sense function $F\colon X \multimap Y$ by putting $F(x) := f(\rho(x))$ for $x \in X$. We will show that $F|_Z\colon Z \to Y$ has the ε-homotopy approximation property with respect to $\mathcal{F}$. Then the claim follows immediately from Theorem 2.9, observing that A must be closed, since it is a retract.

Without loss of generality, we may assume that $\varepsilon < \delta(\varepsilon)$ for all $\varepsilon > 0$. We show first that, if φ is an ε-approximation of $f|_{Z_0}$, then the map $\varphi \circ \rho\colon Z \to Y$ is an $\delta(\varepsilon)$-approximation of $F|_Z$. In fact, the former means that for each $x_0 \in Z_0$ there is some $z_0 \in Z_0$ with $d(x_0, z_0) < \varepsilon$ and $\operatorname{dist}(\varphi(x_0), f(z_0)) < \varepsilon$. Choose some $x \in Z$ with $\rho(x) = x_0$. In view of (2.1), we find some $z \in Z$ with $d(x,z) < \delta(\varepsilon)$ and $\rho(z) = z_0$, i.e. $\operatorname{dist}((\varphi \circ \rho)(x), F(z)) = \operatorname{dist}(\varphi(x_0), f(z_0)) < \varepsilon < \delta(\varepsilon)$. Hence, $\varphi \circ \rho$ is an $\delta(\varepsilon)$-approximation of $F|_Z$, as claimed.

For each $a > 0$, we find some $\varepsilon > 0$ with $\delta(\varepsilon) < a$ and some $\varepsilon_0 > 0$ such that each two ε_0-approximations of $f|_{Z_0}$ from $\mathcal{F}_0$ are connected by some homotopy $h\colon [0,1] \times Z_0 \to Y$ such that $h(t, \cdot\,)$ is an ε-approximation of $f|_{Z_0}$ for each $t \in [0,1]$. Since ρ is uniformly continuous, there is some $a_0 \in (0, \varepsilon_0)$ such that $d(\rho(x), \rho(x_0)) < \varepsilon_0$ whenever $d(x, x_0) < a_0$.

There is some ε-approximation $\varphi \in \mathcal{F}_0$ of $f|_{Z_0}$, and by what we have proved in the beginning, the map $\varphi \circ \rho \in \mathcal{F}$ is an a-approximation of $F|_Z$. Hence, $F|_Z$ has the weak approximation property. Moreover, let $\Phi_i \in \mathcal{F}$ $(i = 1,2)$ be a_0-approximations of $F|_Z$. There are $\varphi_i \in \mathcal{F}_0$ with $\Phi_i = \varphi_i \circ \rho$ $(i = 1,2)$.

We show that φ_i $(i = 1,2)$ must be ε_0-approximations of $f|_{Z_0}$. Indeed, for each $x_0 \in Z_0$, we find some $x \in Z$ with $\rho(x) = x_0$ and some $z \in Z$ with $d(x,z) < a_0$ and $\operatorname{dist}(\Phi_i(x), F(z)) < a_0$. Then $z_0 := \rho(z) \in Z_0$ satisfies $d(x_0,z_0) < \varepsilon_0$, and we have $\operatorname{dist}(\varphi_i(x_0), f(z_0)) = \operatorname{dist}(\Phi_i(x), F(z)) < \varepsilon_0$, as required.

Hence, there is a homotopy $h\colon [0,1]\times Z_0 \to Y$ connecting φ_1 with φ_2 and such that $h(t,\,\cdot\,)$ is an ε-approximation of $f|_{Z_0}$ for each $t \in [0,1]$. By what we have proved in the beginning, it follows that the map $H(t,\,\cdot\,) := h(t,\rho(\,\cdot\,))\colon Z \to Y$ is a $\delta(\varepsilon)$-approximation of $F|_Z$ for each $t \in [0,1]$. Hence, H defines a homotopy connecting Φ_1 with Φ_2 and such that $H(t,\,\cdot\,)$ is an a-approximation of $F|_Z$ for each $t \in [0,1]$. This shows that $F|_Z$ has the ε-homotopy approximation property with respect to $\mathcal{F}$, as claimed. $\square$

Proof of Theorem 2.9. By the weak homotopy approximation property, we find for $n = 1,2,\ldots$ some neighbourhood $W_n \subseteq Z \times Y$ of $\operatorname{Gr}(f|_Z)$ such that each two W_n-approximations can be connected by a homotopy $h\colon [0,1] \times Z \to Y$ where $h(t,\,\cdot\,)$ is a $B_{1/n}(\operatorname{Gr}(f|_Z))$-approximation for each $t \in [0,1]$. In case of the ε-homotopy approximation property, we may and will assume in addition $W_n = B_{\varepsilon_n}(\operatorname{Gr}(f|_Z))$.

