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MULTIPLICITY OF NONRADIAL SOLUTIONS FOR A CLASS OF QUASILINEAR EQUATIONS ON ANNULUS WITH EXPONENTIAL CRITICAL GROWTH

Claudianor O. Alves — Luciana R. de Freitas

Abstract. In this paper, we establish the existence of many rotationally non-equivalent and nonradial solutions for the following class of quasilinear problems

$$\text{(P)} \qquad \begin{cases} -\Delta_N u = \lambda f(|x|, u) & x \in \Omega_r, \\ u > 0 & x \in \Omega_r, \\ u = 0 & x \in \partial\Omega_r, \end{cases}$$

where $\Omega_r = \{x \in \mathbb{R}^N : r < |x| < r+1\}$, $N \geq 2$, $N \neq 3$, $r > 0$, $\lambda > 0$, $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator and f is a continuous function with exponential critical growth.

1. Introduction

This article concerns with the multiplicity of nonradial solutions for the quasilinear problem

$$\text{(P)} \qquad \begin{cases} -\Delta_N u = \lambda f(|x|, u) & x \in \Omega_r, \\ u > 0 & x \in \Omega_r, \\ u = 0 & x \in \partial\Omega_r, \end{cases}$$

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where λ is a positive parameter and Ω_r is an annulus of the form

$$\Omega_r = \{x \in \mathbb{R}^N : r < |x| < r+1\} \quad r > 0,\ N \geq 2,\ N \neq 3.$$

We assume that f is a continuous function with exponential critical growth (see [1], [10], [12]), more precisely:

(H_0) There exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(|x|,s)|}{e^{\alpha|s|^{N/(N-1)}}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \Omega_r$.

We also assume that f satisfies the following conditions:

(H_1) $\displaystyle\lim_{s\to 0} \frac{f(|x|,s)}{|s|^{N-1}} = 0$, uniformly in $x \in \Omega_r$.

(H_2) There exists $\nu > N$ such that

$$0 < \nu F(|x|,s) \leq f(|x|,s)s, \quad \text{for all } |s| > 0 \text{ and all } x \in \Omega_r,$$

where $F(|x|,s) = \int_0^s f(|x|,t)\,dt$.

(H_3) There exist $p > N$ and $C_p > 0$ such that

$$f(|x|,s) \geq C_p s^{p-1}, \quad \text{for all } s \geq 0 \text{ and all } x \in \Omega_r.$$

(H_4) There exist $\sigma \geq N$ and a constant $C_\sigma > 0$ such that

$$\frac{\partial f}{\partial s}(|x|,s)s - (N-1)f(|x|,s) \geq C_\sigma s^\sigma, \quad \text{for all } s \geq 0 \text{ and all } x \in \Omega_r.$$

Since we are looking for positive solutions, hereafter $f(|x|,s) = 0$ in $\Omega_r \times (-\infty,0)$.

Consider the following problem:

$$\begin{cases} -\Delta u + u - u^p = 0 & x \in D, \\ u = 0 & x \in \partial D. \end{cases} \tag{1.1}$$

According B. Gidas, W.N. Ni and L. Nirenberg [15], when $D \subset \mathbb{R}^N$ is the unit ball and $1 < p < 2^* - 1$, if $N \geq 3$ or $p > 1$, if $N = 2$, any positive solution of class C^2 of (1.1) must be radially symmetric. However, if D is an annulus, say

$$D = \{x \in \mathbb{R}^N : r^2 < |x|^2 < (r+d)^2\},$$

we have a phenomenon known as symmetry breaking observed by H. Brezis and L. Niremberg [3]. More precisely, in [3] the authors proved that for $N \geq 3$ the problem (1.1) admits both radial and nonradial positive solutions, for all $p < 2^* - 1$ sufficiently close to $2^* - 1$. C. Coffman in [8] proved that the number of nonradial and rotationally non-equivalent positive solutions of (1.1) in D tends to $+\infty$ as r tends to $+\infty$, if $p > 1$ and $N = 2$ or $1 < p < N/(N-2)$ and $N \geq 3$.

