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STRONG ORBIT EQUIVALENCE AND RESIDUALITY

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Abstract. We consider a class of minimal Cantor systems that up to conjugacy contains all systems strong orbit equivalent to a given system. We define a metric on this strong orbit equivalence class and prove several properties about the resulting metric space including that the space is complete and separable but not compact. If the strong orbit equivalence class contains a finite rank system, we show that the set of finite rank systems is residual in the metric space. The final result shown is that the set of systems with zero entropy is residual in every strong orbit equivalence class of this type.

1. Introduction

The contents of this paper deal primarily with strong orbit equivalence classes of minimal Cantor systems. In the measure-theoretic category, Dye's Theorem states that any two ergodic measure-preserving transformations on nonatomic probability spaces are orbit equivalent. In [8], D.J. Rudolph introduced the idea of restricted orbit equivalence. By defining a notion of the size of an orbit equivalence, Rudolph gave a natural way to more precisely distinguish between measure-theoretic systems. In the topological category, even within the category of minimal Cantor systems, there are several nontrivial systems which are not orbit equivalent. However, serving the same purpose as Rudolph's restricted orbit equivalence in the measure-theoretic setting, strong orbit equivalence provides a more precise way to distinguish between topological systems. Strong orbit

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equivalence was first introduced by T. Giordano, I.F. Putnam and C. Skau in [4] where they proved the following theorem:

Theorem 1.1. *Two minimal Cantor systems are strongly orbit equivalent if and only if their associated dimension groups are order isomorphic by an order isomorphism preserving the distinguished order unit.*

In [6], C. Hochman considered a metric on the space of homeomorphisms of the Cantor set and proved several genericity results about the metric space. In particular, Hochman showed that the universal odometer is residual in the space of transitive systems. Along the same lines, we define a metric on a strong orbit equivalence class of minimal homeomorphisms of a Cantor space. We prove several properties about the resulting metric space including that it is complete and separable but not compact. These results are also related to the work done in [1] where S. Bezuglyi, A.H. Dooley and J. Kwiatkowski considered several different topologies on the space of homeomorphisms of the Cantor set. We go on to show that finite rank systems, as defined in [3] by T. Downarowicz and A. Maass, are residual in any strong orbit equivalence class containing a finite rank system. In particular, we show that odometers are residual in any class containing an odometer. Finally, we show that systems with zero entropy are residual in the strong orbit equivalence class of any minimal Cantor system. These residuality results are related to the measure-theoretic results of D.J. Rudolph found in [9]. To help the reader understand the results in this paper, we begin by introducing much of the needed background information.

2. Background

A *Cantor space* is a nonempty topological space that is perfect, compact, totally disconnected, and metrizable. It is well known that any two such spaces are homeomorphic. A *minimal Cantor system* is an ordered pair (X, T) where X is a Cantor space and $T: X \to X$ is a minimal homeomorphism. The minimality of T means that every *orbit* under T is dense in X, i.e. if for $x \in X$ we define $\mathcal{O}_T(x) = \{T^k x \mid k \in \mathbb{Z}\}$, then for all $x \in X$, $\mathcal{O}_T(x)$ is dense in X. Because X is metrizable, we can define a metric on X that induces the topology of X. We will denote this metric by d_X.

2.1. Notions of equivalence in minimal Cantor systems. There are three notions of equivalence between minimal Cantor systems that we will consider. The strongest notion of equivalence is conjugacy. Two minimal Cantor systems (X, T) and (Y, S) are *conjugate* if there exists a homeomorphism $h: X \to Y$ such that $h \circ T = S \circ h$. A weaker notion of equivalence is orbit equivalence. Two systems (X, T) and (Y, S) are *orbit equivalent* if there exists a homeomorphism $h: X \to Y$ that preserves orbits between the systems. Stated more explicitly, a homeomorphism $h: X \to Y$ is an orbit equivalence if there exist functions $a, b: X \to \mathbb{Z}$ such that for all $x \in X$, $h \circ T(x) = S^{a(x)} \circ h(x)$ and $h \circ T^{b(x)}(x) = S \circ h(x)$. We call a and b the *orbit cocycles* associated to h. If the orbit cocycles associated to h each have at most one point of discontinuity, we say the systems (X, T) and (Y, S) are *strongly orbit equivalent*.

2.2. Tower partitions. Tower partitions provide a visual representation of minimal Cantor systems. Let (X, T) be a minimal Cantor system and let $A \subset X$ be clopen. Then because T is minimal, each $a \in A$ returns to A in a finite number of T-iterations. This allows us to define a function $r_A: A \to \mathbb{N}^+$ where $r_A(a) = \min\{n \geq 1 \mid T^n a \in A\}$. It is easily verified that r_A is a continuous function, and we say that $r_A(a)$ is the *return time* of a to A. Because A is compact, r_A takes on only finitely many values. Therefore, we can partition A into finitely many clopen sets $A_1, \dots, A_k$ such that the return time to A is constant on each A_j, $j = 1, \dots, k$. For $j = 1, \dots, k$, let r_j denote the return time of A_j to A. For each j, we construct a *tower* over A_j by vertically stacking the sets $A_j, TA_j, \dots, T^{r_j - 1}A_j$, which we will call the *floors* of the tower over A_j. An example with A partitioned into three sets A_1, A_2 and A_3 with return times of 4, 3 and 5, respectively, is shown in Figure 1.

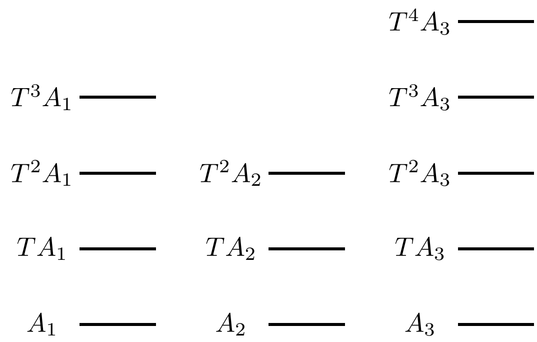

Figure 1. Tower partition

We define the *height of the tower* over A_j to be r_j, the return time of A_j to A. If $0 \leq i \leq r_{j-1}$, we will say that the *height of the tower floor* $T^i(A_j)$ is i. The floors of these towers create a clopen partition of X, and we will call this a *tower partition* of (X, T) over A. If $\mathcal{P}$ is a tower partition of (X, T) over A, notice that the bottom floors of $\mathcal{P}$ partition A. We will denote this partition of A by $\mathcal{P}(A)$. Also notice that the top floors of $\mathcal{P}$ partition the set $T^{-1}(A)$. An important property of a tower partition that we will consider is the minimum height of a tower in the partition. If $\mathcal{P}$ is a tower partition, we will let $\mathcal{H}(\mathcal{P})$ denote the minimum height of a tower in $\mathcal{P}$. For example, if $\mathcal{P}$ is the tower partition shown in Figure 1, then $\mathcal{H}(\mathcal{P}) = 3$.

Let $\{A_n\}$ be a sequence of clopen sets in X such that $A_{n+1} \subset A_n$ for all n. For every n, let $\mathcal{P}_n$ be a tower partition of (X, T) over A_n such that for all $n \geq 1$ the tower partition $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$. We say that the tower partition sequence $\{\mathcal{P}_n\}$ *generates the topology* of X if for any clopen set $C \subset X$, there exists an $N > 0$ such that if $n \geq N$, then C can be written as a finite union of sets in $\mathcal{P}_n$. Suppose the sequence $\{\mathcal{P}_n\}$ generates the topology of X and in addition $\operatorname{diam}(A_n) \to 0$. Then $\bigcap A_n = \{x_1\}$ for some $x_1 \in X$, so we will say that $\{\mathcal{P}_n\}$ is a *generating sequence of tower partitions over* x_1.

Proposition 2.1. *If $\{\mathcal{P}_n\}$ is a sequence of finite clopen partitions of a Cantor space X, then $\{\mathcal{P}_n\}$ generates the topology of X if and only if* $\operatorname{diam}(\mathcal{P}_n) \to 0$.

Proof. This follows from the properties of clopen sets and compactness. $\square$

Proposition 2.2. *Let (X, T) be a minimal Cantor system and let $x_1 \in X$. If $\{\mathcal{P}_n\}$ is a generating sequence of tower partitions over x_1, then $\mathcal{H}(\mathcal{P}_n) \to \infty$.*

Proof. For all n, let A_n be the clopen set in X such that $\mathcal{P}_n$ is a tower partition over A_n, so $\bigcap A_n = \{x_1\}$. Fix $k \in \mathbb{N}^+$ and let B be the clopen set in $\mathcal{P}_k$ with $x_1 \in B$. Since $\operatorname{diam}(A_n) \to 0$, there exists an $N > 0$ such that if $n \geq N$, then $A_n \subset B$. Then for $n \geq N$, the tower height of every tower in $\mathcal{P}_n$ is greater than or equal to the tower height of the tower over B in $\mathcal{P}_k$. Therefore, if we let $\mathcal{P}_n(x_1)$ denote the tower of $\mathcal{P}_n$ that contains x_1, it suffices to show that the height of $\mathcal{P}_n(x_1)$ grows arbitrarily large as $n \to \infty$. The height of the tower $\mathcal{P}_n(x_1)$ is the return time of x_1 to A_n, which we will denote r_n. Because $r_n \leq r_m$ for all $n \leq m$, it suffices to show that for all $n \in \mathbb{N}^+$, there exists an $m > n$ such that $r_m > r_n$. Fix $n \in \mathbb{N}^+$ and let $T^{r_n}(x_1) = y_1 \in A_n$. Let $d_X(x_1, y_1) = p > 0$. Then pick $m > n$ such that $\operatorname{diam}(A_m) < p$. Because $d_X(x_1, y_1) = p$, $T^{r_n}(x_1) = y_1 \notin A_m$, so $r_m \neq r_n$. We must have that $r_m > r_n$ finishing the proof. $\square$

2.3. Bratteli diagrams. Ordered Bratteli diagrams give us another way to visually represent minimal Cantor systems. We refer the reader to [5] for a complete discussion of this topic. A *Bratteli diagram* $B = (V, E)$ consists of a vertex set V and an edge set E, where V and E can be written as the countable union of finite disjoint sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \ldots \quad \text{and} \quad E = E_1 \cup E_2 \cup \ldots$$

where V_k represents the set of vertices at level k and E_k represents the set of edges between V_{k-1} and V_k. The ordering on a Bratteli diagram refers to an ordering on the set of edges that terminate at the same vertex. The first three levels of an ordered Bratteli diagram are shown in Figure 2.

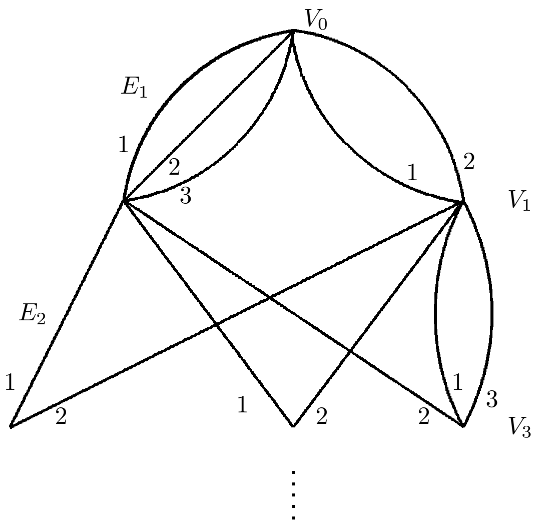

FIGURE 2. Ordered Bratteli diagram

2.4. Telescoping. Given a Bratteli diagram, we can create a new Bratteli diagram by a process called telescoping. Let $B = (V, E, \leq)$ be an ordered Bratteli diagram and remove $E_{k+1}, \ldots, E_l$ and $V_{k+1}, \ldots, V_{l-1}$. We then reconnect V_k and V_l by single edges, one edge for each of the paths between V_k and V_l, beginning and ending at their corresponding source and range, respectively. Ordering these new edges using the reverse lexicographical ordering, we refer this new ordered diagram as a *telescoping* between levels k and l. A telescoping between two levels of a Bratteli diagram is shown in Figure 3.