Without loss of generality, we may assume that $W_{n+1} \subseteq W_n$. There are W_n-approximations $\varphi_n\colon Z \to Y$. We thus find continuous maps $h_n\colon [0,1] \times Z \to Y$ with $h_n(0,\,\cdot\,) = \varphi_n$, $h_n(1,\,\cdot\,) = \varphi_{n+1}$, and

$$\operatorname{Gr}(h_n(t,\,\cdot\,)) \subseteq B_{1/n}(\operatorname{Gr}(f|_Z)) \quad (0 \le t \le 1). \tag{2.2}$$

Define a continuous function $H\colon [1,\infty) \times Z \to Y$ by

$$H(n+t,x) := h_n(t,x) \quad (0 \le t < 1,\ n = 1,2,\ldots).$$

We claim that a required extension is given by

$$F(x) := \begin{cases} f(x) & \text{if } x \in A, \\ \{H(\max\{1/\operatorname{dist}(x,A),1\},x)\} & \text{if } x \notin A. \end{cases}$$

We have to show that F is upper semicontinuous in the ε-sense at each point $x_0 \in \partial A$. Thus, let $\varepsilon > 0$ be given. We find some $r > 0$ such that $f(B_r(x_0)) \subseteq B_{\varepsilon/2}(f(x_0))$, without loss of generality $r < \varepsilon$. Choose some natural number $m > 2/r$ and put $\delta := 1/m$. Then $F(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$. Indeed, for $x \in B_\delta(x_0)$, we distinguish two cases: If $x \in A$, we have $F(x) = f(x) \in f(B_r(x_0)) \subseteq B_{\varepsilon/2}(f(x_0))$. If $x \notin A$, then $\operatorname{dist}(x,A) \le d(x,x_0) < \delta = 1/m$, and so we have $F(x) = \{h_n(t,x)\}$ for some $t \in [0,1)$ and some natural number $n \ge m$. In view of (2.2), there is some $z \in Z$ satisfying $d(z,x) < 1/n$ and $\operatorname{dist}(h_n(t,x), f(z)) < 1/n$. Then $d(z,x_0) < 1/n + \delta = 1/n + 1/m < r$, i.e. $z \in B_r(x_0)$, and so

$f(z) \subseteq B_{\varepsilon/2}(f(x_0))$, i.e. $\operatorname{dist}(h_n(t,x), f(x_0)) < 1/n + \varepsilon/2 < \varepsilon$. This means that $F(x) = \{h_n(t,x)\} \subseteq B_\varepsilon(f(x_0))$, as required. □

3. Single-valued extensions of convex-valued maps

In view of Proposition 2.2 and the discussion in the previous section, it is not surprising that one can obtain results about single-valued extensions of maps with convex values. However, we obtain more general results in a different way, using the idea of the extension theorem of Dugundji [9] and Ma [16]. Moreover, we can even say more about the extending map.

To the authors' surprise, the theorem can be formulated in such a form that it requires neither any form of continuity of the original function $f\colon A \multimap Y$ nor any convexity hypothesis (also no local convexity of Y); nevertheless, the single-valued extension is continuous on $X \setminus A$ and satisfies the "glueing" condition (3.1) on ∂A. Only for the conclusion that this glueing condition really implies upper semicontinuity on ∂A, additional hypotheses are needed.

Theorem 3.1 (AC). *Let X be a metric space, $A \subseteq X$ closed and nonempty, Y be a topological Hausdorff vector space, and $f\colon A \multimap Y$ be such that it assumes on a dense subset of ∂A only nonempty values. Then for each continuous function $\varepsilon\colon X \setminus A \to (0,\infty)$ the map f has an extension $f_\varepsilon\colon X \multimap Y$ which is single-valued, locally compact, continuous* (*and if the uniform structure of* $\operatorname{conv} f(A)$ *is metrizable even locally Lipschitz, independent of the metric*) *on $X \setminus A$ and satisfies in addition*

$$f_\varepsilon(x) \in \operatorname{conv} \bigcup \{f(a) : a \in \partial A \text{ and } d(x,a) \le (1+\varepsilon(x)) \operatorname{dist}(x,A)\} \qquad (x \in X \setminus A). \tag{3.1}$$

In particular, $f_\varepsilon(X) \subseteq \operatorname{conv} f(A)$. Moreover, if f is upper semicontinuous and each value of f on ∂A has a neighbourhood base consisting of convex sets, then (3.1) *implies that f_ε is upper semicontinuous* (*if ε is locally bounded at ∂A*).

Actually, our proof will show that if $D \subseteq \partial A$ is a dense subset on which f has nonempty values and if $g\colon D \to Y$ satisfies $g(a) \in f(a)$ for all $a \in D$, then the function f_ε can be chosen such that even

$$f_\varepsilon(x) \in \operatorname{conv} \{g(a) : a \in \partial A \text{ and } d(x,a) \le (1+\varepsilon(x)) \operatorname{dist}(x,A)\} \qquad (x \in X \setminus A). \tag{3.2}$$

Dugundji's and Ma's extension theorems are contained in the following special case of Theorem 3.1.

Corollary 3.2 (AC). *Let X be a metric space, $A \subseteq X$ be closed and nonempty, Y be a locally convex space and $f\colon A \multimap Y$ be upper semicontinuous and such that for each $a \in \partial A$ the set $f(a)$ is nonempty and convex and either* *open or compact. Then for each continuous function* $\varepsilon\colon X \setminus A \to (0,\infty)$ *the map* f *has an extension to an upper semicontinuous map* $f_\varepsilon\colon X \multimap Y$ *which is single-valued, locally compact, continuous* (*and if the uniform structure of* $\operatorname{conv} f(A)$ *is metrizable even locally Lipschitz, independent of the metric*) *on* $X \setminus A$ *and satisfies* (3.1) *and even* (3.2).

