Topological Methods in Nonlinear Analysis
Volume 41, No. 1, 2013, 191–205

VIETORIS–BEGLE THEOREMS FOR NONCLOSED MAPS

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Abstract. In the paper we provide generalizations of the classical Vietoris–Begle mapping theorem for not necessarily closed maps with respect to the Alexander–Spanier cohomology on paracompact space.

1. Introduction

Let X and Y be paracompact spaces. Let H^* denote the Alexander–Spanier cohomology functor with coefficients in an arbitrary abelian group. A continuous surjection $f\colon X \to Y$ is called a *Vietoris map*, if f is a *closed* map and f has *acyclic fibers* i.e. $H^q(f^{-1}(y)) \approx H^q(pt)$ for any $y \in Y$ and $q \in \mathbb{Z}$, where pt stands for a one point space. The well-known Vietoris–Begle theorem (see [19], [1], [16]) states that *if f is a Vietoris map, then*

$$H^q(f)\colon H^q(Y) \to H^q(X)$$

is a group isomorphism for any $q \in \mathbb{Z}$. The result has been investigated and generalized by many authors. For example, the triviality of cohomology groups of fibers can be assumed in finite number of dimensions and isomorphisms are obtained in finite number of dimensions as well (see [7] and references therein). Moreover it has been considered (co)homotopy variants of the Vietoris–Begle theorem (see [5], [6], [15]).

2010 *Mathematics Subject Classification.* Primary 55N99, 54C60.

Key words and phrases. Vietoris–Begle theorem, set-valued maps, fiberwise topology, taut cohomology theory.

The author was supported in part by KBN Grant NN 201395137.

It is known that, in general, we cannot remove the assumption concerning the closeness or the acyclicity of fibers of the map f (see [17, Example 6.9.16]).

In the paper we introduce the following class of maps $\mathcal{V}$ containing Vietoris maps: for any continuous surjection $f\colon X \to Y$ we write $f \in \mathcal{V}$, if for any $q \in \mathbb{Z}$

$$\varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\} \approx H^q(pt),$$

where the direct system is indexed by the family of all open neighbourhoods V of y. We show that the class $\mathcal{V}$ extends the class of Vietoris maps (see Proposition 3.2) and we prove that the statement of the Vietoris–Begle theorem is still valid (see Theorem 4.5). Moreover, we provide examples of maps of the class $\mathcal{V}$ which are not necessarily closed and their fibers do not have to be acyclic (see Example 3.3).

Motivated by the applications of the Vietoris-Begle theorem in studying homotopical invariants of set-valued maps ([8], [14]) we consider the situation when there are given two subsets G and C of X, G is closed in X. In general if $f|_G$ is a closed map, the surjective map $f|_{G\cap C}\colon G \cap C \to Y$ with the acyclic fibers does not induce isomorphisms on cohomology groups (see Example 4.8). However it turns out that there is the induced isomorphism on the direct limits of cohomology groups of adequate directed systems (see Theorem 4.7). More generally, we introduce a class of Vietoris maps with respect to (G, C) (written $\mathcal{V}(G, C)$, see Definition 3.1) and prove Vietoris–Begle type theorem for this class of maps. Furthermore, we show that if the fibers of a map satisfy the homotopy triviality conditions with respect to the pair (G, C), then the map belongs to the class $\mathcal{V}(G, C)$ (see Proposition 3.4(b)). At the end, the fiberwise version of the Vietoris–Begle theorem is proposed (Theorem 4.10).

2. Preliminaries

From now on by a *space* we mean a *paracompact space* and by a *map* we mean a *continuous transformation* of spaces. A map $f\colon X \to Y$ is *closed* provided that for any closed subset A of X, $f(A)$ is a closed subset of Y. Given spaces X, Y, by a *set-valued map* φ from X into Y (written $\varphi\colon X \multimap Y$) we mean a map assigning to any $x \in X$ a *nonempty* (not necessarily closed) subset $\varphi(x)$ of Y and, by the *graph* of φ, the set $\mathrm{Gr}\,(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$. We say that a set-valued map φ is *upper semicontinuous* (resp. *lower semicontinuous*) if for any open (resp. closed) subset $U \subset Y$, the *preimage* $\varphi^{+1}(U) := \{x \in X : \varphi(x) \subset U\}$ is open (resp. closed). It is easy to see that a surjection $f\colon X \to Y$ is closed if and only if $\varphi\colon Y \multimap X$ given by $\varphi(y) := f^{-1}(y)$ for $y \in Y$, is upper semicontinuous.

Let $H = \{H^q \mid q \in \mathbb{Z}\}$ be a contravariant functor on the category of (paracompact) spaces to the category of graded abelian groups. Let X be a space. For any subsets A and B of X such that $A \subset B$, by $H^q(i_A^B)\colon H^q(B) \to H^q(A)$ we denote the homomorphism of groups induces by the inclusion map $i_A^B\colon A \hookrightarrow B$, where $q \in \mathbb{Z}$. We introduce the notation: for any $h \in H^q(B)$

$$h|_A := H^q(i_A^B)(h) \in H^q(A).$$

Let A be a subset of X. Then the family $\mathcal{A}$ of all open neighbourhoods of A is directed by the *relation of* (*inverse*) *inclusion* i.e. $U \leq V$, if $V \subset U$ for any $U, V \in \mathcal{A}$. Then for any $q \in \mathbb{Z}$ the direct limit of a direct system $\{H^q(U), H^q(i_U^V)\}$ is denoted by $\varinjlim\{H^q(U) \mid U \in \mathcal{A}\}$ or $\varinjlim\{H^q(U) \mid U \supset A\}$. Moreover there is a homomorphism of groups

$$i\colon \varinjlim\{H^q(U) \mid U \supset A\} \to H^q(A), \tag{2.1}$$

defined by the formula: $i([h]) := h|_A$ for any $h \in H^q(U)$. We say that a subset $A \subset X$ is *taut in* X *with respect to* H, if i is an isomorphism for any $q \in \mathbb{Z}$.

Let $H^* = H^*(\cdot; G) = \{H^q \mid q \in \mathbb{Z}\}$ (resp. $H^*_\Delta = H^*_\Delta(\cdot; G) = \{H^q_\Delta \mid q \in \mathbb{Z}\}$) denote Alexander–Spanier (resp. singular) cohomology functor with coefficient in an abelian group G. In view of ([11, Theorem 8.4]) every closed subset A of a space X is taut in X with respect to H^*. Moreover every subset A of a metrizable space X is taut in X with respect to H^*.