Motivated by the above papers, some authors have studied this class of problem. For the subcritical case, we cite the papers of Y.Y. Li [19], T. Suzuki [26], S.S. Lin [20] and therein references.

Related to the critical case, Z. Wang and M. Willem [30] have showed the existence of multiple solutions for the following problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1}, & u > 0, \quad x \in \Omega_r, \\ u = 0 & x \in \partial\Omega_r, \end{cases} \tag{1.2}$$

where $\Omega_r = \{x \in \mathbb{R}^N : r < |x| < r+1\}$, $N \geq 4$. The authors proved that for $0 < \lambda < \pi^2$ and $n \geq 1$, there exists $R(\lambda, n)$ such that for $r > R(\lambda, n)$, the equation (1.2) has at least n nonradial and rotationally non-equivalent solutions. Motivated by [30], D.G. de Figueiredo and O.H. Miyagaki [9] have considered the following problem

$$\begin{cases} -\Delta u = f(|x|, u) + u^{2^*-1}, & u > 0, \quad x \in \Omega_r, \\ u = 0 & x \in \partial\Omega_r, \end{cases} \tag{1.3}$$

where f is a C^1 function with subcritical growth.

Still related to this class of problem, we would like to cite the papers of J. Byeon [4], A. Castro and B.M. Finan [6], F. Catrina and Z.-Q. Wang [7], N. Mizoguchi and T. Suzuki [21], N. Hirano and N. Mizoguchi [16] and references therein.

The present paper was motivated by the fact that we did not find in the literature any article dealing with the existence of multiple nonradial and rotationally non-equivalent positive solutions for problem (P) involving a nonlinearity with exponential critical growth. Here, we adapt some some arguments used in [8] and [30]. However, since we are working with exponential critical growth, we modified the proof of some estimates found in those papers.

Our main result is the following:

Theorem 1.1. *Suppose that f is a function satisfying* (H_0)–(H_4). *Then, for each $n \in \mathbb{N}$, there exist $r_0 = r_0(n) > 0$ and $\lambda_0 = \lambda_0(n) > 0$ such that for $\lambda \geq \lambda_0$ and $r \geq r_0$, the problem* (P) *has at least n nonradial and rotationally non-equivalent solutions.*

For the reader interested in the study of quasilinear problems involving the N-Laplacian operator and nonlinearity with critical exponential growth, we cite the papers of Adimurthi [1], C.O. Alves and G.M. Figueiredo [2], E.A.B. Silva and S.H.M. Soares [25], Bezerra J.M.B. do Ó, E. Medeiros and U. Severo [13], Y. Wang, J. Yang and Y. Zhang [31], E. Tonkes [27], R. Panda [24] and therein references.

2. Technical results involving exponential critical growth

We begin this section recalling the Trudinger–Moser inequality (see N. Trudinger [28] and J. Moser [22]), which will be essential to carry out the proof of our results.

Lemma 2.1 (Trudinger–Moser inequality for bounded domains). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain. Given any $u \in W_0^{1,N}(\Omega)$, we have*

$$\int_\Omega e^{\alpha |u|^{N/(N-1)}}\,dx < \infty, \quad \textit{for every } \alpha > 0.$$

Moreover, there exists a positive constant $C = C(N, |\Omega|)$ such that

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_\Omega e^{\alpha |u|^{N/(N-1)}}\,dx \leq C, \quad \textit{for all } \alpha \leq \alpha_N = N\omega_{N-1}^{1/(N-1)} > 0,$$

where ω_{N-1} is the $(N-1)$-dimensional measure of the $(N-1)$-sphere.

The next result is a version of the Trudinger–Moser inequality for whole $\mathbb{R}^N$, and its proof can be found in D.M. Cao [5], for $N = 2$, and do J.M.B. Ó [11], for $N \geq 2$.