Proof. We have to verify that for each $a \in \partial A$ the set $f(a)$ has a neighbourhood base consisting of convex sets. This is clear if $f(a)$ is open (and convex). Thus, assume that $f(a)$ is compact (and convex), and let $V \subseteq Y$ be an arbitrary neighbourhood of $f(a)$. By the compactness of $f(a)$, we find some neighbourhood $U \subseteq Y$ of 0 with $f(a) + U \subseteq V$. Since Y is locally convex, we may assume that U is convex. Since $f(a)$ is convex, it follows that $f(a) + U$ is a convex neighbourhood of $f(a)$ contained in V. □

Corollary 3.2 implies not only the well-known fact that closed convex subsets of metrizable locally convex spaces are retracts, but moreover that the retraction function can be chosen such that it maps "almost" on the element of best approximation, i.e. up to an arbitrarily small continuous error function ε.

Corollary 3.3 (AC). *Let* X *be a metric subset of a locally convex space* Y (*where the metric is compatible with the uniform structure*). *Then each closed convex subset* $A \subseteq X$ *is a retract of* X *where the retraction is locally compact and locally Lipschitz on* $X \setminus A$. *Moreover, if the metric on* X *has the property that each closed ball in* X *is the intersection of a convex subset of* Y *with* X, *then for each continuous function* $\varepsilon\colon X \to [0,\infty)$ *which is nonzero outside* A *the retraction* ρ_ε *can be chosen such that in addition*

$$d(x,\rho_\varepsilon(x)) \le (1+\varepsilon(x)) \operatorname{dist}(x,A) \quad (x \in X).$$

Proof of Theorem 3.1. For the first claim, we may assume that $\varepsilon\colon X\setminus A \to (0,1]$. For each point $x \in X \setminus A$, let $r(x)$ be the supremum of all numbers $r \in [0, \varepsilon(x)\operatorname{dist}(x,A)/25]$ with the property $\inf_{z \in B_{2r}(x)} \varepsilon(z) \ge \varepsilon(x)/2$. Since ε is continuous and $\varepsilon(x) > 0$, we have $r(x) > 0$, and the ball $B_x := B_{r(x)}(x)$ has the property

$$\varepsilon(z) \ge \varepsilon(x)/2 \quad \text{and} \quad \operatorname{diam} B_x \le \varepsilon(z)\operatorname{dist}(z,A)/6 \quad \text{for each } z \in B_x. \tag{3.3}$$

Indeed, for each $z \in B_x$ we have

$$\operatorname{dist}(x,A) \le d(x,z) + \operatorname{dist}(z,A) \le r(x) + \operatorname{dist}(z,A) \le \operatorname{dist}(x,A)/25 + \operatorname{dist}(z,A).$$

Hence, $25\operatorname{dist}(z,A) \ge 24\operatorname{dist}(x,A)$ which together with

$$\operatorname{diam} B_x \le 2r(x) \le 2\varepsilon(x)\operatorname{dist}(x,A)/25 \le 4\varepsilon(z)\operatorname{dist}(x,A)/25$$

implies (3.3).

The family of all sets B_x ($x \in X \setminus A$) is an open cover of the paracompact space $X \setminus A$, and so it has a subordinate locally finite cover $\mathfrak{U}$ of open sets, i.e. for each $U \in \mathfrak{U}$, we find some $x_U \in X \setminus A$ with $U \subseteq B_{x_U}$. Choose $z_U \in U$ and $a_U \in \partial A$ with

$$d(z_U, a_U) \le \left(1 + \frac{\varepsilon(x_U)}{4}\right) \operatorname{dist}(z_U, A) \tag{3.4}$$

and $f(a_U) \neq \emptyset$ (or $a_U \in D$, respectively). Choose some $y_U \in f(a_U)$ (or put $y_U := g(a_U)$, respectively), and define $\lambda_U(z) := \operatorname{dist}(z, X \setminus U)$ and

$$f_\varepsilon(z) := \begin{cases} f(z) & \text{if } z \in A, \\ \left\{ \dfrac{\sum_{U\in\mathfrak{U}} \lambda_U(z) y_U}{\sum_{U\in\mathfrak{U}} \lambda_U(z)} \right\} & \text{if } z \in X \setminus A. \end{cases}$$

Since $\mathfrak{U}$ is a locally finite open cover of $X\setminus A$, each point $z \in X\setminus A$ has a neighbourhood U_z which belongs to only finitely many $U \in \mathfrak{U}$. On this neighbourhood U_z, f_ε is well-defined, continuous, and in the metrizable case even Lipschitz (since all λ_U are Lipschitz), and $f(U_z)$ is contained in the convex hull of finitely many y_U and thus contained in a compact set. Moreover, for each $U \in \mathfrak{U}$ with $\lambda_U(z) \neq 0$, we have in view of (3.4) and (3.3) that

$$\begin{aligned} d(z, a_U) &\le d(z, z_U) + d(z_U, a_U) \le d(z, z_U) + \left(1 + \frac{\varepsilon(x_U)}{4}\right)\operatorname{dist}(z_U, A) \\ &\le d(z, z_U) + \left(1 + \frac{\varepsilon(z)}{2}\right)(d(z, z_U) + \operatorname{dist}(z, A)) \\ &\le 3 \operatorname{diam} U + \left(1 + \frac{\varepsilon(z)}{2}\right)\operatorname{dist}(z, A) \le (1 + \varepsilon(z)) \operatorname{dist}(z, A). \end{aligned}$$

Hence, $f_\varepsilon(z) \in \operatorname{conv}\{y_U : d(z, a_U) \le (1 + \varepsilon(z)) \operatorname{dist}(z, A)\}$, and so (3.1) holds.