3. Vietoris maps with respect to a pair of sets

Let G and C be subsets of a space X, G is closed in X. Let $f : X \to Y$ be a map and A be a subset of a space Y. We introduce the notation: for any open subset $W \subset X$ we write $W \in \mathcal{W}(A; G)$, if there are open neighbourhoods V of A and U of $f^{-1}(V) \cap G$ such that

$$U \cap f^{-1}(V) \subset W.$$

Note that if $W \in \mathcal{W}(A; G)$, then W is an open neighbourhood of $f^{-1}(A) \cap G$. Moreover, if $f|_G\colon G \to Y$ is a closed map, then

$$\mathcal{W}(A; G) = \{W \mid W \text{ is an open neighbourhood of } f^{-1}(A) \cap G\}. \tag{3.1}$$

Indeed, let W be an open neighbourhood of $f^{-1}(A) \cap G$ in X. If $f|_G$ is a closed map, then a set-valued map $Y \ni y \multimap f^{-1}(y) \cap G \subset G$ is upper semicontinuous. Hence there is an open neighbourhood V of A such that $f^{-1}(V) \cap G \subset W$. Therefore $W \cap f^{-1}(V) \subset W$ and $W \in \mathcal{W}(A; G)$.

Now we introduce the following class of maps.

Definition 3.1. A map $f\colon X \to Y$ is called a *Vietoris map with respect to* (G, C) (written $f \in \mathcal{V}(G, C)$), if $f(G \cap C) = Y$ and for any $y \in Y$, $q \in \mathbb{Z}$

$$\varinjlim\{H^q(W \cap C) \mid W \in \mathcal{W}(\{y\}; G)\} \approx H^q(pt). \tag{3.2}$$

is an isomorphism. In case if $G = C = X$ we write $\mathcal{V} := \mathcal{V}(X, X)$.

The following result provides examples and characterizes the class $\mathcal{V}$.

PROPOSITION 3.2. (a) *$f \in \mathcal{V}$ if and only if f is a surjection and for any $y \in Y$, $q \in \mathbb{Z}$*

$$\varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\} \approx H^q(pt),$$

where V belongs to the family of all open neighbourhoods of y.

(b) *If f is a closed surjection, then $f \in \mathcal{V}$ if and only if f has acyclic fibers.*

(c) *If X, Y are metric spaces, (X, f) is a fiberwise absolute neighbourhood retract over Y, then $f \in \mathcal{V}$* (1).

PROOF. (a) In order to prove (a) it is sufficient to observe that the family $\{f^{-1}(V) \mid V$ is an open neighbourhood $y\}$ is cofinal with $\mathcal{W}(\{y\}; X)$.

(b) Let f be a closed surjection. Then for any open neighbourhood U of the fiber $f^{-1}(y)$ there is an open neighbourhood V of y such that $f^{-1}(V) \subset U$. Hence the family $\{f^{-1}(V) \mid V$ is an neighbourhood $y\}$ is cofinal with the family $\{U \mid U$ is an neighbourhood $f^{-1}(y)\}$. In view of tautness of $f^{-1}(y)$ in X with respect to H^* we obtain isomorphisms

$$\varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\} \approx \varinjlim\{H^q(U) \mid U \supset f^{-1}(y)\} \approx H^q(f^{-1}(y)).$$

Hence by (a) we obtain that $f \in \mathcal{V}$ if and only if $H^q(f^{-1}(y)) \approx H^q(pt)$.

(c) Since (X, f) is an absolute neighbourhood retract over Y, then there are a normed space E and maps $i: X \to E \times Y$ and $r: E \times Y \to X$ over Y such that $r \circ i = \mathrm{id}_X$. Observe that for any open neighbourhood $V \subset Y$, $r(E \times V) = f^{-1}(V)$ and $H^q(r|_{E\times V}): H^q(f^{-1}(V)) \to H^q(E \times V)$ is a monomorphism since $H^q(i|_{f^{-1}(V),E\times V}) \circ H^q(r|_{E\times V}) = \mathrm{id}_{H^q(f^{-1}(V))}$ for any $q \in \mathbb{Z}$. Hence a homomorphism

$$r^*: \varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\} \to \varinjlim\{H^q(E \times V) \mid V \ni y\}$$

defined by a formula: $r^*([h]) := [H^q(r|_{E\times V})(h)]$ for any $h \in H^q(f^{-1}(V))$, is also a monomorphism, since $r^* = \varinjlim\{H^q(r|_{E\times V}) \mid V \ni y\}$. Observe that $H^q(\pi|_{E\times V}): H^q(V) \to H^q(E \times V)$ is an isomorphism for any open $V \subset Y$, and then we obtain the following isomorphism

$$\pi^*: \varinjlim\{H^q(V) \mid V \ni y\} \to \varinjlim\{H^q(E \times V) \mid V \ni y\}$$

given by $\pi^*([h]) := \varinjlim\{H^q(\pi|_{E\times V}) \mid V \ni y\}([h]) = [H^q(\pi|_{E\times V})(h)]$ for any $h \in H^q(V)$. Thus

$$\pi^{*-1} \circ r^* : \varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\} \to \varinjlim\{H^q(V) \mid V \ni y\}$$

(1) Fiberwise absolute neighbourhood retracts were studied e.g. by Dold in [4], see also [12]. Recall that (X, f) is a *fiberwise absolute neighbourhood retract over* Y if there is a normed space E and maps $i: X \to E \times Y$ and $r: E \times Y \to X$ over Y, i.e. $\pi_Y \circ i = f$ and $f \circ r = \pi_Y$, such that $r \circ i = \mathrm{id}_X$, where $\pi_Y: E \times Y \to Y$ is the usual projection onto Y.

is a monomorphism. In view of tautness of $\{y\}$ in Y with respect to H^* we obtain

$$\varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\} \approx H^q(pt). \qquad \square$$

Vietoris maps appears e.g. in theory of set-valued maps ([8]). If $\varphi: X \multimap Y$ is an upper-semicontinuous set-valued map with acyclic values, then the projection $\pi_X: \mathrm{Gr}\,(\varphi) \to X$ is a Vietoris map. In view of Proposition 3.2(b) we obtain that $\pi_X \in \mathcal{V}$. The class $\mathcal{V}$ essentially extends the class of Vietoris maps. In view of [12], if $\varphi: X \multimap Y$ is a lower semicontinuous map with closed and convex values in the Banach space Y, then $(\mathrm{Gr}\,(\varphi), \pi_X)$ is a fiberwise absolute retract over X with the projection $\pi_X: \mathrm{Gr}\,(\varphi) \to X$, and in view of Proposition 3.2(c) π_X is a Vietoris map with respect to $(\mathrm{Gr}\,(\varphi), \mathrm{Gr}\,(\varphi))$. Thus $\pi_X \in \mathcal{V}$ and π_X is not necessarily a closed map. More examples of fiberwise absolute (neighbourhood) retracts can be found in [12].