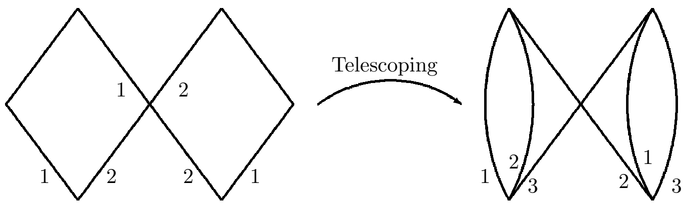

FIGURE 3. Telescoping of a Bratteli diagram

Let $\{n_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{N}$ with $n_0 = 0$ and $n_k < n_{k+1}$ for all k. If we telescope B between levels n_k and n_{k+1} for all k ordering the edges as described above, we have a new ordered Bratteli diagram $B' = (V', E', \leq')$. We say that B' is a *telescoping* of B. We say that the telescoping is finite if only a finite number of levels are telescoped.

2.5. Dimension groups. For a Bratteli diagram $B = (V, E)$, let $V_k = \{v(k, j) \mid 1 \leq j \leq |V_k|\}$. For each k, we define the incidence matrix $M_k = [m_{ij}]$,

$i = 1, \dots, |V_k|$, $j = 1, \dots, |V_{k+1}|$, where m_{ij} is the number of edges between the vertices $v(k,i)$ and $v(k+1,j)$. We can associate a dimension group $K_0(V,E)$ to the Bratteli diagram by taking the inductive limit of groups $\varinjlim(\mathbb{Z}^{|V_k|}, M_k)$. This can be made into an ordered group by declaring that $[v] \in K_0(V,E)^+$ if there is a $w \in [v]$ such that each coordinate of w is non-negative. We distinguish an order unit in $K_0(V,E)$ as the element associated to $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$.

2.6. Bratteli diagrams and minimal Cantor systems.

DEFINITION 2.3. An ordered Bratteli diagram $B = (V, E, \leq)$ is *properly ordered* if

(a) there is a telescoping (not necessarily finite) B' of B such that any two vertices at consecutive levels in B' are connected by an edge;
(b) there are unique infinite edge paths $x_{\max}$ and $x_{\min}$ in B such that each edge of $x_{\max}$ is maximal in $\leq$ and each edge of $x_{\min}$ is minimal in $\leq$.

Given a properly ordered Bratteli diagram $B = (V, E, \leq)$, we let X_B be the set of all infinite paths in B. We topologize X_B by letting the family of cylinder sets be a basis for the topology. A *cylinder set* is the set of paths that begin with a given finite edge path. We will let $[e_1, \dots, e_k]$ represent the cylinder set $\{(x_1, x_2, \dots) \in X_B \mid x_i = e_i \text{ for all } i \leq k\}$. The space X_B along with this topology is a Cantor space. We define the *Vershik map* $V_B \colon X_B \to X_B$ in the following way. If $x = (x_1, x_2, \dots) \in X_B \setminus \{x_{\max}\}$, there is smallest k such that x_k is not maximal. If we let y_k be the successor of x_k and let $(y_1, \dots, y_{k-1})$ be the minimal path from v_0 to the source of y_k, we define $V_B(x) = (y_1, \dots, y_k, x_{k+1}, x_{k+2}, \dots)$. The tails of x and $V_B(x)$ agree past level k, so we say they are *cofinal*. We define $V_B(x_{\max}) = x_{\min}$. The system (X_B, V_B) is a minimal Cantor system and we refer to it as a *Bratteli–Vershik system*. It is shown in [5] that any minimal Cantor system is conjugate to a Bratteli–Vershik system.

3. Results

We will now define a class of minimal Cantor systems that up to conjugacy contains every system strongly orbit equivalent to a given system. We will then define a metric on this strong orbit equivalence class and prove several properties about the metric space. In particular, we will prove some results about residuality in this metric space.

3.1. Definition of $\mathcal{S}(T, x_0)$. If (X, T) is a minimal Cantor system, we define the *future orbit* of x under T, $\mathcal{O}_T^+(x) = \{T^k(x) \mid k \geq 0\}$ and the *past orbit* of x under T, $\mathcal{O}_T^-(x) = \{T^{-k}(x) \mid k > 0\}$. It is easily verified that for all $x \in X$, both sets $\mathcal{O}_T^+(x)$ and $\mathcal{O}_T^-(x)$ are dense in X. If (X, T) and (Y, S) are strongly orbit equivalent minimal Cantor systems with $x_0 \in X$ and $y_0 \in Y$, we will say that $h\colon X \to Y$ is a *pointed strong orbit equivalence* between (X, T, x_0) and (Y, S, y_0) if it is a strong orbit equivalence satisfying the following conditions:

(1) $h(x_0) = y_0$;
(2) $h(Tx_0) = Sy_0$;
(3) the cocycles of h are continuous on $X \setminus \{x_0\}$;
(4) $h(\mathcal{O}_T^-(x_0)) = \mathcal{O}_S^-(y_0)$;
(5) $h(\mathcal{O}_T^+(x_0)) = \mathcal{O}_S^+(y_0)$.

Proposition 3.1. *Let (X, T) and (Y, S) be strongly orbit equivalent minimal Cantor systems. For any points $x_0 \in X$ and $y_0 \in Y$, there exists a pointed strong orbit equivalence between (X, T, x_0) and (Y, S, y_0).*

Proof. This is a consequence of results from [4]. Theorem 3.6 of [4] states that any minimal Cantor system (X, T) with $x_0 \in X$ can be represented as a Bratteli–Vershik system with x_0 being the unique maximal path of the associated ordered Bratteli diagram. In the proof of Theorem 1.1, given two strongly orbit equivalent Bratteli–Vershik systems, Giordano, Putnam and Skau construct a strong orbit equivalence between the systems that preserves the minimal and maximal paths and preserves the cofinality of paths. Moreover, they show that the cocycles of this strong orbit equivalence can be discontinuous only at the maximal path.

So given two strongly orbit equivalent minimal Cantor systems (X, T) and (Y, S), we can find a Bratteli–Vershik representation of (X, T) with maximal path x_0 and a representation of (Y, S) with maximal path y_0. By the proof of Theorem 1.1, we can find a strong orbit equivalence $h\colon X \to Y$ that preserves the minimal and maximal paths, preserves cofinality, and such that the cocycles of h are discontinuous only at x_0. Since x_0 and y_0 are the maximal paths in the diagrams, the points Tx_0 and Sy_0 are the minimal paths. Therefore, h satisfies properties (1) and (2). Since the cocycles of h are discontinuous only at the maximal path, property (3) is satisfied. The points in X that are cofinal with x_0 other than itself are exactly $\mathcal{O}_T^-(x_0)$ and the points cofinal with Tx_0 are exactly $\mathcal{O}_T^+(x_0) \setminus \{x_0\}$, and the analogous statement is true for (Y, S) with y_0 and Sy_0. This along with the fact that h preserves the cofinality of paths guarantees that properties (4) and (5) are satisfied. $\square$

Let (X, T) and (Y, S) be strongly orbit equivalent minimal Cantor systems and let h be a pointed strong orbit equivalence between (X, T, x_0) and (Y, S, y_0). If we let $S' = h^{-1} \circ S \circ h$, (X, S') is a minimal Cantor system conjugate to (Y, S). It can easily be checked that the identity map on X is a strong orbit equivalence between (X, S') and (X, T). Furthermore, S' satisfies the following properties:

(1) $S'(x_0) = T(x_0)$;

(2) $\mathcal{O}^-_{S'}(x_0) = \mathcal{O}^-_T(x_0)$;
(3) $\mathcal{O}^+_{S'}(x_0) = \mathcal{O}^+_T(x_0)$;
(4) the cocycles associated to the identity map are continuous on $X \setminus \{x_0\}$.

We will say that a minimal homeomorphism of X satisfying these four properties is *x_0-id strongly orbit equivalent* to T. We define $\mathcal{S}(T, x_0) = \{P\colon X \to X \mid P$ is $x_0 - $id strongly orbit equivalent to $T\}$. The cocycle property (property (2)) can be stated more explicitly in the following terms. If $P \in \mathcal{S}(T, x_0)$, there exists functions $a, b\colon X \to \mathbb{Z}$ continuous on $X \setminus \{x_0\}$ such that for all $x \in X$, $Tx = P^{a(x)}(x)$ and $Px = T^{b(x)}(x)$. Since a and b depend only on P and T, we will refer to the them as the *cocycles of P relative to T* or just the *cocycles of P* if T is clear by the context. By the preceding arguments, any minimal Cantor system strongly orbit equivalent to (X, T) is conjugate to (X, P) for some $P \in \mathcal{S}(T, x_0)$.

Let (X, T) be a minimal Cantor system with $x_0 \in X$. We will now define a metric m_T on $\mathcal{S}(T, x_0)$. For $S \in \mathcal{S}(T, x_0)$ with cocycles a and b and $S' \in \mathcal{S}(T, x_0)$ with cocyles a' and b', we define

$$m_T(S, S') = \widetilde{m}_T(S, S') + \sup_{x \in X} d_X(Sx, S'x)$$

where

$$\widetilde{m}_T(S, S') = \inf_{\varepsilon > 0}\{a(x) = a'(x) \text{ and } b(x) = b'(x) \text{ for all } x \in X \setminus B(x_0, \varepsilon)\}.$$

The second term in the sum that defines $m_T(S, S')$ is the supremum metric. Because the sum of two metrics defines another metric, in order to show that m_T is a metric on $\mathcal{S}(T, x_0)$, it is sufficient to show that $\widetilde{m}_T$ is a metric on $\mathcal{S}(T, x_0)$. If we can show that $\widetilde{m}_T$ satisfies the triangle inequality, the other metric space properties follow trivially.

For $S_i \in \mathcal{S}(T, x_0)$, $i = 1, 2, 3$, let a_i and b_i be the cocycles of S_i. We will show that $\widetilde{m}_T$ satisfies a stronger form of the triangle inequality, namely $\widetilde{m}_T(S_1, S_3) \le \max\{\widetilde{m}_T(S_1, S_2), \widetilde{m}_T(S_2, S_3)\}$. Assume that $\widetilde{m}_T(S_1, S_3) = p > 0$ and $\widetilde{m}_T(S_1, S_2) = r < p$. Then, by the definition of $\widetilde{m}_T(S_1, S_3)$, if $r < q < p$, there exists an $x_q \in X$ with $q < d_X(x_0, x_q) \le p$ such that either $a_1(x_q) \ne a_3(x_q)$ or $b_1(x_q) \ne b_3(x_q)$. Since $\widetilde{m}_T(S_1, S_2) = r < q$, $a_1(x_q) = a_2(x_q)$ and $b_1(x_q) = b_2(x_q)$. Therefore, either $a_2(x_q) \ne a_3(x_q)$ or $b_2(x_q) \ne b_3(x_q)$, and thus $\widetilde{m}_T(S_2, S_3) \ge d_X(x_0, x_q) > q$. Because this holds for all $r < q < p$, we can conclude that $\widetilde{m}_T(S_2, S_3) \ge p$, finishing the proof. □

3.2. Properties of $\mathcal{S}(T, x_0)$. Here we establish some properties of $\mathcal{S}(T, x_0)$.

Proposition 3.2. *If $S \in \mathcal{S}(T, x_0)$, then $T(\mathcal{O}^+_S(x_0)) = \mathcal{O}^+_S(x_0) \setminus \{x_0\}$ and $T(\mathcal{O}^-_S(x_0)) = \mathcal{O}^-_S(x_0) \cup \{x_0\}$. Furthermore, $S(\mathcal{O}^+_T(x_0)) = \mathcal{O}^+_T(x_0) \setminus \{x_0\}$ and $S(\mathcal{O}^-_T(x_0)) = \mathcal{O}^-_T(x_0) \cup \{x_0\}$.*

Proof. By the definition of $\mathcal{S}(T,x_0)$, if $S \in \mathcal{S}(T,x_0)$, then $\mathcal{O}_T^-(x_0) = \mathcal{O}_S^-(x_0)$. Then we have

$$T(\mathcal{O}_S^-(x_0)) = T(\mathcal{O}_T^-(x_0)) = \mathcal{O}_T^-(x_0) \cup \{x_0\} = \mathcal{O}_S^-(x_0) \cup \{x_0\}.$$

The other statements can be proven by a similar argument. $\square$

Definition 3.3. Let $S \in \mathcal{S}(T,x_0)$ and let C be a clopen set in X. For $x \in C$, define the set C_x in the following way. If $a(x) < 0$, then $C_x = \{S^{a(x)}(x), \dots, S^{-1}(x), x\}$; if $a(x) > 0$, then $C_x = \{x, Sx, \dots, S^{a(x)-1}(x)\}$. We define $C_S = \bigcup_{x \in C} C_x$.