Lemma 2.2 (Trudinger–Moser inequality for unbounded domains). *Given any $u \in W^{1,N}(\mathbb{R}^N)$ with $N \geq 2$, we have*

$$\int_{\mathbb{R}^N} \Big(e^{\alpha |u|^{N/(N-1)}} - S_{N-2}(\alpha, u)\Big)\,dx < \infty,$$

for every $\alpha > 0$. Moreover, if $|\nabla u|_N^N \leq 1$, $|u|_N \leq M < \infty$ and $\alpha < \alpha_N = N\omega_{N-1}^{1/(N-1)}$, then there exists a positive constant $C = C(N, M, \alpha)$ such that

$$\int_{\mathbb{R}^N} \Big(e^{\alpha |u|^{N/(N-1)}} - S_{N-2}(\alpha, u)\Big)\,dx \leq C,$$

where

$$S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |u|^{Nk/(N-1)}$$

and ω_{N-1} is the $(N-1)$-dimensional measure of the $(N-1)-$sphere.

In the sequel, we prove some technical lemmas which will be used in the proof of the some estimates later on.

Lemma 2.3. *Let $\alpha > 0$ and $r > 1$. Then, for every $\beta > r$, there exists a constant $C = C(\beta) > 0$ such that*

$$\Big(e^{\alpha |s|^{N/(N-1)}} - S_{N-2}(\alpha, s)\Big)^r \leq C\Big(e^{\beta\alpha |s|^{N/(N-1)}} - S_{N-2}(\beta\alpha, s)\Big).$$

PROOF. In order to simplify notation, we write $y = |s|^{N/(N-1)}$ and

$$\widetilde{S}(\alpha, y) = \sum_{k=0}^{N-2} \frac{\alpha^k y^k}{k!}.$$

Observing that

$$\frac{\big(e^{\alpha y} - \widetilde{S}(\alpha, y)\big)^r}{\big(e^{\beta\alpha y} - \widetilde{S}(\beta\alpha, y)\big)} = \frac{\bigg(\displaystyle\sum_{k=N-1}^{\infty} \frac{\alpha^k y^k}{k!}\bigg)^r}{\displaystyle\sum_{k=N-1}^{\infty} \frac{(\beta\alpha)^k y^k}{k!}} = \frac{y^{r(N-1)}\bigg(\displaystyle\sum_{k=N-1}^{\infty} \frac{\alpha^k y^{k-N+1}}{k!}\bigg)^r}{y^{N-1}\displaystyle\sum_{k=N-1}^{\infty} \frac{(\beta\alpha)^k y^{k-N+1}}{k!}},$$

we deduce that

$$\lim_{y\to 0} \frac{\Big(e^{\alpha y} - \widetilde{S}(\alpha, y)\Big)^r}{\Big(e^{\beta\alpha y} - \widetilde{S}(\beta\alpha, y)\Big)} = 0.$$

Furthermore,

$$\lim_{y\to\infty} \frac{\big(e^{\alpha y} - \widetilde{S}(\alpha, y)\big)^r}{\big(e^{\beta\alpha y} - \widetilde{S}(\beta\alpha, y)\big)} = \frac{e^{\alpha r y}\bigg(1 - \dfrac{\widetilde{S}(\alpha, y)}{e^{\alpha y}}\bigg)^r}{e^{\beta\alpha y}\bigg(1 - \dfrac{\widetilde{S}(\beta\alpha, y)}{e^{\alpha y}}\bigg)} = 0,$$

and the lemma follows. □

LEMMA 2.4. *Let (u_n) be a sequence in $W^{1,N}(\mathbb{R}^N)$ with*

$$\limsup_{n\to+\infty} \|u_n\|^N < \bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{N-1}.$$

Then, there exist $\alpha > \alpha_0$, $t > 1$ and $C > 0$ independent of n, such that

$$\int_{\mathbb{R}^N} \Big(e^{\alpha|u_n|^{N/(N-1)}} - S_{N-2}(\alpha, u_n)\Big)^t \,dx \le C, \quad \textit{for all } n \ge n_0,$$

for some n_0 sufficiently large.