As we have seen above, maps of the class $\mathcal{V}$ does not have to be closed. Moreover, the acyclicity of fibers is not inevitable as well.

EXAMPLE 3.3. Let $X := ([0,1/2)\times\{0\})\cup(\{1/2\}\times\{0,1\})\cup((1/2,1]\times[0,1])$, $Y := [0,1]$ and $f: X \to Y$ defined by the formula $f(x_1,x_2) := x_1$ for $(x_1,x_2) \in X$. Notice that for any $y \in Y$, $n \geq 1$, the preimage $f^{-1}(B(y,1/n))$ is contractible. Therefore $f \in \mathcal{V}$. However f is not a closed map since $f(A) := (1/2,1]$ is not closed in Y for a closed $A := (1/2,1] \times [1/3,2/3]$ in X. Moreover the fiber $f^{-1}(1/2) = \{0,1\}$ is not acyclic.

Now we consider a more general situation.

PROPOSITION 3.4. *Let $f|_G: G \to Y$ be closed map such that $f(G \cap C) = Y$.*

(a) *$f \in \mathcal{V}(G,C)$ if and only if for any $y \in Y$, $q \in \mathbb{Z}$*

$$\varinjlim\{H^q(W \cap C) \mid W \supset G \cap f^{-1}(y)\} \approx H^q(pt),$$

where W belongs to the family of all open neighbourhoods of $G \cap f^{-1}(y)$.

(b) *Let C be homologically locally connected. Assume that for any $n \geq 0$, any open neighbourhood U of $G \cap f^{-1}(y)$ contains an open neighbourhood V of $G \cap f^{-1}(y)$ such that the inclusion $V \cap C \hookrightarrow U \cap C$ is homotopy n-trivial* (2).

Then $f \in \mathcal{V}(G,C)$.

PROOF. (a) Since $f|_G$ is closed, then in view of (3.1) we get

$$\mathcal{W}(\{y\}; G) = \{W \mid W \text{ is an open neighbourhood of } G \cap f^{-1}(y)\}$$

(2) If $A \subset B$, then $A \hookrightarrow B$ is *homotopy n-trivial* provided that for any $0 \leq k \leq n$, every continuous map $f_0: S^k \to A$ admits a continuous extension $f: D^{k+1} \to B$, i.e. $f(x) = f_0(x)$ for any $x \in S^k$, where S^k and D^{k+1} stands for a unit sphere and a closed ball in $\mathbb{R}^{k+1}$.

for any $y \in Y$, which completes the proof of (a).

(b) Let $y \in Y$ and $n \in \mathbb{N}$. It is sufficient to show that for any $q \leq n$

$$\varinjlim\{H^q(W \cap C) \mid W \supset G \cap f^{-1}(y)\} \approx H^q(pt). \tag{3.2}$$

In view of the homotopy 0-triviality we obtain that for any open neighbourhood W of $G \cap f^{-1}(y)$ there is an open neighbourhood $W' \subset W$ of $G \cap f^{-1}(y)$ such that, $W' \cap C$ is connected. Hence (3.2) is proven for $q = 0$. Suppose that $q > 0$. Let W be an open neighbourhood of A. Then there are sets $A \subset W' := W_0 \subset \ldots \subset W_{n+1} \subset W$ such that the inclusion $W_q \cap C \hookrightarrow W_{q+1} \cap C$ is homotopy n-trivial for $1 \leq q \leq n$. Then the homomorphism

$$\alpha_q := H_q^{\Delta}(i_{W'\cap C}^{W\cap C}) : H_q^{\Delta}(W' \cap C) \to H_q^{\Delta}(W \cap C)$$

induced by the inclusion $i_{W'\cap C}^{W\cap C}$ on singular homology with integer coefficients is trivial for $1 \leq q \leq n$ (comp. proof of Theorem 4.1 in [9]). Similarly, we find an open subset W'' such that $A \subset W'' \subset W'$ and the homomorphism

$$\beta_q := H_q^{\Delta}(i_{W''\cap C}^{W'\cap C}) : H_q^{\Delta}(W'' \cap C) \to H_q^{\Delta}(W' \cap C)$$

is trivial for $1 \leq q \leq n$. In view of the universal coefficient theorem the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{q-1}^{\Delta}(W'' \cap C), G) \arrow[r] & H^q_{\Delta}(W'' \cap C; G) \arrow[r] & \mathrm{Hom}(H_q^{\Delta}(W'' \cap C), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{q-1}^{\Delta}(W' \cap C), G) \arrow[r, "i"] \arrow[u, "\mathrm{Ext}(\beta_{q-1})"] & H^q_{\Delta}(W' \cap C; G) \arrow[r, "j"] \arrow[u, "H^q_{\Delta}(i_{W''\cap C}^{W'\cap C})"] & \mathrm{Hom}(H_q^{\Delta}(W' \cap C), G) \arrow[r] \arrow[u, "\mathrm{Hom}(\beta_q)"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{q-1}^{\Delta}(W \cap C), G) \arrow[r] \arrow[u, "\mathrm{Ext}(\alpha_{q-1})"] & H^q_{\Delta}(W \cap C; G) \arrow[r] \arrow[u, "H^q_{\Delta}(i_{W'\cap C}^{W\cap C})"] & \mathrm{Hom}(H_q^{\Delta}(W \cap C), G) \arrow[r] \arrow[u, "\mathrm{Hom}(\alpha_q)"] & 0
\end{tikzcd}$$

is commutative with exact rows. Since $\mathrm{Hom}(\alpha_q)$ is trivial, then

$$\mathrm{Im}(H^q_{\Delta}(i_{W'\cap C}^{W\cap C})) \subset \mathrm{Ker}(j) = \mathrm{Im}(i).$$