The last claim follows from (3.1). In fact, since f_ε is continuous on $X\setminus A$, it remains to verify the upper semicontinuity on ∂A. However, if $a_0 \in \partial A$ and $V \subseteq Y$ is a neighbourhood of $f(a_0)$, then we find by hypothesis a convex neighbourhood $V_0 \subseteq V$ of $f(a_0)$. There is some $\delta > 0$ with $f(B_\delta(a_0) \cap A) \subseteq V_0$. Shrinking δ if necessary, we may assume that the function $1 + \varepsilon$ is bounded on $B_\delta(a_0) \setminus A$ by some constant $M > 1$. Then for each $x \in X\setminus A$ with $d(x, a_0) < \delta/(1+M)$ the following holds: For each $a \in \partial A$ satisfying $d(x, a) \le (1 + \varepsilon(x)) \operatorname{dist}(x, A)$, we have $d(a, a_0) \le d(a, x) + d(x, a_0) \le M \operatorname{dist}(x, A) + d(x, a_0) \le (1 + M)d(x, a_0) < \delta$, and so $f(a) \subseteq V_0$. Hence, (3.1) implies $f(x) \subseteq \operatorname{conv} V_0 = V_0 \subseteq V$, and so $f(B_{\delta/(1+M)}(a_0)) \subseteq V$. This shows that f is upper semicontinuous at a_0, as claimed. $\square$

Remark 3.4. Theorem 3.1 and Corollaries 3.2 and 3.3 depend essentially on the axiom of choice which is not only needed to select the function $U \mapsto y_U$ but moreover for the proof of Stone's theorem that metric spaces are paracompact. However, if $X \setminus A$ is separable, then the countable axiom of choice suffices for the proof. In fact, each open covering of a separable metric space has a locally finite countable subordinate open covering $\mathfrak{U}$: This can be seen as in the proof of [17, 10.7], observing that each separable metric space has a countable base of open sets.

In view of Remark 3.4, it appears important to have a more constructive approach to the above results. Hence, although the following result is a special case of the above results, it is of independent interest, since its proof uses a different method which requires only a countable form of the axiom of choice. In finite-dimensional normed spaces, a simpler version of this proof was used in [8].

Theorem 3.5. *Assume only the countable axiom of choice. Consider one of the situations of Theorem* 3.1*, Corollary* 3.2*, or Corollary* 3.3 *and assume that ∂A is separable and that Y is locally compact, metrizable and equipped with a metric which is compatible with the uniform structure of Y.*

(a) *If $\dim \operatorname{span} f(\partial A) < \infty$, then all statements of the above results remain true* (*except for the local compactness and the local Lipschitz property*)*.*

(b) *Let for some countable dense subset $\{a_1, a_2, \dots\} \subseteq \partial A$ on which f assumes nonempty values be a selection $y_n \in f(a_n)$ such that each of the sets*

$$C_x := \overline{\operatorname{conv}}\{y_n : d(x, a_n) \leq (1 + \varepsilon(x)) \operatorname{dist}(x, A),\ n = 1, 2, \dots\} \quad (x \in X \setminus A)$$

is complete. (*This holds in particular if Y is complete*)*.*

Then all statements of the above results remain true (*except for the local compactness and the local Lipschitz property*) *provided that we replace everywhere* conv *by* $\overline{\operatorname{conv}}$*. Moreover, in this case*

$$f_\varepsilon(x) \subseteq C_x \quad (x \in X \setminus A). \tag{3.5}$$

Proof. We consider first the second statement. Without loss of generality, we assume that ε is bounded by 1. Let the metric in Y be generated by the countable family $\|\cdot\|_k$ of seminorms. By our assumption, a sequence is a Cauchy sequence (converges, or is bounded) in Y if and only if it is a Cauchy sequence (converges, or is bounded, respectively) with respect to each seminorm $\|\cdot\|_k$.