Proposition 3.4. *If C is a clopen set in X with $x_0 \notin C$, then the set C_S defined above is clopen in X and $x_0 \notin C_S$.*

Proof. Since $x_0 \notin C$, the function $a|_C\colon C \to \mathbb{Z}$ is continuous. Then because C is compact, $a|_C$ takes on only finitely many values. Therefore, there exists an integer $M > 0$ such that $a|_C(C) \subset [-M, M]$. For $k \in \mathbb{Z}$, $|k| \le M$, the set $a|_C^{-1}\{k\}$ is clopen in C, and because C is clopen in X, $a|_C^{-1}\{k\}$ is also clopen in X. Because S is a homeomorphism, the set $S^j(a|_C^{-1}\{k\})$ is clopen in X for all $j \in \mathbb{Z}$. If $0 < k \le M$, we let $C_k = \bigcup_{j=0}^{k-1} S^j(a|_C^{-1}\{k\})$ and if $-M \le k < 0$, we let $C_k = \bigcup_{j=0}^{|k|} S^{-j}(a|_C^{-1}\{k\})$. Each C_k is clopen in X, and moreover $C_S = \bigcup_{k=-M}^{M} C_k$. Since C_S is the finite union of clopen sets, C_S is clopen as claimed.

To show $x_0 \notin C_S$, we will argue by contradiction. Assume $x_0 \in C_S$. Then there exists $x \in C$ such that $x_0 = S^j(x)$ where $0 < j < a(x)$ if $a(x) > 0$ or $0 < j \le a(x)$ if $a(x) < 0$. If we assume $a(x) > 0$, then $x_0 = S^j(x)$ for $0 < j < a(x)$. Then $x = S^{-j}x_0$ and we have

$$T(S^{-j}x_0) = Tx = S^{a(x)}(x) = S^{a(x)-j}S^j(x) = S^{a(x)-j}(x_0).$$

Since $a(x) - j > 0$, T is mapping a point in $\mathcal{O}_S^-(x_0)$ to a point in $\mathcal{O}_S^+(x_0) \setminus \{x_0\}$ contradicting Proposition 3.2. If $a(x) < 0$, then $x_0 = S^{-j}x$ with $a(x) \le -j < 0$, and we have $S^j x_0 = x$. By an argument similar to the one above, $T(S^j x_0) = S^{a(x)+j}(x_0)$. Since $a(x) + j \le 0$, T is mapping a point in $\mathcal{O}_S^+(x_0)$ to a point in $\mathcal{O}_S^+(x_0) \cup \{x_0\}$, which again contradicts Proposition 3.2. This proves $x_0 \notin C_S$.$\square$

Proposition 3.5. *Suppose $S \in \mathcal{S}(T,x_0)$ with cocycles a and b and C is a clopen set in X with $x_0 \notin C$. If $S' \in \mathcal{S}(T,x_0)$ with cocycles a' and b' such that $Sx = S'x$ for all $x \in C_S$, then $a(x) = a'(x)$ and $b(x) = b'(x)$ for all $x \in C$.*

Proof. Since $C \subset C_S$, we have that $Sx = S'x$ for all $x \in C$. Then because $Sx = T^{b(x)}(x)$ and $S'x = T^{b'(x)}(x)$ for all $x \in X$, $b(x) = b'(x)$ for all $x \in C$. Fix $x \in C$. If $a(x) > 0$, then S and S' agree on the set $\{x, Sx, \dots, S^{a(x)-1}(x)\}$. In particular, $S'^{a(x)}(x) = S^{a(x)}(x) = Tx$, so $a'(x) = a(x)$. If $a(x) < 0$, then S and S' agree on the set $\{S^{a(x)}(x) \dots S^{-1}(x), x\}$. Since $S^{a(x)}(x) = Tx$, we have $x = S^{|a(x)|}(Tx) = S'^{|a(x)|}(Tx)$. So $S'^{a(x)}(x) = S^{a(x)}(x) = Tx$, which again shows that $a'(x) = a(x)$ finishing the proof. $\square$

PROPOSITION 3.6. *If $S \in \mathcal{S}(T, x_0)$, then $\mathcal{S}(T, x_0) = \mathcal{S}(S, x_0)$.*

PROOF. Let a and b be the cocycles of S relative to T and suppose $P \in \mathcal{S}(T, x_0)$ with cocycles a' and b' relative to T. It is easily seen that P satisfies properties (1)–(3) of $\mathcal{S}(S, x_0)$ as $Px_0 = Tx_0 = Sx_0$, $\mathcal{O}_P^-(x_0) = \mathcal{O}_T^-(x_0) = \mathcal{O}_S^-(x_0)$, and $\mathcal{O}_P^+(x_0) = \mathcal{O}_T^+(x_0) = \mathcal{O}_S^+(x_0)$. We will now show that P satisfies property (4).

Let $x \in X$ with $x \neq x_0$. If we assume $b(x) = k > 0$, then we have:

$$Sx = T^k(x) = T(T^{k-1}(x)) = P^{a'(T^{k-1}(x))}(T^{k-1}(x)).$$

If we repeat this process until we get x as the argument on the right hand side, we get that $Sx = P^{p(x)}(x)$ where

$$p(x) = \sum_{j=0}^{k-1} a'(T^j x).$$

An argument similar to that in the proof of Proposition 3.4 shows that $x_0 \neq T^j x$ for $j = 0, \dots, k-1$. Therefore, p is continuous on $X \setminus \{x_0\}$. If $b(x) < 0$, the proof is done similarly. If $b'(x) = k > 0$, we have that $Px = S^{q(x)}(x)$ where

$$q(x) = \sum_{j=0}^{k-1} a(T^j x).$$

As stated above, we have that $x_0 \neq T^j x$ for $j = 0, \dots, k-1$, so q is continuous on $X \setminus \{x_0\}$. The proof is done similarly if $b'(x) < 0$. The preceding arguments have shown that the cocyles of P relative to S are the functions p and q. Since p and q are continuous on $X \setminus \{x_0\}$, P satisfies property (4) of $\mathcal{S}(S, x_0)$. This establishes that $\mathcal{S}(T, x_0) \subset \mathcal{S}(S, x_0)$. By symmetry, $\mathcal{S}(T, x_0) = \mathcal{S}(S, x_0)$. $\square$

THEOREM 3.7. *Suppose (X, T) and (Y, S) are strongly orbit equivalent minimal Cantor systems with $x_0 \in X$ and $y_0 \in Y$. Then $(\mathcal{S}(T, x_0), m_T)$ and $(\mathcal{S}(S, y_0), m_S)$ are uniformly homeomorphic metric spaces.*

PROOF. By Proposition 3.1, there exists a pointed strong orbit equivalence h between (X, T, x_0) and (Y, S, y_0). Define the function $f \colon \mathcal{S}(T, x_0) \to \mathcal{S}(S, y_0)$ by $f(P) = h \circ P \circ h^{-1}$. Throughout this proof, we will use P' to denote $f(P) = h \circ P \circ h^{-1}$. In particular, we have that $T' = h \circ T \circ h^{-1}$. We will begin by showing that $T' \in \mathcal{S}(S, y_0)$. Clearly $T' \colon Y \to Y$ is a minimal homeomorphism and

$$T'(y_0) = h \circ T \circ h^{-1}(h(x_0)) = h \circ T(x_0) = Sy_0.$$

Furthermore, we have that

$$\begin{aligned}\mathcal{O}^+_{T'}(y_0) &= \{(h\circ T\circ h^{-1})^k(y_0)) \mid k\geq 0\} = \{(h\circ T^k\circ h^{-1})(h(x_0)) \mid k\geq 0\}\\ &= \{(h\circ T^k)(x_0) \mid k\geq 0\} = h(\mathcal{O}^+_T(x_0)) = \mathcal{O}^+_S(y_0).\end{aligned}$$

With a similar calculation, we can show that $\mathcal{O}^-_{T'}(y_0) = \mathcal{O}^-_S(y_0)$. It remains to be shown that T' satisfies property (4) of $\mathcal{S}(S, y_0)$.

Let m and n be the cocycles of h, so for all $x\in X$,

$$h\circ T(x) = S^{m(x)}\circ h(x) \quad \text{and} \quad h\circ T^{n(x)}(x) = S\circ h(x),$$

and m and n are continuous on $X\setminus\{x_0\}$. Then for $y\in Y$, we have

$$\begin{aligned}(T')^{n(h^{-1}(y))}(y) &= (h\circ T\circ h^{-1})^{n(h^{-1}(y))})(y)\\ &= h\circ T^{n(h^{-1}(y))}(h^{-1}(y)) = S\circ h(h^{-1}(y)) = Sy,\\ S^{m(h^{-1}(y))}(y) &= S^{m(h^{-1}(y))}(h(h^{-1}(y)) = h\circ T(h^{-1}(y)) = T'y.\end{aligned}$$

This shows that the cocycles of T' relative to S are the functions $m\circ h^{-1}$ and $n\circ h^{-1}$. These functions are continuous as long as $h^{-1}(y)\neq x_0$, i.e. if $y\neq h(x_0) = y_0$. Therefore the cocycles of T' relative to S are continuous on $Y\setminus\{y_0\}$. This establishes that $T'\in\mathcal{S}(S, y_0)$. By Proposition 3.6, we have that $\mathcal{S}(T', y_0) = \mathcal{S}(S, y_0)$. We will now show that if $P\in\mathcal{S}(T, x_0)$, then $P'\in\mathcal{S}(T', y_0)$.

If $P\in\mathcal{S}(T, x_0)$ with cocycles a and b, then for $y\in Y$,

$$\begin{aligned}(P')^{a(h^{-1}(y))}(y) &= h\circ P^{a(h^{-1}(y))}(h^{-1}(y)) = h\circ T\circ h^{-1}(y) = T'y,\\ (T')^{b(h^{-1}(y))}(y) &= h\circ T^{b(h^{-1}(y))}(h^{-1}(y)) = h\circ R\circ h^{-1}(y) = P'y.\end{aligned}$$

This shows that the cocycles of P' relative to T' are the functions $a\circ h^{-1}$ and $b\circ h^{-1}$. These functions are continuous on $Y\setminus\{y_0\}$, so by an argument similar to the one above, $P'\in\mathcal{S}(T', y_0) = \mathcal{S}(S, y_0)$. We have established that f is a well-defined map from $\mathcal{S}(T, x_0)$ to $\mathcal{S}(S, y_0)$.