Moreover, the homomorphism $\mathrm{Ext}(\beta_{q-1})$ is trivial and

$$\mathrm{Im}(i) \subset \mathrm{Ker}(H^q_{\Delta}(i_{W''\cap C}^{W'\cap C})).$$

Therefore the homomorphism $H^q_{\Delta}(i_{W''\cap C}^{W\cap C})$ is trivial as well. Taking into account the homologically locally connectivity of spaces $W'' \cap C$ and $W \cap C$, we obtain natural isomorphisms of groups $H^q_{\Delta}(W'' \cap C)$, $H^q(W'' \cap C)$, and of groups $H^q_{\Delta}(W \cap C)$, $H^q(W \cap C)$. Hence $H^q(i_{W''\cap C}^{W\cap C})$ is trivial. By the arbitrariness of the open neighbourhood W of $G \cap f^{-1}(y)$ we obtain that

$$\varinjlim\{H^q(W \cap C) \mid W \supset G \cap f^{-1}(y)\}$$

is trivial for $1 \leq q \leq n$, which completes the proof of (b). □

4. Vietoris–Begle type theorems

Let G and C be subsets of a space X such that G is closed in X. Let $f\colon X \to Y$ be a map such that $f(G \cap C) = Y$. For any $A \subset Y$ we define a group

$$\widetilde{H}^q_f(A) := \varinjlim\{H^q(W \cap C) \mid W \in \mathcal{W}(A;G)\}.$$

It is clear that $f \in \mathcal{V}(G,C)$ if and only if $\widetilde{H}^q_f(\{y\}) \approx H^q(pt)$.

Note that the inclusion $i^A_B\colon B \hookrightarrow A$ induces the homomorphism

$$\widetilde{H}^q_f(i^A_B)\colon \widetilde{H}^q_f(A) \to \widetilde{H}^q_f(B)$$

defined as follows:

$$\widetilde{H}^q_f(i^A_B)([h]) := [h] \in \widetilde{H}^q_f(B)$$

for $h \in H^q(W \cap C)$, $W \in \mathcal{W}(A;G)$.

If $Y = X$, $f = \mathrm{id}_X$, then $W \in \mathcal{W}(A;X)$ if and only if W is an open neighbourhood of A. In this case we omit the subindex f, i.e.

$$\widetilde{H}^q(A) := \widetilde{H}^q_{\mathrm{id}_X}(A) = \varinjlim\{H^q(W) \mid W \supset A\}.$$

Let Z be a space, $A \subset Y$, $A' \subset Z$ and $W \in \mathcal{W}(A;G)$. We say that a map $g\colon W \cap C \to Z$ is *conforming* to the pair (A, A'), if the following condition is satisfied

(4.1) for any open neighbourhood U' of A' there is $W' \in \mathcal{W}(A;G)$ such that $W' \subset W$ and $g(W' \cap C) \subset U'$.

If $g\colon W \cap C \to Z$ is conforming to a pair (A, A'), then we define a group homomorphism:

$$\widetilde{H}^q(g_{A,A'})\colon \widetilde{H}^q(A') \to \widetilde{H}^q_f(A),$$

by the formula: $\widetilde{H}^q(g_{A,A'})([h]) := [H^q(g|_{W'\cap C,U'})(h)]$ for $h \in H^q(U')$, where $A' \subset U' \subset Z$, and W' is chosen according to condition (4.1), and $H^q(g|_{W'\cap C,U'})$ is the homomorphism induced by the map $g|_{W'\cap C,U'}\colon W' \cap C \to U'$.

Lemma 4.1. *$\widetilde{H}^q(g_{A,A'})$ is a well-defined group homomorphism and does not depend on choice of neighbourhoods U', W'.*

Proof. Proof is a direct consequence of the above definition. □

Note that the map $g := f|_C : C = X \cap C \to Y$ is conforming to the pair (A, A) for any subset $A \subset Y$. Indeed, if U' is an open neighbourhood of A, then we obtain $W' := f^{-1}(U') \in \mathcal{W}(A;G)$ such that $g(W' \cap C) \subset U'$. Hence for any $q \in \mathbb{Z}$ and $A \subset Y$ we get the following homomorphism

$$\widetilde{f}_A := \widetilde{H}^q(g_{A,A}) : \widetilde{H}^q(A) \to \widetilde{H}^q_f(A).$$

Now we formulate the main result of the paper.

Theorem 4.2. *Let G and C be subsets of a space X such that G is closed in X, $f\colon X \to Y$ be a map such that $f(G \cap C) = Y$. Then $f \in \mathcal{V}(G,C)$ if and only if for any locally closed subset A of Y and, for any $q \in \mathbb{Z}$,*

$$\widetilde{f}_A : \widetilde{H}^q(A) \to \widetilde{H}^q_f(A)$$

is an isomorphism.

For the need of the proof of Theorem 4.2, we introduce an auxiliary notation: for any topological space X by $\mathrm{Top}(X)$ we denote the category of all closed subspaces of X with morphisms being inclusions $i^B_A\colon A \hookrightarrow B$, where $A \subset B$, $A, B \in \mathrm{Top}(X)$ ([3]). Moreover, for any closed subsets A of Y and K of X, we write $K \in \mathcal{W}_d(A;G)$, provided that there are closed neighbourhoods N ([4]) of A and M of $f^{-1}(N) \cap G$ such that $M \cap f^{-1}(N) \subset \mathrm{int}(K)$.