Choose $\{a_1, a_2, \dots\} \subseteq \partial A$ and $y_n \in f(a_n)$ as in the hypothesis, and choose numbers $c_n > 0$ such that

$$\sum_{n=1}^{\infty} c_n \max\{1, \|y_n\|_1, \dots, \|y_n\|_n\} < \infty.$$

For $x \in X \setminus A$, we define $\lambda_n(x) := \max\{1 + \varepsilon(x) - d(x, a_n)/\operatorname{dist}(x, A), 0\}$ and put

$$f_\varepsilon(x) := \begin{cases} f(x) & \text{if } x \in A, \\ \left\{ \dfrac{\sum_{n=1}^{\infty} c_n \lambda_n(x) y_n}{\sum_{n=1}^{\infty} c_n \lambda_n(x)} \right\} & \text{if } x \in X \setminus A. \end{cases} \tag{3.6}$$

Then f_ε is defined and continuous on $X \setminus A$. Indeed, since $\{a_1, a_2, \dots\}$ is dense in ∂A, the function

$$w_N(x) := \frac{\sum_{n=1}^N c_n \lambda_n(x) y_n}{\sum_{n=1}^N c_n \lambda_n(x)}$$

is defined for sufficiently large N, and $w_N(x) \in C_x$. Moreover, w_N forms a locally uniform Cauchy sequence with respect to each seminorm $\|\cdot\|_k$, since λ_n is bounded by 2 and $\sum_n 2c_n \|y_n\|_k$ and $\sum_n 2c_n$ converge. In particular, w_N is a Cauchy sequence in C_x and thus convergent to an element of C_x. By the local uniform convergence, the function f_ε is also continuous on $X \setminus A$. Since we have already proved (3.5) the last claim of Theorem 3.1 follows as in the proof of Theorem 3.1.

For the first claim, we use the same proof observing that we have in this case even

$$f_\varepsilon(x) \subseteq \operatorname{conv}\{y_n : d(x, a_n) \le (1 + \varepsilon(x)) \operatorname{dist}(x, A)\} \quad (x \in X \setminus A)$$

(i.e. conv instead of $\overline{\operatorname{conv}}$) in view of the following Lemma 3.6. □

Lemma 3.6. *Let M be a convex (not necessarily closed) subset of a finite-dimensional vector space Y. Let $x_n \in M$ and $\lambda_n \ge 0$ be such that $\sum_{n=1}^\infty \lambda_n = 1$ and $x := \sum_{n=1}^\infty \lambda_n x_n$ converges in Y. Then $x \in M$.*

Proof. We prove the claim by induction on the (real) dimension N of Y. For $N = 0$, the claim is trivial. For the induction step, we assume without loss of generality that $x = 0$. Assume by contradiction that $x = 0 \notin M$. Then there is a nontrivial linear functional f on Y with $f(y) \le f(0) = 0$ for all $y \in M$. Then $\lambda_n f(x_n) = 0$ for all n. In fact, if this were not true, we would find some N with $\lambda_N f(x_N) < 0$ which by the continuity of f yields the contradiction

$$0 = \sum_{n=1}^\infty \lambda_n f(x_n) < \sum_{\substack{n=1 \\ n \ne N}}^\infty \lambda_n f(x_n) \le 0.$$

Hence, we may assume that all x_n belong to the null space of f. The contradiction follows by applying the induction hypothesis to this null space which has dimension $N - 1$ (and to the intersection of M with this null space). □

Using the Tietze extension theorem, we can also obtain a constructive proof for the single-valued extension of real-valued multivalued maps when X is not necessarily metric but only normal.

Recall that $f : X \multimap Y$ is called *lower semicontinuous at* $x_0 \in X$ if for each $y \in f(x_0)$ and each neighbourhood $V \subseteq Y$ of y there is some neighbourhood $U \subseteq X$ of x_0 such that $f(x) \cap V \ne \emptyset$ for each $x \in U$.

THEOREM 3.7. *Let X be a T_4-space, $A \subseteq X$ closed and nonempty, and $f: A \multimap \mathbb{R}$ upper semicontinuous* (*or upper semicontinuous in the ε-sense*). *Suppose that $f|_{\partial A}$ is lower semicontinuous and has one of the following two properties:*

(a) *For each $a \in \partial A$ the set $f(a)$ is nonempty and has a minimum.*
(b) *For each $a \in \partial A$ the set $f(a)$ is nonempty and has a maximum.*

Then f has an extension to an upper semicontinuous (*or upper semicontinuous in the ε-sense, respectively*) *map $F: X \multimap \operatorname{conv} f(A)$ which single-valued and continuous and in case of metrizable X even locally Lipschitz on $X \setminus A$. Moreover, if $\partial A \neq \emptyset$, then*

$$f(X \setminus A) \subseteq \operatorname{conv}\{\min f(a) : a \in \partial A\}$$

or

$$f(X \setminus A) \subseteq \operatorname{conv}\{\max f(a) : a \in \partial A\},$$

respectively.

Of course, there is no chance that the extension is lower semicontinuous, in general:

EXAMPLE 3.8. Let $X = \mathbb{R}$, $A := \{0\}$, and $f(0) := [0,1]$. Then f has no lower semicontinuous extension to $\mathbb{R}$ which is single-valued on $X \setminus A$.

PROOF OF THEOREM 3.7. We may assume that $\partial A \neq \emptyset$. Replacing f by $-f$ if necessary, we may assume without loss of generality that for each $a \in \partial A$ the set $f(a)$ is nonempty and has a minimum. Our hypothesis implies that the function $g: \partial A \to \mathbb{R}$, defined by $g(a) := \min f(a)$, is continuous. By Tietze's extension theorem, we can extend g to a continuous function $G: X \to \mathbb{R}$ with $G(X) \subseteq \operatorname{conv} g(A)$. Then

$$F(x) := \begin{cases} f(x) & \text{if } x \in A, \\ \{G(x)\} & \text{if } x \in X \setminus A. \end{cases}$$

is the required extension. If X is metrizable, it may be arranged that G is locally Lipschitz by the subsequent Lemma 3.9. □

The following lemma is essentially an examination of one of the standard proofs of Tietze's theorem for metric spaces. Since we found no reference containing the details, we provide them for the reader's convenience.