We have left to show that f is a uniformly continuous homeomorphism. First, f is clearly invertible as $f^{-1}\colon\mathcal{S}(S, y_0)\to\mathcal{S}(T, x_0)$ is defined by $f^{-1}(Q) = h^{-1}\circ Q\circ h$. Moreover, h^{-1} is a pointed strong orbit equivalence between (Y, S, y_0) and (X, T, x_0), so if we show that f is uniformly continuous, by the same argument we will have that f^{-1} is uniformly continuous. We will now show that f is a uniformly continuous function.

Fix $\varepsilon > 0$. Because h is uniformly continuous on X, there exists a $\delta > 0$ such that if $x, x'\in X$ with $d_X(x, x') < \delta$, then $d_Y(h(x), h(x')) < \varepsilon$. Pick $P, R\in\mathcal{S}(T, x_0)$ with $\sup_{x\in X} d_X(Px, Rx)\leq m_T(P, R) < \delta$. Then we have

$$\begin{aligned}\sup_{y\in Y} d_Y(P'y, R'y) &= \sup_{y\in Y} d_Y(h\circ P\circ h^{-1}(y), h\circ R\circ h^{-1}(y))\\ &= \sup_{x\in X} d_Y(h(P(x)), h(R(x))) < \varepsilon.\end{aligned}$$

We only have left to show that by making $m_T(P,R)$ small enough, we can make the cocyles of P' and R' agree everywhere on Y except in in an ε-ball around y_0. Since $P, R \in \mathcal{S}(T, x_0)$, for all $x \in X$,

$$Tx = P^{a(x)}(x) \ \& \ Px = T^{b(x)}(x) \quad \text{and} \quad Tx = R^{c(x)}(x) \ \& \ Rx = T^{d(x)}(x)$$

where a, b, c and d are each continuous functions on $X \setminus \{x_0\}$. Since $P', R' \in \mathcal{S}(S, y_0)$, for all $y \in Y$,

$$Sy = (P')^{a'(y)}(y) \ \& \ P'y = S^{b'(y)}(y) \quad \text{and} \quad Sy = (R')^{c'(y)}(y) \ \& \ R'y = S^{d'(y)}(y)$$

where a', b', c' and d' are each continuous functions on $Y \setminus \{y_0\}$.

Fix $\varepsilon > 0$ and let C be a clopen set containing $Y \setminus B(y_0, \varepsilon)$ with $y_0 \notin C$. Since $T' \in \mathcal{S}(S, y_0)$, we define the set $C_{T'}$ analogously as done in Definition 3.3. By Proposition 3.4, $C_{T'}$ is clopen in Y with $y_0 \notin C_{T'}$, so there exists a $\delta' > 0$ such that $B(y_0, \delta') \subset Y \setminus C_{T'}$. Since h is uniformly continuous on X, we can find a $\delta > 0$ such that if $x, x' \in X$ with $d_X(x, x') < \delta$, then $d_Y(h(x), h(x')) < \delta'$. Now, suppose $m_T(P, R) < \delta$. Fix $y \in Y \setminus B(y_0, \varepsilon)$, so $y \in C_{T'}$ and thus $y \notin B(y_0, \delta')$. Suppose $h^{-1}(y) \in B(x_0, \delta)$. Then $d(y, h(x_0)) < \delta'$, but $h(x_0) = y_0$, so $y \in B(y_0, \delta')$ which is a contradiction. Therefore, $h^{-1}(y) \notin B(x_0, \delta)$. Since $m_T(P, R) < \delta$, $b(h^{-1}(y)) = d(h^{-1}(y))$ and so $P(h^{-1}(y)) = R(h^{-1}(y))$. From this, we can conclude $P'y = R'y$ and thus $b'(y) = d'(y)$ for all $y \in y \setminus B(y_0, \varepsilon)$. We will now show that the same is true for a' and c'. Fix $y \in Y \setminus B(y_0, \varepsilon)$, and suppose $Sy = T'^k y$, $k > 0$. Using that fact shown above that the cocycles of P' relative to T' are $a \circ h^{-1}$ and $b \circ h^{-1}$, we have

$$Sy = (T')^k(y) = T'((T')^{k-1}(y)) = (P')^{a \circ h^{-1}((T')^{k-1}(y))}((T')^{k-1}(y).$$

Repeating this procedure k times, we get

$$a'(y) = \sum_{j=0}^{k-1} a(h^{-1}((T')^j(y))).$$

Similarly we get that

$$c'(y) = \sum_{j=0}^{k-1} c(h^{-1}((T')^j(y))).$$

But for each $j = 0, \ldots, k-1$, $(T')^j(y) \in C_{T'}$, so $h^{-1}((T')^j y) \notin B(x_0, \delta)$. Since $m_T(P, R) < \delta$, $a(h^{-1}((T')^j(y))) = c(h^{-1}((T')^j(y)))$ for $j = 0, \ldots, k-1$, so $a'(y) = c'(y)$.

Now suppose $Sy = (T')^{-k}y$, $k > 0$. Then, we have

$$\begin{aligned} y &= (T')^k(Sy) = T'((T')^{k-1}(Sy)) \\ &= (P')^{a \circ h^{-1}((T')^{k-1}(Sy))}((T)'^{k-1}(Sy)) = (P')^{a \circ h^{-1}((T')^{-1}(y))}((T')^{k-1}(Sy)). \end{aligned}$$

Repeating this procedure k times, we get that $y = (P')^{q(y)}(Sy)$ where

$$q(y) = \sum_{j=1}^{k} a \ \circ h^{-1}((T')^{-j}(y)).$$

Therefore,

$$a'(y) = -q(y) = -\sum_{j=1}^{k} a \ \circ h^{-1}((T')^{-j}(y)).$$

Similarly we get that

$$c'(y) = -\sum_{j=1}^{k} c \circ h^{-1}((T')^{-j}(y)).$$

Again, for each $j = 1, \dots, k-1$, $(T')^{-j}(y) \in C_{T'}$, so by the same argument as above we have that $a \circ h^{-1}((T')^{-j}(y)) = c \circ h^{-1}((T')^{-j}(y))$ for each $j = 1, \dots, k$. This establishes $a'(y) = c'(y)$ for all $y \in Y \setminus B(y_0, \varepsilon)$. In both of the preceding arguments, the choice of δ was independent of P and R, so we can conclude that f is uniformly continuous. □

Corollary 3.8. *For $S \in \mathcal{S}(T, x_0)$, the identity map $\mathcal{S}(T, x_0) \to \mathcal{S}(S, x_0)$ is a uniformly continuous homeomorphism.*

Proof. It is easily verified that the identity map on X is a pointed strong orbit equivalence between (X, T, x_0) to (X, S, x_0). Then, by Proposition 3.6, the identity map from $(\mathcal{S}(T, x_0), m_T)$ to $(\mathcal{S}(S, x_0), m_S)$ is a bijection. By Theorem 3.7, the identity map is a uniformly continuous homeomorphism. □

Theorem 3.7 shows that the resulting metric space is independent of map chosen from the strong orbit equivalence class and independent of the point chosen from the space. From this point forward we will only consider one Cantor space X and one special point $x_0 \in X$, and we will let $\mathcal{S}(T)$ denote $\mathcal{S}(T, x_0)$. We will now establish some properties of $(\mathcal{S}(T), m_T)$.

Proposition 3.9. *$(\mathcal{S}(T), m_T)$ is a complete metric space.*

Proof. Let $\{S_n\}$ be an m_T-Cauchy sequence in $\mathcal{S}(T)$. For all n, let a_n and b_n be the cocycles of S_n. For all $x \in X$, each of the sequences $\{S_n(x)\}$, $\{a_n(x)\}$, and $\{b_n(x)\}$ are eventually fixed. This holds for $x \in X \setminus \{x_0\}$ because there exists an $N > 0$ such that if $n, m \geq N$, then $a_n(x) = a_m(x)$ and $b_n(x) = b_m(x)$. Since $b_n(x) = b_m(x)$, for all $n, m \geq N$, this also means $S_n(x) = S_m(x)$ for all $n, m \geq N$. Furthermore, $S_n(x_0) = Tx_0$ for all n, so $a_n(x_0) = 1 = b_n(x_0)$ for all n. This argument can be generalized to show that for any $j \in \mathbb{Z}$ and $x \in X$, the sequence $\{S_n^j(x)\}$ is eventually fixed. So we can define $Sx = \lim_{n\to\infty} S_n(x)$, $a(x) = \lim_{n\to\infty} a_n(x)$, and $b(x) = \lim_{n\to\infty} b_n(x)$ for all $x \in X$. We will show that $S \in \mathcal{S}(T)$ with cocycles a and b and $\{S_n\}$ is m_T-convergent to S proving the proposition.

We begin by showing that S is a homeomorphism. Because for every $x \in X$, the sequence $\{S_n(x)\}$ is eventually fixed, S must be one-to-one and onto since each S_n is one-to-one and onto. Since $\sup_{x\in X} d_X(S_n(x), S_m(x)) \to 0$ and $\{S_n\}$ converges pointwise to S, by the Cauchy criterion for uniform convergence $\{S_n\}$ converges uniformly to S. Since S is the uniform limit of continuous functions, S is continuous. Furthermore, it is a well known theorem that a continuous bijection between compact metric spaces has a continuous inverse.

We will now show that S satisfies the properties of $\mathcal{S}(T)$. It is easily seen that S satisfies property (1) of $\mathcal{S}(T)$ because for all $n \in \mathbb{N}^+$, $S_n(x_0) = Tx_0$ and thus $Sx_0 = Tx_0$. We will now show that the cocycles of S are the functions a and b, and they satisfy property (4) of $\mathcal{S}(T)$. Fix $x \in X$. By the argument above, there exists an $N > 0$ such that if $n \geq N$, $b_n(x) = b(x)$. This also means for $n \geq N$, $S_n(x) = S(x)$. So for $n \geq N$,

$$Sx = S_n(x) = T^{b_n(x)}(x) = T^{b(x)}(x).$$

To see that b is continuous on $X \setminus \{x_0\}$, we fix $x \neq x_0$ and find a clopen neighbourhood D of x with $x_0 \notin D$. If N is chosen large enough such that for $n \geq N$, b and b_n agree on D, since b_n is continuous on D, b is also continuous on D. Because $x \in D$, b is continuous at x.

We will now show that a satisfies the desired properties. Fix $x \in X$ and suppose $a(x) > 0$. Pick N large enough so that for $n \geq N$, $S^j(x) = (S_n)^j(x)$ for all $j = 1, \dots, a(x)$ and $a_n(x) = a(x)$. Then, for $n \geq N$,

$$Tx = (S_n)^{a_n(x)}(x) = (S_n)^{a(x)}(x) = S^{a(x)}(x).$$

We can argue in a similar fashion if $a(x) < 0$. Furthermore, by a similar argument to that above, a is continuous on $X \setminus \{x_0\}$. This shows that S satisfies property (4) of $\mathcal{S}(T)$.

To see that S satisfies properties (2) and (3) of $\mathcal{S}(T)$, fix $j \in \mathbb{Z}$ and pick N such that if $n \geq N$, then $(S_n)^j(x_0)$ is fixed. Then, for $n \geq N$, $S^j(x_0) = (S_n)^j(x_0)$. Since $\mathcal{O}^-_{S_n}(x_0) = \mathcal{O}^-_T(x_0)$ and $\mathcal{O}^+_{S_n}(x_0) = \mathcal{O}^+_T(x_0)$, this means $\mathcal{O}^-_S(x_0) \subset \mathcal{O}^-_T(x_0)$ and $\mathcal{O}^+_S(x_0) \subset \mathcal{O}^+_T(x_0)$. However, we know that $\mathcal{O}_S(x_0) = \mathcal{O}_T(x_0)$ because the functions a and b are the cocycles S. So we must have that $\mathcal{O}^-_S(x_0) = \mathcal{O}^-_T(x_0)$ and $\mathcal{O}^+_S(x_0) = \mathcal{O}^+_T(x_0)$. This establishes that $S \in \mathcal{S}(T)$.