Lemma 4.3. *Let A and B be closed subsets of a space Y.*

(a) *Then the families $\mathcal{W}_d(A;G)$ and $\mathcal{W}(A;G)$ are cofinal with $\mathcal{W}_d(A;G) \cup \mathcal{W}(A;G)$.*

(b) *If $K_A \in \mathcal{W}_d(A;G)$, $K_B \in \mathcal{W}_d(B;G)$, $K_{A\cap B} \in \mathcal{W}_d(A\cap B;G)$ and $K_{A\cup B} \in \mathcal{W}_d(A \cup B;G)$, then there are sets $K'_A \in \mathcal{W}_d(A;G)$ and $K'_B \in \mathcal{W}_d(B;G)$ such that*

$$K'_A \subset K_A,\ K'_A \subset K_A,\ K'_A \cap K'_B \subset K_{A\cap B},\ K'_A \cup K'_B \subset K_{A\cup B},$$
$$K'_A \cup K'_B = \mathrm{int}_{K'_A \cup K'_B}(K'_A) \cup \mathrm{int}_{K'_A \cup K'_B}(K'_B).$$

(c) *If $\mathcal{A}$ is a discrete family of closed subsets of Y ([5]), $A_0 := \bigcup_{A\in\mathcal{A}} A$ and $K_A \in \mathcal{W}_d(A;G)$ where $A \in \mathcal{A}$, $K_0 \in \mathcal{W}_d(A_0;G)$, then there is a discrete family $\{K'_A \mid A \in \mathcal{A}\}$ such that for any $A \in \mathcal{A}$, $K'_A \in \mathcal{W}_d(A;G)$, $K'_A \subset K_A$, and $\bigcup_{A\in\mathcal{A}} K'_A \subset K_0$.*

Proof. Note that if $K \in \mathcal{W}_d(A;G)$, then $\mathrm{int}(K) \in \mathcal{W}(A;G)$. Let $W \in \mathcal{W}(A;G)$. Then there are open neighbourhoods V of A and U of $f^{-1}(V) \cap G$ such that $U \cap f^{-1}(V) \subset W$. In view of the normality of the spaces X and Y we obtain closed subsets N of A and M of $f^{-1}(N) \cap G$ such that $N \subset V$ and $M \subset U$. Let K be a closed neighbourhood of $M \cap f^{-1}(N)$ such that $K \subset W$. Thus $K \in \mathcal{W}_d(A;G)$, which completes the proof of (a).

Statements (b) and (c) follows from the normality of X and the paracompactness of Y, analogously as in the proof of [18, Theorem 3.1]). □

([3]) $\mathrm{Top}(X)$ denotes the family of all closed subsets of X, too.

([4]) N is a closed neighbourhood of A if N is closed and $A \subset \mathrm{int}(N)$.

([5]) A family $\mathcal{A}$ of closed subsets of X is *discrete*, if $\mathcal{A}$ consists of sets pairwise disjoint such that the sum of members of any subfamily of $\mathcal{A}$ is closed, or equivalently, for any $x \in X$ there is an open neighbourhood W of x such that the family $\{A \in \mathcal{A} \mid A \cap W \neq \emptyset\}$ is finite.

PROOF OF THEOREM 4.2. If $\widetilde{f}_A\colon \widetilde{H}^q(A) \to \widetilde{H}^q_f(A)$ is an isomorphism for any $A = \{y\}$, $y \in Y$, then $\widetilde{H}^q_f(\{y\}) \approx \widetilde{H}^q(\{y\}) \approx H^q(\{y\})$, since $\{y\}$ is taut in Y with respect H^*. Therefore $f \in \mathcal{V}(G, C)$.

Now suppose that $f \in \mathcal{V}(G, C)$. In view of Lemma 4.3(a), for any $A \in \mathrm{Top}(Y)$, $q \in \mathbb{Z}$ we obtain that

$$\begin{aligned} \widetilde{H}^q_f(A) &\approx \varinjlim\{H^q(K \cap C) \mid K \in \mathcal{W}_d(A; G) \cup \mathcal{W}(A; G)\} \\ &\approx \varinjlim\{H^q(K \cap C) \mid K \in \mathcal{W}_d(A; G)\}. \end{aligned}$$

Therefore we may identify

$$\widetilde{H}^q_f(A) = \varinjlim\{H^q(K \cap C) \mid K \in \mathcal{W}_d(A; G)\}. \tag{4.2}$$

Similarly, taking into account the normality of Y we identify

$$\widetilde{H}^q(A) = \varinjlim\{H^q(N) \mid A \subset \mathrm{int}(N),\ N \in \mathrm{Top}(Y)\}. \tag{4.3}$$

We prove that $\widetilde{H}$, $\widetilde{H}_f$ are additive cohomology theories on space Y in the sense of Lawson [10]. We provide argumentations for $\widetilde{H}_f$. Note that if $A, B \in \mathrm{Top}(Y)$ such that $B \subset A$, the inclusion map $i^A_B\colon B \hookrightarrow A$ induces the group homomorphism $\widetilde{H^q_f}(i^A_B)\colon \widetilde{H}^q_f(A) \to \widetilde{H}^q_f(B)$ defined by the formula

$$\widetilde{H^q_f}(i^A_B)([h]) := [h] \in \widetilde{H}^q_f(B)$$

for $h \in H^q(K \cap C)$, $K \in \mathcal{W}_d(A; G) \subset \mathcal{W}_d(B; G)$. Thus it is easy to check that $\widetilde{H_f} := \{\widetilde{H}^q_f\}$ is a contravariant functor on category $\mathrm{Top}(Y)$ to the category of graded abelian groups. Let A and B be closed subsets of Y. In view of Lemma 4.3(b) the family of sets of the form $(K'_A, K'_B, K'_A \cap K'_B, K'_A \cup K'_B)$ is cofinal with the family $\mathcal{W}_d(A; G) \times \mathcal{W}_d(B; G) \times \mathcal{W}_d(A \cap B; G) \times \mathcal{W}_d(A \cup B; G)$, where the directed relation is the relation of (inverse) inclusion of sets. Moreover, the following sets $C_A := K'_A \cap C$, $C_B := K'_B \cap C$ are closed in C and $C_A \cup C_B = \mathrm{int}_{C_A \cup C_B} C_A \cup \mathrm{int}_{C_A \cup C_B} C_B$. Hence we obtain the Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots \xrightarrow{\Delta} H^q(C_A \cup C_B) \xrightarrow{J} H^q(C_A) \oplus H^q(C_B) \xrightarrow{I} \\ \longrightarrow H^q(C_A \cap C_B) \xrightarrow{\Delta} H^{q+1}(C_A \cup C_B) \longrightarrow \cdots \end{aligned}$$

Passing to the direct limit we get the following exact sequence

$$\cdots \xrightarrow{\widetilde{\Delta}_f} \widetilde{H}^q_f(A \cup B) \xrightarrow{\widetilde{J}_f} \widetilde{H}^q_f(A) \oplus \widetilde{H}^q_f(B) \xrightarrow{\widetilde{I}_f} \widetilde{H}^q_f(A \cap B) \xrightarrow{\widetilde{\Delta}_f} \widetilde{H}^{q+1}_f(A \cup B) \longrightarrow \cdots$$

where $\widetilde{\Delta}_f$, $\widetilde{J}_f$, $\widetilde{I}_f$ are appropriate homomorphisms. Therefore $\widetilde{H}_f$ is a cohomology theory on Y (see [10]).