LEMMA 3.9. *Let X be a metric space, $A \subseteq X$ closed and nonempty and $f: A \to \mathbb{R}$ be continuous. Then f has a continuous extension $F: X \to \operatorname{conv} f(A)$ which is locally Lipschitz on $X \setminus A$ and satisfies $F(X \setminus A) \subseteq \operatorname{conv} f(\partial A)$ if $\partial A \neq \emptyset$.*

PROOF. We may assume that $\partial A \neq \emptyset$. We assume first that f is bounded from below on ∂A by some constant $M > -\infty$. For $c > 0$, define $F_0: X \setminus A \to \mathbb{R}$

by

$$F_0(x) := \inf\left\{f(a) + c\left(\frac{d(x,a)}{\operatorname{dist}(x,A)} - 1\right) : a \in \partial A\right\} \quad (x \in X \setminus A). \tag{3.7}$$

In view of Lemma 2.5, F_0 is locally Lipschitz, the local Lipschitz constant even depending linear on c.

We prove now that $F_0(x) \to f(a_0)$ as $x \to a_0 \in \partial A$. For $\varepsilon > 0$ choose $r > 0$ with $f(B_r(a_0) \cap A) \subseteq B_\varepsilon(f(a_0))$ and $\delta > 0$ such that

$$(2+\varepsilon)\delta < r \quad \text{and} \quad M + c\left(\frac{r}{\delta} - 2\right) > f(a_0) - \varepsilon. \tag{3.8}$$

For $x \in X \setminus A$ with $d(x, a_0) < \delta$, note that $\operatorname{dist}(x,A) \le d(x,a_0) < \delta$ and choose some $a_x \in \partial A$ with $d(x,a_x) \le (1+\varepsilon)\operatorname{dist}(x,A) < (1+\varepsilon)\delta$; the choice $a = a_x$ in (3.7) implies $F_0(x) \le f(a_x) + c\varepsilon$. Since $d(a_x,a_0) \le d(a_x,x) + d(x,a_0) < (2+\varepsilon)\delta < r$, we conclude $F_0(x) \le f(a_0) + (1+c)\varepsilon$.

For $a \in \partial A$, we have in the case $d(a,a_0) \ge r$ that $d(x,a) \ge r - d(x,a_0) \ge r - \delta$; in view of (3.8), we obtain in this case

$$M + c\left(\frac{d(x,a)}{\operatorname{dist}(x,A)} - 1\right) \ge M + c\left(\frac{r-\delta}{\delta} - 1\right) > f(a_0) - \varepsilon.$$

In the opposite case $d(a,a_0) < r$, we have $f(a) > f(a_0) - \varepsilon$. Hence, in all cases, we obtain from (3.7) that $F_0(x) \ge f(a_0) - \varepsilon$.

By what we have proved, we can conclude that the function

$$F_1(x) := \begin{cases} f(x) & \text{if } x \in A, \\ \min\{F_0(x), \sup f(A)\} & \text{if } x \in X \setminus A, \end{cases}$$

has almost the required properties: It is continuous, locally Lipschitz on $X \setminus A$, and $F_1(X \setminus A) \subseteq \overline{\operatorname{conv}}\, f(\partial A)$. We still must get rid of the closure in the last inclusion by modifying F_1. To this end, let $B_{\min}, B_{\max} \subseteq X$ be the (possibly empty) sets where F_1 attains its minimum or maximum, respectively. Let $B := B_{\min}$, $B := B_{\max}$, $B := B_{\min} \cup B_{\max}$, or $B := \emptyset$ depending whether $f(\partial A)$ has no minimum, no maximum, neither, or both. Then $B_0 := B \cap (X \setminus A)$ is closed and disjoint from A. Define $\lambda\colon A \cup B_0 \to [0,1]$ as 1 on A and as 0 on B_0. Then λ is bounded from below, and by what we have proved so far, we can extend λ to a locally Lipschitz function $\lambda_1\colon X \to [0,1]$. Fix some $y \in f(\partial A)$. Then $F(x) := y + \lambda_1(x)(F_1(x) - y)$ has all required properties.

For general f, we can extend the function $\exp\circ f$ to a continuous function $F\colon X \to \mathbb{R}$ which is locally Lipschitz on $X \setminus A$ and satisfies $F(X \setminus A) \subseteq \operatorname{conv}\exp(f(\partial A))$. In particular, $F(x) > 0$ for each $x \in X$ (here it is important that we were able to replace $\overline{\operatorname{conv}}$ by conv), and so $\log F$ is the required extension of f. $\square$

4. Extensions of compact maps

We call a topological space Y a *convex* (*neighbourhood*) *retract* if it is homeomorphic to a retract (or neighbourhood retract, respectively) of a convex subset of a locally convex space. Recall that if Y is metric, then such a set is called an AR (or ANR). A topological space Y is called an AE (or ANE) for metric spaces, if for each metric space X, each closed subset $A \subseteq X$, and each continuous map $f: A \to Y$ there is a continuous extension $f: X \to Y$ (or $f: U \to Y$ for some neighbourhood $U \subseteq X$ of Y, respectively).