It remains to be shown that $\{S_n\}$ is m_T-convergent to S. Above we argued that $\{S_n\}$ converges uniformly to S, so to prove that $\{S_n\}$ is m_T-convergent to S, we only have left to show $\widetilde{m}_T(S, S_n) \to 0$. Let $\varepsilon > 0$. Pick a clopen set C with $X \setminus B(x_0, \varepsilon) \subset C$ and $x_0 \notin C$. Let C_S the set defined in Definition 3.3. Since $x_0 \notin C_S$, there exists a $\delta > 0$ such that $B(x_0, \delta) \subset X \setminus C_S$. Pick N such that if $n, m \geq N$, $m_T(S_n, S_m) < \delta$. Then, for $n \geq N$, $S_n(x) = Sx$ for all $x \in C_S$. By Proposition 3.5, $a(x)$ and $b(x)$ agree with $a_n(x)$ and $b_n(x)$, respectively, for all $x \in C$, so $\widetilde{m}_T(S, S_n) < \varepsilon$. $\square$

Because $(\mathcal{S}(T), m_T)$ is a complete metric space, the Baire Category Theorem applies. We can now ask questions similar to those addressed by M. Hochman and D.J. Rudolph in [6] and [9], respectively, about what systems are typical in these spaces. We will begin by showing that $\mathcal{S}(T)$ is separable for any minimal Cantor system (X, T). This along with Proposition 3.9 shows that $(\mathcal{S}(T), m_T)$ is a *Polish metric space*, i.e. it is complete and separable. Before proving that $\mathcal{S}(T)$ is separable, we need some definitions.

Let $\mathcal{P}$ be a tower partition of a minimal Cantor system (X, T) over a clopen set A such that $\mathcal{P}$ partitions A into finitely many clopen sets $A_1, \ldots, A_k$. For each $1 \leq j \leq k$, let r_j denote the return time of A_j to A and let $f_j : \{0, \ldots, r_j - 1\} \to \{0, \ldots, r_j - 1\}$ be a permutation with the properties that $f_j(0) = 0$ and $f_j(r_j - 1) = r_j - 1$. Then each f_j defines a reordering of the tower over A_j that fixes the top and bottom floors of the tower. Define $\phi : X \to X$ in the following way. If $x \in T^i(A_j)$ for $1 \leq j \leq k$ and $0 \leq i \leq r_j - 1$, we define $\phi(x) = T^{f_j(i) - i}(x)$. We will say that ϕ is a *tower permutation* of $\mathcal{P}$ with corresponding permutations $f_1, \ldots, f_k$. We will denote the set of all tower permutations of $\mathcal{P}$ by $\Pi(\mathcal{P})$. If $\{\mathcal{P}_n\}$ is a sequence of tower partitions of (X, T), we let $\Pi\{\mathcal{P}_n\} = \bigcup \Pi(\mathcal{P}_n)$.

If $\mathcal{P}$ is a tower permutation of a minimal Cantor system (X, T) and $\phi \in \Pi(\mathcal{P})$, then the map $\phi T \phi^{-1} : X \to X$ moves points of X through the towers of $\mathcal{P}$ according to the corresponding permutations of ϕ. For example, suppose $B \subset X$ is a bottom tower floor of $\mathcal{P}$ and the height of the tower over B is 5. Let $\phi \in \Pi(\mathcal{P})$ be a tower permutation whose corresponding permutation f on the tower over B is given by the following:

$$f : \begin{cases} 0 \to 0 \quad 1 \to 3 \quad 2 \to 1 \\ 3 \to 2 \quad 4 \to 4. \end{cases}$$

Then the maps ϕ and $\phi T \phi^{-1}$ are as shown in Figure 4.

Definition 3.10. For $S \in \mathcal{S}(T)$, let $\mathcal{C}(S) = \{P \in \mathcal{S}(T) \mid (X, P) \text{ is conjugate to } (X, S)\}$.

Theorem 3.11. *$\mathcal{S}(T)$ is separable. In fact, for all $S \in \mathcal{S}(T)$, there exists a countable subset of $\mathcal{C}(S)$ that is dense in $\mathcal{S}(T)$.*

Before we proving this theorem, we need a lemma.

Lemma 3.12. *Suppose $S \in \mathcal{S}(T)$ and C is a clopen set in X with $x_0 \notin C$. If $\{\mathcal{P}_n\}$ is generating sequence of tower partitions over Tx_0, there exists $\phi \in \Pi\{\mathcal{P}_n\}$ such that the cocycles of $\phi T \phi^{-1}$ agree with the cocycles of S for all $x \in C$.*

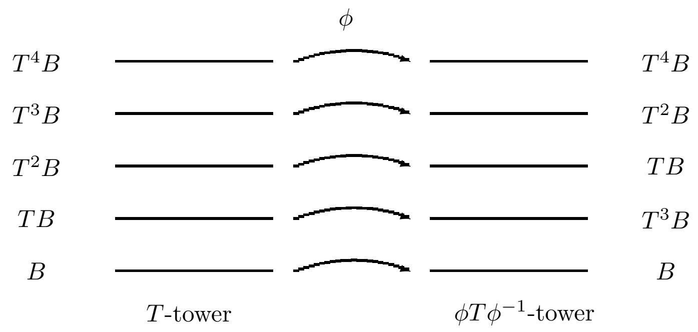

Figure 4. T-tower to $\phi T\phi^{-1}$-tower

Proof. Let a, b be the cocycles of S such that $Tx = S^{a(x)}(x)$ and $Sx = T^{b(x)}(x)$ for all $x \in X$ and let C_S be as in Definition 3.3. There exists an $M > 0$ such that $b(C_S) \subset [-M, M]$. Since every forward orbit is dense in X, there exists a $K > 0$ such that $S^K(Tx_0) \in X \setminus C_S$. Since S^K is continuous, there is a clopen neighbourhood D of Tx_0 with $S^K(D) \subset X \setminus C_S$. Let $\{\mathcal{P}_n\}$ be a sequence of generating tower partitions over Tx_0, and for all n, let A_n be the clopen set such that $\mathcal{P}_n$ is a tower partition over A_n. By Proposition 2.2, $\mathcal{H}(\mathcal{P}_n)$ grows arbitrarily large, so we can pick N' large enough such that $\mathcal{P}_{N'}$ satisfies the following:

(1') C_S is the finite union of tower floors in $\mathcal{P}_{N'}$;
(2') a and b are constant on each of the C_S tower floors;
(3') the towers of $\mathcal{P}_{N'}$ that contain x_0 and Tx_0 each have height greater than M.

Now, we pick $N > N'$ such that $\mathcal{P}_N$ has the following properties:

(1) A_N is contained in the tower floor of $\mathcal{P}_{N'}$ that contains Tx_0 and $T^{-1}(A_N)$ is contained in the tower floor of $\mathcal{P}_{N'}$ that contains x_0;
(2) $\mathcal{H}(\mathcal{P}_N) > KM$;
(3) $A_N \subset D$;
(4) $T^{-1}(A_N) \cap C_S = \emptyset$.

We will find $\phi \in \Pi(\mathcal{P}_N) \subset \Pi\{\mathcal{P}_n\}$ so that $\phi T\phi^{-1}$ agrees with S on C_S. By Proposition 3.5, this will prove the lemma. We consider a fixed tower in $\mathcal{P}_N$ whose bottom floor we will denote by F. Suppose the height of the tower over F is $L+1$. Then the floors of the tower over F are the sets $F, TF, \dots, T^L(F)$. Fix $i \in \{0, \dots, L-1\}$ such that $T^i(F) \subset C_S$. We claim that $S(T^i(F))$ is another floor in the tower over F other than F. By condition (2'), b is constant on $T^i(F)$, so for all $x \in T^i(F)$, let $b(x) = m \in [-M, M]$. Then $S(T^i(F)) = T^{i+m}(F)$, and therefore if $0 < i + m \le L$, $S(T^i(F))$ is another tower floor in the tower over F other than F. We have three cases to consider.

Case 1. If $0 \leq i \leq M$, by conditions (3') and (1), there exists a tower floor $\widetilde{P} \in \mathcal{P}_{N'}$ with height i such that $T^i(F) \subset \widetilde{P}$ and $\widetilde{P}$ is in the same tower of $\mathcal{P}_{N'}$ that contains Tx_0. So there exists an $x \in \widetilde{P}$ such that $x = T^i(Tx_0)$. By property (1'), $\widetilde{P} \subset C_S$. Therefore b is constant on $\widetilde{P}$, so $b(x') = m$ for all $x' \in \widetilde{P}$. By Proposition 3.2, because $x \in \mathcal{O}_T^+(x_0)$, $m > -i$. Because $m > -i$ and $0 \leq i \leq M$, we have $0 < i + m \leq 2M \leq L$. The last inequality holds by properties (3') and (1).

Case 2. If $M < i \leq L - M$, then because $-M \leq m \leq M$, we have $0 < m + i \leq L$.

Case 3. If $L - M < i < L$, the argument is similar to that in Case 1. By conditions (3') and (1), there exists a tower floor $\widetilde{P} \in \mathcal{P}_{N'}$ with height i such that $T^i(F) \subset \widetilde{P}$ and $\widetilde{P}$ is in the same tower of $\mathcal{P}_{N'}$ that contains x_0. So there exists an $x \in \widetilde{P}$ such that $x_0 = T^{L-i}(x)$ or equivalently $T^{-(L-i)}(x_0) = x$. Since b is constant on $\widetilde{P}$, $b(x') = m$ for all $x' \in \widetilde{P}$. Because $x \in \mathcal{O}_T^-(x_0)$, by Proposition 3.2 $m \leq L - i$. Because $m \leq L - i$ and $L - M < i < L$, we have $0 \leq L - 2M < i + m \leq L$.

Because this tower was chosen arbitrarily, we have shown that for every tower floor of $\mathcal{P}_N$ that is a subset of C_S, there is a unique tower floor other than the bottom floor in the same tower that is its image under S. We will now show how to permute the tower floors of the tower over F so that if $\phi \in \Pi(\mathcal{P}_N)$ is a map that corresponds to this permutation, then $\phi T \phi^{-1}$ agrees with S on C_S. Because the height of the tower over F is $L+1$, we need to define a permutation f on the set $\{0, \ldots, L\}$ such that $f(0) = 0$ and $f(L) = L$. We define f in the following way. First, we let $f(0) = i_0 = 0$. If $F \subset C_S$, $S(F) = T^{i_1}(F)$ for some $0 < i_1 < L$, and we define $f(1) = i_1$. If $T^{i_1}(F) \subset C_S$, then $S(T^{i_1}(F)) = T^{i_2}(F)$ for some $0 < i_2 < L$, $i_2 \neq i_1$. We define $f(2) = i_2$. For $j > 2$, we continue defining $f(j) = i_j$ recursively so that $S(T^{i_{j-1}}(F)) = T^{i_j}(F)$ until we reach a $k \geq 0$ such that $T^{i_k}(F) \notin C_S$. From conditions (2) and (3) above, we have that $T^{i_j}(F) \neq T^L(F)$ for any $j = 1, \ldots, k$. Now we define $f(L) = i_L = L$. If $T^L(F) \subset S(C_S)$, there exists a $0 < i_{L-1} < L$ such that $S(T^{i_{L-1}}(F)) = T^L(F)$. We define $f(L-1) = i_{L-1}$. We continue defining $f(j) = i_j$ recursively so that $S(T^{i_j}(F)) = T^{i_{j+1}}(F)$ until we reach an $l \geq 0$ such that $T^{i_{L-l}}(F)$ that is not a subset of $S(C_S)$.