Let $\mathcal{A}$ be a discrete family of closed subsets of Y, $A_0 := \bigcup_{A\in\mathcal{A}} A$. Then by Lemma 4.3(c) we get a family $\mathcal{K}$ of sets of the form $\Big((K'_A \mid A \in \mathcal{A}), \bigcup_{A\in\mathcal{A}} K'_A\Big)$ cofinal with $\Big(\prod_{A\in\mathcal{A}} \mathcal{W}_d(A;G)\Big) \times \mathcal{W}_d(A_0;G)$. Moreover, the family $\{K'_A \cap C \mid A \in \mathcal{A}\}$ is discrete family of subsets of the topological space C. In view of the additivity of cohomology theory H^* on C and by the commutativity of the direct limit with the Cartesian product we obtain:

$$\begin{aligned}
\widetilde{H}^q_f(A_0) &= \varinjlim\{H^q(K\cap C) \mid K \in \mathcal{W}_d(A_0;G)\}\\
&\approx \varinjlim\left\{H^q(K\cap C) \mid ((K_A \mid A\in\mathcal{A}),K) \in \left(\prod_{A\in\mathcal{A}}\mathcal{W}_d(A;G)\right)\times\mathcal{W}_d(A_0;G)\right\}\\
&\approx \varinjlim\left\{H^q\left(\left(\bigcup_{A\in\mathcal{A}}K'_A\right)\cap C\right)\,\middle|\,\left((K'_A \mid A\in\mathcal{A}),\ \bigcup_{A\in\mathcal{A}}K'_A\right)\in\mathcal{K}\right\}\\
&\approx \varinjlim\left\{\prod_{A\in\mathcal{A}}H^q(K'_A\cap C)\,\middle|\,\left((K'_A \mid A\in\mathcal{A}),\ \bigcup_{A\in\mathcal{A}}K'_A\right)\in\mathcal{K}\right\}\\
&\approx \prod_{A\in\mathcal{A}}\varinjlim\left\{H^q(K'_A\cap C)\,\middle|\,\left((K'_A \mid A\in\mathcal{A}),\ \bigcup_{A\in\mathcal{A}}K'_A\right)\in\mathcal{K}\right\}\\
&\approx \prod_{A\in\mathcal{A}}\varinjlim\{H^q(K_A\cap C)\mid K_A\in\mathcal{W}_d(A;G)\} = \prod_{A\in\mathcal{A}}\widetilde{H}^q_f(A).
\end{aligned}$$

Therefore $\widetilde{H}_f$ is additive.

Now we show that any closed subset A of Y is taut in Y with respect to $\widetilde{H}_f$. Let $A \in \mathrm{Top}(Y)$. We show that the homomorphism

$$i\colon \varinjlim\{\widetilde{H}^q_f(L) \mid A \subset \mathrm{int}(L),\ L \in \mathrm{Top}(X)\} \to \widetilde{H}^q_f(A)$$

defined as follows: $i([h]) := \widetilde{H}^q_f(i^L_A)(h) = h|_A$ for $h \in \widetilde{H}^q_f(L)$, is a isomorphism for any $q \in \mathbb{Z}$. Let $h \in \widetilde{H}^q_f(A)$. Then there is $g \in H^q(K \cap C)$ such that $h = [g]$ and $K \in \mathcal{W}_d(A;G)$. Thus there are closed neighbourhoods N of A and M of $f^{-1}(N) \cap G$ such that $M \cap f^{-1}(N) \subset \mathrm{int}(K)$. Let L be a closed neighbourhood of A such that $L \subset \mathrm{int}(N)$. Then N is a closed neighbourhood of L and $K \in \mathcal{W}_d(L;G)$. Hence $[g] \in \widetilde{H}^q_f(L)$, which proves that i is an epimorphism.

Let $h \in \widetilde{H}^q_f(L)$ and $i([h]) = 0$, where L is a closed neighbourhood of A in Y. Then there is $g \in H^q(K \cap C)$ such that $h = [g]$, where $K \in \mathcal{W}_d(L;G)$ and N, M are closed neighbourhoods of L, $f^{-1}(N) \cap G$, respectively. It is clear that $i([h]) = [g] = 0 \in \widetilde{H}^q_f(A)$. Hence there is $K' \in \mathcal{W}_d(A;G)$ such that $K' \subset K$ and $g|_{K'\cap C} = 0$. We find a closed neighbourhood $L' \subset L$ of A such that $K' \in \mathcal{W}_d(L';G)$. Hence $[g] = 0 \in \widetilde{H}^q_f(L')$ and $[h] = 0 \in \varinjlim\{\widetilde{H}^q_f(L) \mid A \subset \mathrm{int}(L), L \in \mathrm{Top}(X)\}$, which completes the proof of the tautness of A.

Similarly as above, we obtain that $\widetilde{H}$ together with the homomorphism $\widetilde{\Delta}$ defined analogously, is additive and any closed subset of Y is taut in Y with respect to $\widetilde{H}$.

Now we show that there is an isomorphism $\tau\colon(\widetilde{H},\widetilde{\Delta})\to(\widetilde{H}_f,\widetilde{\Delta}_f)$ of cohomology theories. Let $A\in\mathrm{Top}(Y)$, then $\tau^q_A\colon\widetilde{H}^q(A)\to\widetilde{H}^q_f(A)$ defined by the formula:

$$\tau^q_A([h]) := [H^q(f|_{(f^{-1}(N)\cap C,N)})(h)]$$

for any $q\in\mathbb{Z}$, $h\in H^q(N)$, where N is a closed neighbourhood of A. Then τ^q_A is a group homomorphism for any $q\in\mathbb{Z}$.