Remark 4.1. Since by the Arens–Eells embedding theorem [4], each metric space Y is isometric to a closed subset A of a normed space X, it follows that each metrizable AE/ANE is an AR/ANR. Moreover, using Dugundji's extension theorem (Corollary 3.2) it is easy to see that conversely each AR/ANR is an AE/ANE. However, it is good to distinguish the notions anyway, since the proof of Corollary 3.2 requires the axiom of choice, and moreover, e.g. each non-metrizable convex retract is an AE which is not an AR.

Although the following observation about extension of multivalued maps is rather trivial, it appears very useful, since the results of Section 2 usually only provide an extension to a neighbourhood of A:

Proposition 4.2. *Let X be a metric space, $A \subseteq X$ be closed, and suppose that a multivalued map $f: A \multimap Y$ has an extension to a map $F_0: U \multimap Y$ on a neighbourhood $U \subseteq X$ of A which is single-valued and continuous on $U \setminus A$. If Y is an* AE *for metric spaces, then f even has an extension $F: X \multimap Y$ which is single-valued and continuous on $X \setminus A$ and coincides with F_0 in some neighbourhood of A.*

Proof. Since the metric space X is normal, U contains some closed neighbourhood $A_0 \subseteq X$ of A. Since Y is an AE, the restriction of F_0 to ∂A_0 (considered as a continuous map) has an extension to a continuous map $F_1: X \to Y$, and so the map

$$F(x) := \begin{cases} F_0(x) & \text{if } x \in A_0, \\ \{F_1(x)\} & \text{if } x \in X \setminus A_0, \end{cases}$$

has the required properties by the glueing lemma. □

However, in particular in connection with degree theory, one not only needs an upper semicontinuous extension of a certain map f but even a *compact* such extension (provided, of course, that f is compact). If we understand compactness relatively to the image space Y, we can prove the expected answer, using a deep result of Girolo [11].

Theorem 4.3 (AC). *Let X be a metric space, $A \subseteq X$ be closed and Y be an* ANR. *Suppose that a multivalued map $f: A \multimap Y$ has an extension to an upper* *semicontinuous* (*in the ε-sense*) *map* $F_0: U \multimap Y$ *on a neighbourhood* $U \subseteq X$ *of* A *which is single-valued on* $U \setminus A$. *Suppose in addition that* $f(A)$ *is contained in a compact subset of* Y.

(a) *Then* f *has an extension to an upper semicontinuous* (*in the* ε*-sense, respectively*) *map* $F: U_0 \multimap Y$ *to some neighbourhood* $U_0 \subseteq X$ *of* A *which is single-valued on* $U_0 \setminus A$ *and such that* $F(U_0)$ *is contained in a compact subset of* Y.

(b) *If* Y *is even an* AR, *then the above holds with* $U_0 = X$.

Proof. In view of the Arens–Eells embedding theorem, we may assume that Y is a closed subset of a normed vector space Z. By [11], there is a compact retract $K \subseteq Z$ of Z containing $f(A)$. We do not know whether K must be contained in Y, but the identity map on $K \cap Y$ has a continuous extension $i: Y \to K$ and a continuous extension $j: V \to Y$ where $V \subseteq K$ is some closed neighbourhood of $K \cap Y$ (in the space K) or even $V := K$ if Y is an AR. By Proposition 4.2, the map $i \circ F_0: U \multimap K$ has an extension to an upper semicontinuous (in the ε-sense) map $F_1: X \multimap K$ which is single-valued on $X \setminus A$. Since the set $K \cap Y$ is compact, the neighbourhood $V \subseteq K$ contains a set of the form $B_\varepsilon(K \cap Y) \cap K$ with some $\varepsilon > 0$. Since this set contains $F_1(A) = f(A)$, the upper semicontinuity (in the ε-sense) of F_1 implies that there is some neighbourhood $U_0 \subseteq X$ of A with $F_1(U_0) \subseteq V$; in case $V = K$ (if Y is an AR), one may even choose $U_0 := X$. Then $F := j \circ F_1: U_0 \multimap Y$ has all required properties. Indeed, since $V \subseteq K$ is closed and thus compact, $j(V) \subseteq Y$ is compact and contains $F(U_0)$. □

Remark 4.4. Our proof shows in view of Remark 3.4 and the fact that the compact metric space K is separable that the general axiom of choice is not needed for Theorem 4.3 if $X \setminus A$ is separable.