We have defined f on two disjoint subsets $\{0, 1, \ldots, k\}$ and $\{L - l, \ldots, L\}$ where $k < L - 1$. If $k + 1 = L - l$, we have completely defined f on the set $\{0, \ldots, L\}$. However, if $k+1 < L-l$, we need to define f on $\{k+1, \ldots, L-l-1\}$. Because $T^{i_k}(F)$ is not a subset of C_S, as long as $f(k+1)$ is the height of a tower floor that is not a subset of $S(C_S)$, it will not affect whether this rearrangement is an S-tower on C_S. Let $I = \{1, \ldots, L\} \setminus \{i_1, \ldots, i_k, l_{L-l}, \ldots i_L\}$ and let $B = \bigcup_{i \in I} T^i(F)$. We want to find $i_{k+1} \in I$ such that $T^{i_{k+1}}(F)$ is not a subset of $S(C_S)$. Suppose no such i_{k+1} exists. This means that every tower floor contained in B is a subset of $S(C_S)$. Every tower floor in the tower over F that is not a subset of B is either not a subset of C_S or has an image under S that is a tower floor not contained in B. So for every $i' \in I$, we must have that $(T)^{i'}(F) = S(T^i(F))$ for some $i \in I$, $i \neq i'$. However, this means that $S(B) = B$ contradicting the minimality of S. Therefore, there must exist $i_{k+1} \in B$ such that $T^{i_{k+1}}(F)$ is not a subset of $S(C_S)$, and we define $f(k+1) = i_{k+1}$. In general for $k+1 < j < L-l$, we defined $f(j) = i_j$ recursively in the following way. If $T^{i_{j-1}}(F) \subset C_S$, then $S(T^{i_{j-1}}(F)) = T^{i_j}(F)$ for some $i_j \in \{1, \dots, L-1\}$ and we define $f(j) = i_j$. If $T^{i_{j-1}}(F)$ is not a subset of C_S, using the minimality argument as above, we can find $i_j \in \{1, \dots, L\} \setminus \{i_1, \dots, i_{j-1}, i_{L-l}, \dots, L\}$ such that $T^{i_j}(F)$ is not a subset of $S(C_S)$ and we define $f(j) = i_j$. We continue to define f recursively in this manner until it is defined on all of $\{0, \dots, L\}$.

For each $j \in \{0, \dots, L\}$, we have defined $f(j) = i_j$ where i_j is defined so that if $i_j \in \{0, \dots, L-1\}$ and $T^{i_j}(F) \subset C_S$, then $S(T^{i_j}(F)) = T^{i_{j+1}}(F) = T^{f(j+1)}(F)$. Furthermore, note that $f(0) = 0$ and $f(L) = L$. Let $\phi \in \Pi\{\mathcal{P}_n\}$ be a map that corresponds to the permutation f on the tower over F, so if $0 \leq i \leq L$ and $x \in T^i(F)$, $\phi(x) = T^{f(i)-i}(x)$. Note that for $x \in T^i(F)$, $\phi^{-1}(x) = T^{f^{-1}(i)-i}(x)$. Fix $i \in \{0, \dots, L-1\}$ such that $T^i(F) \subset C_S$. We claim that $\phi T \phi^{-1}(x) = Sx$ for all $x \in T^i(F)$. Find j with $0 \leq j \leq L-1$ such that $i = i_j$. Fix $x \in T^i(F) = T^{i_j}(F)$, so $x = T^{i_j}(x')$ for some $x' \in F$. Because $T^{i_j}(F) \subset C_S$, $S(T^{i_j}(F)) = T^{f(j+1)}(F)$ and so $Sx = S(T^{i_j}(x')) = T^{f(j+1)}(x')$. Then we have:

$$\begin{aligned}\phi T \phi^{-1}(x) &= \phi T \phi^{-1}(T^{i_j}(x')) = \phi T(T^{f^{-1}(i_j)-i_j}(T^{i_j}(x'))) = \phi T^{f^{-1}(i_j)+1}(x') \\ &= \phi T^{j+1}(x') = T^{f(j+1)-(j+1)}(T^{j+1}(x')) = T^{f(j+1)}(x') = Sx.\end{aligned}$$

Therefore, we have shown that there exists $\phi \in \Pi(\mathcal{P}_N)$ such that $\phi T \phi^{-1}$ agrees with S on every tower floor of the tower over F that is a subset of C_S. If we repeat the construction of the permutation f for every tower in $\mathcal{P}_N$ and let $\phi \in \Pi(\mathcal{P}_N)$ be the map associated to this set of permutations, then $\phi T \phi^{-1}$ will agree with S on every tower floor of $\mathcal{P}_N$ that is a subset of C_S. By Proposition 3.5, this finishes the proof. $\square$

Proof of Theorem 3.11. Let $\{\mathcal{P}_n\}$ be a sequence of generating tower partitions over Tx_0. Because there are only finitely many ways to permute tower floors in each $\mathcal{P}_n$, $\Pi\{\mathcal{P}_n\}$ is countable. Therefore, the set $\mathcal{D}(T, \{\mathcal{P}_n\}) = \{\phi T \phi^{-1} \mid \phi \in \Pi\{\mathcal{P}_n\}\}$ is a countable subset of $\mathcal{S}(T)$. If we can show $\mathcal{D}(T, \mathcal{P}_n\})$ is dense, the theorem is proven. Let $S \in \mathcal{S}(T)$ and fix $\varepsilon > 0$. Since S is continuous at x_0, there is a $\delta' > 0$ such that $S(B(x_0, \delta')) \subset B(Sx_0, \varepsilon/4)$. Let $\delta = \min\{\delta', \varepsilon/2\}$ and find a clopen set C such that $X \setminus B(x_0, \delta) \subset C$ and $x_0 \notin C$. By the previous lemma, we can find a $\phi T \phi^{-1} \in \mathcal{D}(T, \{\mathcal{P}_n\})$ whose cocycles agree with the cocycles of S

on C. Now, we will show that $m_T(\phi T\phi^{-1}, S) < \varepsilon$ proving the theorem. Since the cocycles of these two maps agree on C, clearly $\widetilde{m}_T(\phi T\phi^{-1}, S) < \varepsilon/2$. Thus, we only need to show that $\sup_{x\in X} d_X(\phi T\phi^{-1}(x), Sx) < \varepsilon/2$. Since the cocycles of $\phi T\phi^{-1}$ and S agree on $X \setminus B(x_0, \delta)$, $\phi T\phi^{-1}(x) = Sx$ for all $x \in X \setminus B(x_0, \delta)$. Fix $x \in B(x_0, \delta)$ and assume $y = \phi T\phi^{-1}(x) \notin B(Sx_0, \varepsilon/4)$. Then $S^{-1}(y) \notin B(x_0, \delta)$, so $\phi T\phi^{-1}(S^{-1}(y)) = S(S^{-1}y) = y$. Since $\phi T\phi^{-1}$ is a homeomorphism, $S^{-1}y = x$. This means $x \notin B(x_0, \delta)$, which is a contradiction. So, for $x \in B(x_0, \delta)$ we have

$$d_X(\phi T\phi^{-1}(x), Sx) \leq d_X(\phi T\phi^{-1}(x), Sx_0) + d_X(Sx_0, Sx) < \varepsilon/2.$$

If $\{\mathcal{P}_n\}$ is a sequence of generating tower partitions over Tx_0, $\mathcal{D}(T, \{\mathcal{P}_n\})$ is a countable dense subset of $\mathcal{S}(T)$ and clearly $\mathcal{D}(T, \{\mathcal{P}_n\}) \subset \mathcal{C}(T)$. By the preceding arguments, for any $S \in \mathcal{S}(T)$ there exists a countable dense subset $\mathcal{D}(S)$ of $\mathcal{S}(S)$ with $\mathcal{D}(S) \subset \mathcal{C}(S)$. By Corollary 3.8, the identity map from $\mathcal{S}(T)$ to $\mathcal{S}(S)$ is a uniformly continuous homeomorphism. Because $\mathcal{D}(S)$ is dense in $\mathcal{S}(S)$, it must also be dense in $\mathcal{S}(T)$. $\square$

Corollary 3.12. *For any $S \in \mathcal{S}(T)$, $\mathcal{C}(S)$ is dense in $\mathcal{S}(T)$.*

Proposition 3.13. *$(\mathcal{S}(T), m_T)$ is not compact.*

Proof. Let $\{\mathcal{P}_n\}$ be a sequence of generating partitions over Tx_0. By Proposition 2.2, $\mathcal{H}(\mathcal{P}_n)$ grows arbitrarily large as $n \to \infty$, so we can find a subsequence $\{\mathcal{P}_{n_k}\}$ such that for all k, $\mathcal{H}(\mathcal{P}_{n_k}) \geq k + 3$. For all k, let B_k be the tower floor in $\mathcal{P}_{n_k}$ such that $Tx_0 \in B_k$. We define a sequence $\{\phi_k\}$ in $\Pi\{\mathcal{P}_{n_k}\}$ by

$$\phi_k(x) = \begin{cases} T^k x & \text{if } x \in T(B_k), \\ T^{-k} x & \text{if } x \in T^{k+1}(B_k), \\ x & \text{otherwise.} \end{cases}$$

Then for all k, we have

$$\phi_k T\phi_k^{-1}(Tx_0) = \phi_k T(Tx_0) = T^k(Tx_0).$$

If b_k is the cocycle of $\phi_k T\phi_k^{-1}$ such that $\phi_k T\phi_k^{-1}(x) = T^{b_k(x)}(x)$ for all $x \in X$, by the equation above, $b_k(Tx_0) = k$ for all k. For a sequence to converge in $\mathcal{S}(T)$, its cocycles values at Tx_0 need to stabilize to a fixed integer. Therefore the sequence $\{\phi_k T\phi_k^{-1}\} \subset \mathcal{S}(T)$ has no converging subsequence proving the proposition. $\square$

3.3. Finite rank systems. As defined in [3], a minimal Cantor system (X, T) has *finite rank* if there exists a $K > 0$ such that (X, T) can be represented as a Bratteli–Vershik system with K or fewer vertices at each level. If K is the smallest such integer, we say that (X, T) has *rank K*. We will let $\mathcal{F}(T)$ denote the set of maps in $\mathcal{S}(T)$ that have finite rank. An *odometer* is a system that has rank 1. We say that (X, T) has *x_0-finite rank* if there exists a $K > 0$ such that (X, T) can be represented as a Bratteli–Vershik system with fewer than K

vertices at each level and x_0 is the maximal path in the diagram. If K is the smallest such integer, we will say that (X,T) has x_0*-rank* K.

DEFINITION 3.13. Let $\varepsilon > 0$ and let $K \in \mathbb{N}^+$. We will say that (X,T) satisfies the (x_0, ε)*-rank* K *condition* if there exists a clopen set $A \subset X$ with the following properties:

(a) $Tx_0 \in A$;
(b) A partitions into $L \leq K$ clopen sets $A_1, \ldots, A_L$ each with constant return time r_j to A;
(c) for each $j = 1, \ldots, L$, $\operatorname{diam}(T^i Aj) < \varepsilon$ for $i = 0, \ldots, r_j - 1$;
(d) $\operatorname{diam}(A) < \varepsilon$.

PROPOSITION 3.14. *A minimal Cantor system* (X,T) *has* x_0*-rank less than or equal to* K *if and only if it satisfies the* (x_0, ε)*-rank* K *condition for all* $\varepsilon > 0$.