Let $A,B\in\mathrm{Top}(Y)$, $A\subset B$. Then for any $q\in\mathbb{Z}$, $h\in H^q(N)$, where N is a closed neighbourhood of B, we get

$$\widetilde{H}^q_f(i^B_A)\circ\tau^q_B([h]) = [H^q(f|_{(f^{-1}(N)\cap C,N)})(h)] = \tau^q_A\circ\widetilde{H}^q(i^B_A)([h]),$$

which means that τ is a natural transformation of functors $\widetilde{H}$ and $\widetilde{H}_f$. If $A\cup B=\mathrm{int}_{A\cup B}(A)\cup\mathrm{int}_{A\cup B}(B)$, then we easily show that

$$\widetilde{\Delta}_f\circ\tau^q_{A\cap B}=\tau^{q+1}_{A\cup B}\circ\widetilde{\Delta}.$$

Then $\tau\colon(\widetilde{H},\widetilde{\Delta})\to(\widetilde{H}_f,\widetilde{\Delta}_f)$ is a homomorphism of cohomology theories.

In view of (3.2) we get for any $y\in Y$ and $q\in\mathbb{Z}$

$$\widetilde{H}^q_f(\{y\})\approx H^q(pt).$$

Moreover, taking into account the tautness of $\{y\}$ in Y with respect H^* we obtain group isomorphisms:

$$\widetilde{H}^q(\{y\})\approx H^q(pt)\approx\widetilde{H}^q_f(\{y\})$$

for any $y\in Y$ and $q\in\mathbb{Z}$. Hence

$$\tau^q_{\{y\}}\colon\widetilde{H}^q(\{y\})\to\widetilde{H}^q_f(\{y\})$$

is an isomorphism for any $y\in Y$ and $q\in\mathbb{Z}$. In view of 10, Theorem 3.2

$$\tau^q_A\colon\widetilde{H}^q(A)\to\widetilde{H}^q_f(A)$$

is an isomorphism for any $q\in\mathbb{Z}$ and $A\in\mathrm{Top}(Y)$. Taking into account the identifications (4.2) and (4.3) observe that $\widetilde{f}_A\colon\widetilde{H}^q(A)\to\widetilde{H}^q_f(A)$ is an isomorphism.

Now let A be locally closed subset in A and $f\in\mathcal{V}(G,C)$. Then A is closed in some open neighbourhood $W\subset Y$ of itself. Let $X_W:=f^{-1}(W)$, $G_W:=G\cap X_W$, $C_W:=C\cap X_W$, $Y_W:=W$, $f_W:=f|_{(X_W,Y_W)}\colon X_W\to Y_W$. Then we may consider the cohomology theories $\widetilde{H}$ and $\widetilde{H}_{f_W}$ defined on Y_W. Then $f_W\in\mathcal{V}(G_W,C_W)$ and $\widetilde{f}_A\colon\widetilde{H}^q(A)\to\widetilde{H}^q_{f_W}(A)=\widetilde{H}^q_f(A)$ is an isomorphism for any $q\in\mathbb{Z}$, which completes the proof of the theorem. $\square$

Remark 4.4. (a) Note that Theorem 4.2 is also valid if we consider singular cohomology theory H^*_Δ instead of Alexander–Spanier cohomology theory H^* theory provided that Y is homologically locally connected (e.g. Y is an absolute neighbourhood retract). Indeed in the proof of Theorem 4.2 we used only the pointwise tautness of H^* in Y, i.e. any singleton $\{y\}$ is taut in Y and by ([17]) singular cohomology theory is pointwise taut in a homologically locally connected space.

(b) For any subsets A, B of Y, $B \subset A$ and for any $q \in \mathbb{Z}$ we can define the homomorphism induced by f in the relative case

$$\widetilde{f}_{(A,B)} := \widetilde{H}^q(f|_{(A,B),(A,B)})\colon \widetilde{H}^q(A,B) \to \widetilde{H}^q_f(A,B)$$

similarly as above. Moreover, in view of Theorem 4.2 and by the standard application of long exact sequences of pairs of spaces and the five lemma, we obtain that if $f \in \mathcal{V}(G,C)$, A, B are locally closed, then $\widetilde{f}_{(A,B)}$ is an isomorphism.

Now we are ready to prove the following generalization of the Vietoris–Begle Theorem (see Proposition 3.2(b)).

Theorem 4.5. *If $f \in \mathcal{V}$, then $H^q(f)\colon H^q(Y) \to H^q(X)$ is an isomorphism for any $q \in \mathbb{Z}$.*

Proof. If $f \in \mathcal{V}$, then assuming that $G = C = X$, in view of Theorem 4.2

$$\widetilde{f}_Y\colon \widetilde{H}^q(Y) \to \widetilde{H}^q_f(Y)$$

is an isomorphism for any $q \in \mathbb{Z}$. In order to complete the proof it is sufficient to note that we may identify $\widetilde{H}^q(Y) = H^q(Y)$, $\widetilde{H}^q_f(Y) = H^q(X)$ and $\widetilde{f}_Y = H^q(f)$.□

Remark 4.6. (a) Let $f\colon X \to Y$ be a surjection. Then the above theorem can be expressed more general: $f \in \mathcal{V}$ *if and only if* $H^q(f|_{f^{-1}(V)}) : H^q(V) \to H^q(f^{-1}(V))$ *is an isomorphism for any open subset* $V \subset Y$ *and* $q \in \mathbb{Z}$. Indeed, if $f \in \mathcal{V}$, then $f|_{f^{-1}(V)} \in \mathcal{V}$ for any open $V \subset Y$. Then in view of Theorem 4.5, $H^q(f)\colon H^q(V) \to H^q(f^{-1}(V))$ is an isomorphism for any $q \in \mathbb{Z}$. In the other hand, let V be an open neighbourhood of $y \in Y$. Then $H^q(f|_{f^{-1}(V)})\colon H^q(V) \to H^q(f^{-1}(V))$ is an isomorphism and hence

$$\varinjlim\{H^q(V) \mid V \ni y\} \approx \varinjlim\{H^q(f^{-1}(V)) \mid V \ni y\},$$

is an isomorphism for any $y \in Y$. In view of the tautness of $\{y\}$ in Y with respect to H^* we obtain that $f \in \mathcal{V}$.

(b) In general, if $f \in \mathcal{V}$ and A is a closed subset of Y, then

$$H^q(f|_{f^{-1}(A)})\colon H^q(A) \to H^q(f^{-1}(A))$$

does not have to be an isomorphism.

Recall Example 3.3 and note that $H^q(f|_{f^{-1}(A)})$ is not an isomorphism provided that $A = [0,1/2]$.