PROOF. Since

$$\limsup_{n\to\infty} ||u_n||^N < \bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{N-1},$$

there are $m > 0$ and $n_0 \in \mathbb{N}$ such that

$$||u_n||^{N/(N-1)} < m < \frac{\alpha_N}{\alpha_0}, \quad \text{for all } n \ge n_0.$$

Choose $\alpha > \alpha_0$, $t > 1$ and $\beta > t$ satisfying $\alpha m < \alpha_N$ and $\beta\alpha m < \alpha_N$. From Lemma 2.3, there exists $C = C(\beta)$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} &\Big(e^{\alpha|u_n|^{N/(N-1)}} - S_{N-2}(\alpha, u_n)\Big)^t \,dx \\ &\le C \int_{\mathbb{R}^N} \bigg(e^{\beta\alpha m(|u_n|/||u_n||)^{N/(N-1)}} - S_{N-2}\bigg(\beta\alpha m, \frac{|u_n|}{||u_n||}\bigg)\bigg) \,dx,\end{aligned}$$

for every $n \geq n_0$. Hence, by Lemma 2.2, there exists $C > 0$ independent of n such that

$$\int_{\mathbb{R}^N} \left(e^{\alpha|u_n|^{N/(N-1)}} - S_{N-2}(\alpha, u_n)\right)^t dx \leq C, \quad \text{for all } n \geq n_0,$$

which completes the proof. □

The same arguments used in the proof of the last lemma can be used to prove the following corollary:

Corollary 2.5. *Let B a bounded domain in $\mathbb{R}^N$ and (u_n) be a sequence in $W_0^{1,N}(B)$ with*

$$\limsup_{n\to+\infty} \|u_n\|^N < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Then, there exist $\alpha > \alpha_0$, $t > 1$ and $C > 0$ independent of n, such that

$$\int_B e^{t\alpha|u_n|^{N/(N-1)}} \, dx \leq C, \quad \textit{for all } n \geq n_0,$$

for some n_0 sufficiently large.

3. Preliminares

In what follows, $O(N)$ denotes the group of $N \times N$ orthogonal matrices. For any integer $k \geq 1$, let us consider the finite rotational subgroup O_k of $O(2)$ given by

$$O_k := \left\{ g \in O(2) : g(x) = \left(x_1 \cos\frac{2\pi l}{k} + x_2 \mathrm{sen}\frac{2\pi l}{k}, -x_1 \mathrm{sen}\frac{2\pi l}{k} + x_2 \cos\frac{2\pi l}{k}\right)\right\},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $l \in \{0, \dots, k-1\}$. We define the subgroups of $O(N)$

$$G_k := O_k \times O(N-2), \quad 1 \leq k < \infty \quad \text{and} \quad G_\infty := O(2) \times O(N-2).$$

Associated with the above subgroups, we set the subspaces

$$W_{0,G_k}^{1,N}(\Omega_r) := \{u \in W_0^{1,N}(\Omega_r) : u(x) = u(g^{-1}x), \text{ for all } g \in G_k\}, \quad 1 \leq k \leq \infty,$$

endowed with the usual norm of $W_0^{1,N}(\Omega_r)$, that is,

$$\|u\| = \left(\int_{\Omega_r} |\nabla u|^N \, dx\right)^{1/N}, \quad u \in W_{0,G_k}^{1,N}(\Omega_r), \ 1 \leq k \leq \infty.$$

The above subspaces verify the following compact embeddings, whose proof can be found in [32]

$$W_{0,G_k}^{1,N}(\Omega_r) \hookrightarrow L^t(\Omega_r), \quad 1 \leq t < \infty, \quad 1 \leq k \leq \infty$$

and

$$W_{G_\infty}^{1,N}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N), \quad N < t < \infty.$$

Hereafter, we denote by $I_\lambda \colon W^{1,N}_{0,G_k}(\Omega_r) \to \mathbb{R}$ the functional given by

$$I_\lambda(u) = \frac{1}{N}\int_{\Omega_r} |\nabla u|^N \, dx - \lambda \int_{\Omega_r} F(|x|, u)\, dx$$

and by $J_{k,r}$ the following real number

$$J_{k,r} := \inf_{u \in \mathcal{M}_{k,r}} I_\lambda(u),$$

where $\mathcal{M}_{k,r} := \{u \in W^{1,N}_{0,G_k}(\Omega_r) \setminus \{0\} : I'_\lambda(u)u = 0\}$.