PROOF. Suppose (X,T) has x_0-rank less than or equal to K. Then it can be represented as a Bratteli–Vershik system with fewer than K vertices at each level and so that x_0 is the maximal path in the diagram, i.e. $x_{\max} = x_0$. For all n, let $\mathcal{P}_n$ denote the partition of X into the cylinder sets of paths that begin with a particular path down to level n and let A_n denote the union of cylinders sets in $\mathcal{P}_n$ that correspond to minimal paths down to level n. Since $\{\mathcal{P}_n\}$ generates the topology of X, we have that $\operatorname{diam}(\mathcal{P}_n) \to 0$. Because (X,T) has a unique minimal path in its Bratteli diagram, we also have that $\operatorname{diam}(A_n) \to 0$. Fix $\varepsilon > 0$ and pick an $N > 0$ such that if $n \geq N$, then $\operatorname{diam}(\mathcal{P}_n) < \varepsilon$ and $\operatorname{diam}(A_n) < \varepsilon$. Fix $n \geq N$ and let $A = A_n$. Since $Tx_0 = x_{\min}$, $Tx_0 \in A$. We partition A the same way it is partitioned in $\mathcal{P}_n$, and we denote this partition by $\mathcal{P}_n(A)$. This partition of A will have fewer than K sets since the number of sets in $\mathcal{P}_n(A)$ is equal to the number of vertices at level n in the Bratteli diagram. Each set in $\mathcal{P}_n(A)$ will have a constant return time to A since each set corresponds to a minimal path cylinder set in the diagram. Condition (c) is satisfied because $\operatorname{diam}(\mathcal{P}_n) < \varepsilon$ and condition (d) is satisfied because $\operatorname{diam}(A) = \operatorname{diam}(A_n) < \varepsilon$.

Conversely for $n \in \mathbb{N}^+$, pick a sequence of sets $A_n \subset X$ such that A_n satisfies the $(x_0, 1/n)$-rank K condition and so that $A_{n+1} \subset A_n$ for all n. We then consider the tower partitions of (X,T) over each A_n. Because each A_n can be partitioned into fewer than K clopen sets each with constant return time to A_n, we can construct a Bratteli–Vershik representation of (X,T) with fewer than K vertices at each level. Because $Tx_0 \in A_n$ for all n, Tx_0 is the minimal path in the diagram; therefore, x_0 is the maximal path in the diagram. Therefore (X,T) has x_0-rank less than or equal to K. $\square$

PROPOSITION 3.15. *A minimal Cantor system* (X,T) *has finite rank if and only if it has* x_0*-finite rank. Moreover, if* (X,T) *has rank* K*, then* (X,T) *has* x_0*-rank less than or equal to* K^2.

Proof. If (X,T) has x_0-finite rank, then by definition (X,T) has finite rank. Conversely, if (X,T) has rank K, then it must have x_1-rank K for some $x_1 \in X$. For $\varepsilon > 0$, we will find a set B containing Tx_0 satisfying the (x_0,ε)-rank K^2 condition. Because $\mathcal{O}_T^+(Tx_1)$ is dense in X, there exists an $m \geq 0$ such that $T^m(Tx_1) \in B(Tx_0,\varepsilon/4)$. Since T^m is continuous at Tx_1, there exists a $\delta' > 0$ such that if $d_X(x,Tx_1) < \delta'$, then $T^m(x) \in B(T^m(Tx_1),\varepsilon/4)$. Set $\delta = \min\{\delta',\varepsilon/4\}$. Since (X,T) has x_1-rank K, by Proposition 3.14, there exists a clopen set A satisfying the (x_1,δ)-rank K condition. Furthermore, if we let $\mathcal{P}$ denote the tower partition over A given by the definition of the (x_1,δ)-rank K condition, then by Proposition 2.2, A can be chosen so that $\mathcal{H}(\mathcal{P}) > m$.

Let A and $\mathcal{P}$ be as described in the preceding paragraph with $\mathcal{H}(\mathcal{P}) > m$. We pick one tower floor from each tower of $\mathcal{P}$ in the following way. Supppose that $\mathcal{P}$ partitions A into $L \leq K$ clopen sets $A_1,\dots,A_L$. For each $i \in \{1,\dots,L\}$, let the tower over A_i in $\mathcal{P}$ have height $r_i > 0$. Let $i_0 \in \{1,\dots,L\}$ such that Tx_0 is in the same tower as A_{i_0}. Let B_{i_0} be the tower floor in the tower over A_{i_0} that contains Tx_0. For all $i \in \{1,\dots,L\}$, $i \neq i_0$, let B_i be the tower floor of height $m+1$ in the tower over A_i.

Set $B = \bigcup_{i=1}^{L} B_i$. For each $i = 1,\dots,L$, we will partition B_i into L subsets determined by which A_j it intersects when it first returns to A under T, i.e. for a fixed $i \leq L$, set $B_{ij} = \{x \in B_i \mid$ the first time x returns to A under T, it returns to $A_j\}$ with $j \in \{1,\dots,L\}$. This partitions B into $L^2 \leq K^2$ clopen sets. We will now show that B satisfies the properties desired. By the definition of the B_{ij} sets, clearly each one has a constant T-return time to B. Each iteration of a B_{ij} set under T before returning to B is a subset of some $T^l(A_k) \in \mathcal{P}$ with $k \leq L$ and $l \leq r_k - 1$. Because $\operatorname{diam}(\mathcal{P}) < \delta < \varepsilon$, for all $i,j \in \{1,\dots,L\}$, $\operatorname{diam}(B_{ij}) < \varepsilon$. We only have left to show that $\operatorname{diam}(B) < \varepsilon$. Fix $x,y \in B$. There are three cases that need to consider.

Case 1. Suppose $x,y \in B_{i_0}$. Since B_{i_0} a tower floor in $\mathcal{P}$ and $\operatorname{diam}(\mathcal{P}) < \delta \leq \varepsilon/4$, we have $d_X(x,y) < \varepsilon/4$.

Case 2. Suppose $x \in B_i$ and $y \in B_j$ where $i,j \neq i_0$. Then $B_i, B_j \subset T^m(A)$, so there exist $x',y' \in A$ such that $T^m(x') = x$ and $T^m(y') = y$. Since $\operatorname{diam}(A) < \delta$, we have

$$\begin{aligned} d_X(x,y) &= d_X(T^m(x'),T^m(y')) \\ &\leq d_X(T^m(x'),T^m(Tx_1)) + d_X(T^m(Tx_1),T^m(y')) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

Case 3. Suppose $x \in B_{i_0}$ and $y \in B_j$ with $j \neq i_0$. Since $B_j \subset T^m(A)$, there exists a $y' \in A$ such that $T^m(y') = y$. Then, we have that

$$\begin{aligned} d_X(x,y) &= d_X(x,T^m(y')) \\ &\leq d_X(x,T^m(Tx_1)) + d_X(T^m(Tx_1),T^m(y')) \end{aligned}$$

$$\le d_X(x, Tx_0) + d_X(Tx_0, T^m(Tx_1)) + d_X(T^m(Tx_1), T^m(y'))$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3}{4}\varepsilon.$$

This shows that diam $(B) < \varepsilon$ and thus (X,T) satisfies the (x_0,ε)-rank K^2 property. Since ε was chosen arbitrarily, by Proposition 3.14 (X,T) has x_0-rank less than or equal to K^2. □

3.4. Residuality and finite rank systems.

THEOREM 3.16. *If (X,T) has finite rank, then the set of finite rank systems $\mathcal{F}(T)$ is residual in $\mathcal{S}(T)$, i.e. $\mathcal{F}(T)$ contains a dense G_δ.*

Before we prove this theorem, we need following lemma.

LEMMA 3.17. *Let $S \in \mathcal{S}(T)$ and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a clopen partition of X. There exists an $\varepsilon > 0$ such that if $m_T(S', S) < \varepsilon$, then $S'(P_i) = S(P_i)$ for $i = 1, \dots, n$.*

PROOF. Since S is a homeomorphism the set $\{S(P_1), \dots, S(P_n)\}$ is a clopen partition of X, so for $i \ne j$, $d_X(S(P_i), S(P_j)) > 0$.

Define $\varepsilon = \min_{i \ne j} d_X(S(P_i), S(P_j))$. If $m_T(S', S) < \varepsilon$, then

$$\sup_{x \in X} d_X(S'x, Sx) < \varepsilon,$$

and so $S'(P_i) \subset S(P_i)$ for $i = 1, \dots, n$. Since S' is a homeomorphism, we have $S'(P_i) = S(P_i)$ for $i = 1, \dots, n$ finishing the proof. □

PROOF OF THEOREM 3.16. Let $\mathcal{F}_K(T,\varepsilon)$ denote the systems that satisfy the (x_0,ε)-rank K condition. By Proposition 3.14,

$$\bigcap_{n=1}^{\infty} \mathcal{F}_K(T, 1/n) = \mathcal{F}_K(T, x_0)$$

where $\mathcal{F}_K(T, x_0)$ is the set of systems that have x_0-rank less than or equal to K. Since $\mathcal{F}_K(T, x_0) \subset \mathcal{F}(T)$, if we can show that each $\mathcal{F}_K(T, 1/n)$ is an open dense set in $\mathcal{S}(T)$, by the Baire Category Theorem, we will have that $\mathcal{F}(T)$ is residual in $\mathcal{S}(T)$.

We will show that for all $\varepsilon > 0$, the set $\mathcal{F}_K(T,\varepsilon)$ is dense in $\mathcal{S}(T)$. By Proposition 3.15, (X,T) has x_0-finite rank. Therefore, there exists a $K > 0$ such that (X,T) can be has a Bratteli diagram representation B with K or fewer vertices at each level and with maximal path x_0. For all n, let $\mathcal{P}_n$ denote the tower partition of X over the union of minimal path cylinders sets in B down to level n. Then $\{\mathcal{P}_n\}$ is a generating sequence of tower partitions, so by Theorem 3.11, $\mathcal{D}(T, \{\mathcal{P}_n\})$ is dense in $\mathcal{S}(T)$. We claim that $\mathcal{D}(T, \{\mathcal{P}_n\}) \subset \mathcal{F}_K(T, x_0)$. If $\phi T \phi^{-1} \in \mathcal{D}(T, \{\mathcal{P}_n\})$, then there exists some $k \in \mathbb{N}^+$ such that the map $\phi T \phi^{-1}$ is created by rearranging the tower floors of $\mathcal{P}_k$ (excluding the top and bottom floors of $\mathcal{P}_k$). But a rearrangement of the tower floors of $\mathcal{P}_k$ is equivalent to reordering paths of B down to level k (excluding the minimal and maximal paths). Therefore, by reordering paths of B down to level k, we can obtain a Bratteli diagram representation B' of $(X, \phi T \phi^{-1})$. Since the number of vertices at each level of B' is equal to the number of vertices at each corresponding level of B and the maximal path of B' is x_0 (since no minimal or maximal paths were reordered), we have that $(X, \phi T \phi^{-1})$ has x_0-rank less than or equal to K. Therefore, $\phi T \phi^{-1} \in \mathcal{F}(T, x_0)$ proving the claim. Since for all $\varepsilon > 0$, $\mathcal{F}_K(T, x_0) \subset \mathcal{F}_K(T, \varepsilon)$, we have that $\mathcal{F}_K(T, \varepsilon)$ is dense in $\mathcal{S}(T)$.