(c) In view of Remark 4.4, the statement of Theorem 4.5 is still valid if we consider singular cohomology H^*_Δ instead of H^* provided that Y is homologically locally connected.

Theorem 4.7. *Let $f \in \mathcal{V}(G,C)$ such that $f|_G$ is a closed map. Then, for any $q \in \mathbb{Z}$,*

$$\widetilde{f}\colon H^q(Y) \to \varinjlim\{H^q(W \cap C) \mid W \text{ is an open neighbourhood of } G\}$$

is an isomorphism defined by the formula:

$$\widetilde{f}(h) := [H^q(f)(h)|_C] \quad \text{for any } h \in H^q(Y).$$

Proof. The map $f|_G$ is closed, then in view of (3.1) we obtain that

$$\mathcal{W}(Y;G) = \{W \mid W \text{ is an open neighbourhood of } G\}.$$

Hence, identifying $\widetilde{H}^q(Y) = H^q(Y)$ and by Theorem 4.2 we obtain that

$$\widetilde{f} = \widetilde{f}_Y : \widetilde{H}^q(Y) \to \widetilde{H}^q_f(Y) = \varinjlim\{H^q(W \cap C) \mid W \supset G\}$$

is an isomorphism. □

Note that if $f \in \mathcal{V}(G,C)$ such that $f|_G$ is a closed map, then in general

$$H^q(f|_{G\cap C})\colon H^q(Y) \to H^q(G \cap C)$$

is not an isomorphism for $q \in \mathbb{Z}$ (see the following example).

Example 4.8. Let $X := [0,1] \times [0,+\infty)$, $C := [0,1/2) \times [0,+\infty) \cup [1/2,1] \times [1,+\infty)$, $G := [0,1/2) \times \{0\} \cup [1/2,1] \times [0,1]$, $f\colon X \to Y := [0,1]$ be defined by the formula: $f(x,y) = x$ for $(x,y) \in X$. Then $G \cap C = [0,1/2) \times \{0\} \cup [1/2,1) \times \{1\}$, f is surjective, $f|_G$ is closed and $f \in \mathcal{V}(G,C)$ (see Proposition 3.4(a)). In view of Theorem 4.7, $\widetilde{f}$ is an isomorphism. However $H^q(f|_{G\cap C})\colon H^q(Y) \to H^q(G\cap C)$ is not an isomorphism for any $q \geq 0$.

Remark 4.9. We may consider the following class of maps: $f \in \mathcal{V}_n(G,C)$, if $f(G \cap C) = Y$ and for any $y \in Y$, $q = 0, \dots, n$

$$\varinjlim\{H^q(W \cap C) \mid W \in \mathcal{W}(\{y\};G)\} \approx H^q(pt)$$

is an isomorphism. Then using the similar arguments of the paper it is possible to prove the following generalization of Theorem 4.2: $f \in \mathcal{V}_n(G,C)$ *if and only if for any locally closed subset A of Y*

$$\widetilde{f}_A\colon \widetilde{H}^q(A) \to \widetilde{H}^q_f(A)$$

is an isomorphism for $q = 0, \dots, n$ and monomorphism for $q = n+1$.

Now we present a version of the Vietoris-Begle theorem for not necessarily closed maps over a space B.

Theorem 4.10. *Let X, Y, B be spaces. Assume that that $p\colon X \to B$, $q\colon Y \to B$ are closed surjections and $f\colon X \to Y$ is a map over B (i.e. $q(f(x)) = p(x)$ for any $x \in X$) such that*

$$H^k(f|_{p^{-1}(b),q^{-1}(b)})\colon H^k(q^{-1}(b)) \to H^k(p^{-1}(b))$$

is an isomorphism for any $k \in \mathbb{Z}$ and $b \in B$. Then

$$H^k(f)\colon H^k(Y) \to H^k(X)$$

is an isomorphism for any $k \in \mathbb{Z}$.

Proof. Similarly as in proof of Theorem 4.2, for any $A \in \mathrm{Top}(B)$, $k \in \mathbb{Z}$

$$\begin{aligned}\widetilde{H}^k_p(A) &= \varinjlim\{H^k(p^{-1}(N)) \mid A \subset \mathrm{int}(N),\ N \in \mathrm{Top}(B)\},\\ \widetilde{H}^k_q(A) &= \varinjlim\{H^k(q^{-1}(N)) \mid A \subset \mathrm{int}(N),\ N \in \mathrm{Top}(B)\},\end{aligned}$$

and it can be shown that $\widetilde{H}_p$, $\widetilde{H}_q$ are additive cohomology theories on space B and any closed subset of B is taut with respect to $\widetilde{H}_p$, $\widetilde{H}_q$. Note that f induces the homomorphism of cohomology theories $\tau\colon(\widetilde{H}_q,\Delta) \to (\widetilde{H}_p,\Delta)$ for any $A \in \mathrm{Top}(B)$, $k \in \mathbb{Z}$

$$\tau^k_A([h]) := [H^k(f|_{p^{-1}(N),q^{-1}(N)})(h)],$$

where $h \in H^k(q^{-1}(N))$, $A \subset \mathrm{int}(N)$, $N \in \mathrm{Top}(B)$. Moreover, $\tau^k_{\{b\}}$ is an isomorphism for any $b \in B$ and $k \in \mathbb{Z}$. Hence in view of [10, Theorem 3.2] we easily get the isomorphism

$$H^k(f) = \tau^k_B\colon H^k(Y) = \widetilde{H}^k_q(B) \to H^k(X) = \widetilde{H}^k_p(B)$$

for any $k \in \mathbb{Z}$. □

Note that the composition of closed surjections is a closed surjection. Hence Theorem 4.10 is a generalization of the following Biaynicki–Birula result (see [2], [16], [7]): *If $f\colon X \to Y$ and $g\colon Y \to Z$ are closed surjections and*

$$H^k(f|_{f^{-1}(g^{-1}(z))})\colon H^k(g^{-1}(z)) \to H^k(f^{-1}(g^{-1}(z)))$$

is an isomorphism for any $z \in Z$ and $k \in \mathbb{Z}$, then $H^k(f)\colon H^k(Y) \to H^k(X)$ is an isomorphism for any $k \in \mathbb{Z}$.

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Manuscript received May 16, 2011

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