However, in some applications of degree theory, one would like to consider also a subset Y of some larger space Z which is not necessarily closed, and one knows only that $f(A)$ is contained in a compact subset of Z (but then not necessarily in a compact subset of Y). One is then looking for an extension F as in Theorem 4.3 with the difference that one wants to require only that $F(U_0)$ is contained in a compact subset of Z. Even if f is single-valued and Z is a Banach space, it seems to be unknown whether a corresponding extension of Theorem 4.3 is true. However, at least there *are* nontrivial situations in which this is the case:

Theorem 4.5 (AC). *Let* X *be a metric space,* $A \subseteq X$ *be closed,* Z *be a locally convex space, and* $Y \subseteq Z$ *possess some interior point and a metrizable closure. Suppose that a multivalued map* $f: A \multimap Y$ *has an extension to an upper* *semicontinuous* (*in the* ε*-sense*) *map* $F_0: U \multimap Y$ *on a neighbourhood* $U \subseteq X$ *of* A *which is single-valued on* $U \setminus A$. *Suppose in addition that* $f(A)$ *is contained in a compact subset of* Z. *Then* f *has an extension to an upper semicontinuous* (*in the* ε*-sense, respectively*) *map* $F: X \multimap Y$ *which is single-valued on* $X \setminus A$ *and such that* $F(X)$ *is contained in a compact subset of* Z.

Proof. Note that $\overline{Y}$ (closure in Z) is an AR. Hence, by Theorem 4.3, we find an extension of f to an upper semicontinuous (in the ε-sense) map $F_1: X \multimap \overline{Y}$ which is single-valued on $X \setminus A$ and such that $F_1(X)$ is contained in a compact subset $K \subseteq \overline{Y}$. Let y_0 be an interior point of Y. Define a homotopy $h: [0,1] \times K \to \overline{Y}$ by $h(t,y) := (1-t)y + ty_0$. Then we have for each $t \in [0,1]$ and each $y_1 \in K$ that $h(t, y_1) \in Y$. In fact, since y_0 is an interior point of Y, we find a finite family P of seminorms in Y such that the set

$$V_0 := \{y \in Y : p(y - y_0) \leq 1 \text{ for each } p \in P\}$$

is contained in Y. Since $y_1 \in \overline{Y}$ and $t > 0$, we find some $y_2 \in Y$ such that $p(y_2 - y_1)(1-t) \leq t$ for all $p \in P$. Then $y_3 := y_0 + (1-t)(y_1 - y_2)/t \in V_0$, because $p(y_3 - y_0) = (1-t)p(y_1 - y_2)/t \leq 1$ for all $p \in P$. Since Y is convex, we conclude that $ty_3 + (1-t)y_2 = h(t, y_1)$ belongs to Y, as claimed. We conclude that for each $x \in X \setminus A$ the value

$$F(x) := h(\min\{\operatorname{dist}(x, A), 1\}, F_1(x))$$

is contained in Y. Moreover, $F(X)$ is contained in the compact set $h([0,1] \times K)$. For $x \in A$, we have $F(x) = F_1(x) = f(x) \subseteq Y$, and so F has all required properties. □

Remark 4.6. In view of Remark 4.4, the general axiom of choice is not needed for Theorem 4.5 if $X \setminus A$ is separable.

For the case that X is not necessarily metrizable but f is lower semicontinuous on ∂A, we can use the multivalued Tietze theorem (Theorem 3.7) to obtain the following constructive extension result.

Theorem 4.7. *Let* X *be a* T_4*-space,* $A \subseteq X$ *closed, and* Y *be a convex* (*neighbourhood*) *retract. Let* $f: A \multimap Y$ *be upper semicontinuous, and suppose that* $f(A)$ *is contained in a compact metrizable subset* $K \subseteq Y$. *Suppose that* f *is lower semicontinuous on* ∂A *and that* $f(a)$ *is nonempty and compact for each* $a \in \partial A$. *Then for* $U := X$ (*or some neighbourhood* $U \subseteq X$ *of* A*, respectively*), *there is an upper semicontinuous extension* $F: U \multimap Y$ *of* f *which is single-valued on* $U \setminus A$ *and whose range is contined in a compact subset of* Y. *An analogous result holds if* Y *is an* AE (ANE) *for metric spaces.*

Proof. Since K is a separable metric space, it is homeomorphic to a subset K_0 of the Hilbert cube $H := [0,1]^{\mathbb{N}}$. Let $g: K \to K_0$ be the corresponding homeomorphism. By the compactness of K, it follows that K_0 is compact and thus closed in H. Using Theorem 3.1 (more precisely, Remark 3.4, observing that H is separable), it can easily be checked that we can extend the map $g^{-1}: K_0 \to K$ to a continuous map $G: \overline{V} \to Y$ where $V \subseteq H$ is some open neighbourhood of K_0 (or where $V := H$, respectively, if Y is a convex retract or an AE). For each $n \in \mathbb{N}$, let $\pi_n: H \to [0,1]$ be the projection of the n-th component. We can use Theorem 3.7 to extend the map $\pi_n \circ g \circ f: A \multimap [0,1]$ to an upper semicontinuous map $f_n: X \multimap [0,1]$ which is single-valued on $X \setminus A$. Then $F_0 := \prod_{n\in\mathbb{N}} f_n: X \to H$ is upper semicontinuous on X and single-valued on $X \setminus A$. Put $U := \{x \in X : F_0(x) \subseteq V\}$. Since V is open and F_0 is upper semicontinuous, we conclude that U is open. Hence, $F := G \circ F_0|_U: U \to Y$ has the required properties. Note that $F(U)$ is contained in the compact subset $G(\overline{V})$ of Y. $\square$

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