It remains to be shown that for all $\varepsilon > 0$, the set $\mathcal{F}_K(T, \varepsilon)$ is open in $\mathcal{S}(T)$. Let $S \in \mathcal{F}_K(T, \varepsilon)$. Let $\mathcal{P}$ be the tower partition of (X, S) given by the definition of the (x_0, ε)-rank K condition. By Lemma 3.17, there exists an $\varepsilon' > 0$ such that if $m_T(S, S') < \varepsilon'$, then $S(P) = S'(P)$ for all $P \in \mathcal{P}$. Therefore if $m_T(S, S') < \varepsilon$, then S' also satisfies the (x_0, ε)-rank K condition with the same partition $\mathcal{P}$. This shows that $\mathcal{F}_K(T, \varepsilon)$ is open in $\mathcal{S}(T)$ finishing the proof. □

Corollary 3.18. *If (X, T) is an odometer, then odometers are residual in $\mathcal{S}(T)$.*

Proof. In the proof of Theorem 3.16, it was shown that if (X, T) has rank K, then the systems with x_0-rank less than or equal to K are residual in $\mathcal{S}(T)$. If (X, T) is an odometer, it has rank 1 and thus odometers are residual in $\mathcal{S}(T)$. □

3.5. Entropy. We will define entropy as done in [10]. Let (X, T) be a minimal Cantor system (this definition is the same for any topological dynamical system). If α and β are open covers of X, their *join* $\alpha \vee \beta$ is the open cover containing sets of the form $A \cap B$ where $A \in \alpha$ and $B \in \beta$. The join of any finite number of open covers $\bigvee_{i=1}^{n} \alpha_i$ is defined similarly. If α is an open cover of X, $T^{-1}\alpha$ will denote the open cover of X containing sets of the form $T^{-1}A$ where $A \in \alpha$. Let $N(\alpha)$ denote the number of sets in a subcover of α with minimal cardinality. If we let $H(\alpha) = \log N(\alpha)$, the *entropy of* (X, T) *relative to* α is given by

$$h(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right).$$

In [10], it is shown that this limit exists and $h(T, \alpha) \leq H(\alpha)$. The *topological entropy* of (X, T) is defined as $h(T) = \sup_\alpha h(T, \alpha)$ where α ranges over all open covers of X. Topological entropy is an invariant under conjugacy.

If $\mathcal{P} = \{P_1, \ldots, P_n\}$ is a clopen partition of X, then $N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P})$ is the number of T-itineraries of length n through $\mathcal{P}$. Let $\pi_T(\mathcal{P})$ denote the shift space of itineraries through $\mathcal{P}$, i.e. for $x \in X$ and $i \in \mathbb{Z}$, set $x_i = j \in \{1, \ldots, n\}$ where $T^i x \in P_j$. Then $\pi_T(\mathcal{P})$ is system consisting of the space $\{\ldots x_{-2}x_{-1}.x_0x_1x_2 \ldots \mid$

$x \in X\}$ along with the shift map. Theorem 7.13 of [10] shows that if (Y, S) is a shift space, then

$$h(S) = \lim_{n\to\infty} \frac{\log|\mathcal{W}_n(Y)|}{n}$$

where $\mathcal{W}_n(Y)$ is the set of words of length n in Y. By the preceding statements, we have that $h(T, \mathcal{P}) = h(\pi_T(\mathcal{P}))$, or equivalently

$$h(T, \mathcal{P}) = \lim_{n\to\infty} \frac{\log|\mathcal{W}_n(\pi_T(\mathcal{P}))|}{n}.$$

3.6. Residuality and entropy. Fix a sequence of clopen sets $\{A_k\}$ contained in X such that $A_{k+1} \subset A_k$ and $\operatorname{diam}(A_k) \to 0$. Let $\{\mathcal{P}_l\}$ be a sequence of clopen partitions (not necessarily tower partitions) of X that generates the topology of X. It follows from Theorem 7.6 of [10] that $\lim_{l\to\infty} h(S, \mathcal{P}_l) = h(S)$ for any $S \in \mathcal{S}(T)$. For each pair $k, l \in \mathbb{N}^+$ and $S \in \mathcal{S}(T)$, we will define a shift space that describes how points of A_k move through the partition $\mathcal{P}_l$. Fix $k, l \in \mathbb{N}^+$ and let $x \in A_k$ with T-return time $r > 0$ to A_k. If $\mathcal{P}_l = \{P_1, \dots, P_n\}$. We define $w_S(k,l)(x) = x_0 \dots x_{r-1}$ where $x_i = j \in \{1, \dots, n\}$ if and only if $T^i x \in P_j$. Let $\mathcal{W}_S(k,l) = \{w_S(k,l)(x) \mid x \in A_k\}$, and we define $\pi_S(k,l)$ to be the shift space of all bi-infinite words that can be formed by concatenating words in $\mathcal{W}_S(k,l)$.

Proposition 3.19. *Let $S \in \mathcal{S}(T)$. For all $k > 0$, there exists an $\varepsilon > 0$ such that if $m_T(S', S) < \varepsilon$, then $\mathcal{W}_S(k,l) = \mathcal{W}_{S'}(k,l)$.*

Proof. This follows directly from Lemma 3.17. □

Theorem 3.20 (Lind and Marcus from [7]). *Let $\pi_1 \supset \pi_2 \supset \pi_3$ be shift spaces whose intersection is π. Then $\lim_{k\to\infty} h(\pi_k) = h(\pi)$.*

Lemma 3.21. *The sequence $\{h(\pi_S(k,l))\}_{k=1}^{\infty}$ is decreasing and*

$$\lim_{k\to\infty} h(\pi_S(k,l)) = h(S, \mathcal{P}_l).$$

proof. If $k' > k$, the words in $\mathcal{W}_S(k', l)$ are concatenations of the words in $\mathcal{W}_S(k,l)$, so $\pi_S(k', l) \subset \pi_S(k,l)$. Therefore, $h(\pi_S(k', l)) \le h(\pi_S(k,l))$. Since $h(S, \mathcal{P}_l) = h(\pi_S(\mathcal{P}_l))$, if we can show that $\bigcap_k \pi_S(k,l) = \pi_S(\mathcal{P}_l)$, the limit statement holds by Theorem 3.20.

If $\mathcal{P}_l = \{P_1, \dots, P_n\}$, then $\bigcap_k \pi_S(k,l)$ and $\pi_S(\mathcal{P}_l)$ are both closed subspaces of the full shift $\{1, \dots, n\}^{\mathbb{Z}}$. Therefore, in order to show that $\bigcap_k \pi_S(k,l) = \pi_S(\mathcal{P}_l)$, it suffices show that any finite word appearing in one space also appears in the other. It is clear that any finite word appearing in $\pi_S(\mathcal{P}_l)$ also appears $\bigcap_k \pi_S(k,l)$ because if some point in X follows a particular S-itinerary through $\mathcal{P}_l$, then that same point follows the same S-itinerary through every tower partition of (X, S).

We will now show that any finite word appearing in $\bigcap_k \pi_S(k,l)$ also appears in $\pi_S(\mathcal{P}_l)$. Let $w = w_0 \dots w_{n-1}$ be a finite word that appears in $\bigcap_k \pi_S(k,l)$.

Pick $K > 0$ such that if $k \geq K$, then each of the sets $A_k, S(A_k), \ldots, S^{n-1}(A_k)$ is contained in only one element of the partition $\mathcal{P}_l$. For $j = 0, \ldots, n-1$, say $S^j(A_k) \subset P_{i_j} \in \mathcal{P}_l$. Because of the way K was chosen, every word in $\mathcal{W}_S(K,l)$ must begin with the subword $i_0 i_1 \ldots i_{n-1}$. Since w appears in $\bigcap_k \pi_S(k,l)$, in particular it is a subword of some concatenation of words in $\mathcal{W}_S(K,l)$. If w is a subword of a single word in $\mathcal{W}_S(K,l)$, then clearly w appears in $\pi_S(\mathcal{P}_l)$. If w is a subword of the concatenation of multiple words in $\mathcal{W}_S(K,l)$, let m be the minimal positive integer such that w_m is the first symbol of a new word in $\mathcal{W}_S(K,l)$. Because w_m is the first symbol of a word in $\mathcal{W}_S(K,l)$, we have that $w_j = i_{j-m}$ for $j = m, \ldots, n-1$. Since $w_0 \ldots w_{m-1}$ is a subword of a single word in $\mathcal{W}_S(K,l)$, there exists $x \in X$ with S-itinerary $w_0 w_1 \ldots w_{m-1}$ through $\mathcal{P}_l$, i.e. $S^j(x) \in P_{w_j}$ for $j = 0, \ldots, m-1$. Because w_{m-1} is the last symbol of a word in $\mathcal{W}_S(K,l)$, we also have that $S^m(x) \in A_K$. Then for $j = m, \ldots, n-1$, $S^j(x) \in S^{j-m}(A_K) \subset P_{i_{j-m}} = P_{w_j}$. Therefore, x has exactly the S-itinerary $w_0 \ldots w_{n-1}$ through the partition $\mathcal{P}_l$ showing that w does appear in $\pi_S(\mathcal{P}_l)$ and finishing the proof. □

LEMMA 3.22. *Let $l \in \mathbb{N}^+$ and $p > 0$, then the set $\mathcal{S}(p,l) = \{S \in \mathcal{S}(T) \mid h(S,\mathcal{P}_l) < p\}$ is open in $\mathcal{S}(T)$.*

PROOF. Let $S \in \mathcal{S}(T)$ with $h(S,\mathcal{P}_l) < p$. By Lemma 3.21, there exists a K such that if $k \geq K$, then $h(\pi_S(k,l)) < p$. By Proposition 3.19, there exists $\varepsilon > 0$ such that if $m_T(S',S) < \varepsilon$, then $\mathcal{W}_{S'}(K,l) = \mathcal{W}_S(K,l)$. Then $h(\pi_{S'}(K,l)) = h(\pi_S(K,l)) < p$. Since $\{h(\pi_{S'}(k,l))\}_{k=1}^{\infty}$ is decreasing and converges to $h(S',\mathcal{P}_l)$, $h(S',\mathcal{P}_l) < p$. □

THEOREM 3.23 (Boyle and Handelman from [2]). *Any minimal Cantor system is strongly orbit equivalent to a system with zero entropy.*

THEOREM 3.24. *For any minimal Cantor system (X,T), the set of maps in $\mathcal{S}(T)$ with zero entropy is residual.*

PROOF. By Theorem 3.23, $\mathcal{S}(T)$ contains a system with zero entropy. By Corollary 3.12, the conjugacy class of this zero entropy dense is dense in $\mathcal{S}(T)$. Since entropy is invariant under conjugacy, the systems with zero entropy are dense in $\mathcal{S}(T)$. It follows from the definition of entropy that if $S \in \mathcal{S}(T)$ with $h(S) = 0$, then $h(S,\mathcal{P}) = 0$ for any clopen partition $\mathcal{P}$ of X. Therefore, if l is a positive integer and $p > 0$, $\mathcal{S}(p,l)$ contains all systems in $\mathcal{S}(T)$ with zero entropy; therefore, $S(p,l)$ is dense in $\mathcal{S}(T)$. Define

$$\mathcal{S}(l) = \bigcap_{n=1}^{\infty} \mathcal{S}(n^{-1},l).$$

From the previous statement and Lemma 3.22, we can conclude that $\mathcal{S}(l)$ is residual in $\mathcal{S}(T)$. Furthermore, $\mathcal{S}(l) = \{S \in \mathcal{S}(T) \mid h(S,\mathcal{P}_l) = 0\}$. Since

$\lim_{l\to\infty} h(S,\mathcal{P}_l) = h(S)$ for all $S \in \mathcal{S}(T)$, we have that $\bigcap_{l=1}^{\infty} \mathcal{S}(l) = \{S \in \mathcal{S}(T) \mid h(S) = 0\}$. Because the countable intersection of residual sets is residual, the theorem is proven. $\square$

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