The next lemma is a version of Poincaré's inequality, which is a key point in our study.

Lemma 3.1 (Poincaré's inequality).

$$\int_{\Omega_r} |u(z)|^N dz \leq \left(\frac{r+1}{r}\right)^{N-1} \int_{\Omega_r} |\nabla u(z)|^N \, dz, \quad \textit{for all } u \in W^{1,N}_0(\Omega_r).$$

Proof. Note that for $\psi \in C_0^\infty((r, r+1))$,

$$\psi(t) = \int_r^t \psi'(s) ds, \quad r \leq t \leq r+1.$$

Thus, applying the Hölder's inequality

$$|\psi(t)| \leq \int_r^{r+1} |\psi'(s)|\, ds \leq \left(\int_r^{r+1} |\psi'(s)|^N \, ds\right)^{1/N} \left(\int_r^{r+1} ds\right)^{(N-1)/N},$$

that is

$$|\psi(t)|^N \leq \int_r^{r+1} |\psi'(s)|^N \, ds,$$

which implies,

$$\int_r^{r+1} |\psi(t)|^N \, dt \leq \int_r^{r+1} |\psi'(t)|^N \, dt, \quad \text{for all } \psi \in C_0^\infty((r, r+1)). \tag{3.1}$$

Consider the hyperspherical coordinates $z = (\rho, \theta_1, \ldots, \theta_{N-1})$ of the $z \in \Omega_r$, which consists of a radial coordinate $r < \rho < r+1$ and $N-1$ angular coordinates $\theta_1, \ldots, \theta_{N-1}$, with $0 \leq \theta_j \leq \pi$, $j = 1, \ldots, N-2$ and $0 \leq \theta_{N-1} \leq 2\pi$. If $z = (z_1, \ldots, z_N)$ is written in the cartesian coordinates, we have

$$\begin{aligned}
z_1 &= \rho \cos\theta_1 \\
z_2 &= \rho \operatorname{sen}, \theta_1 \cos\theta_2 \\
z_3 &= \rho \operatorname{sen}\theta_1 \operatorname{sen}\theta_2 \cos\theta_3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
z_{N-1} &= \rho \operatorname{sen}\theta_1 \ldots \operatorname{sen}\theta_{N-2} \cos\theta_{N-1}, \\
z_N &= \rho \operatorname{sen}\theta_1 \ldots \operatorname{sen}\theta_{N-2} \operatorname{sen}\theta_{N-1}.
\end{aligned}$$

For simplicity, we denote $\theta := (\theta_1, \dots, \theta_{N-1})$, $d\theta := d\theta_1 \dots d\theta_{N-1}$ and

$$\text{sen}(\theta_1, \dots, \theta_{N-1}) = \text{sen}^{N-2}\theta_1 \text{sen}^{N-3}\theta_2 \dots \text{sen}\theta_{N-2}.$$

For each $\varphi \in C_0^\infty(\Omega_r)$, $\varphi(z) = \varphi(\rho, \theta)$ and

$$\int_{\Omega_r} |\varphi(z)|^N \, dz = \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} \int_r^{r+1} |\varphi(\rho,\theta)|^N \rho^{N-1} \text{sen}(\theta_1, \dots, \theta_{N-1}) \, d\rho \, d\theta,$$

from where it follows that

$$\int_{\Omega_r} |\varphi(z)|^N \, dz \leq (r+1)^{N-1} \int \int_r^{r+1} |\varphi(\rho,\theta)|^N \text{sen}(\theta_1, \dots, \theta_{N-1}) \, d\rho \, d\theta. \tag{3.2}$$

For each θ, the function $\psi(\rho) := \varphi(\rho, \theta)$ belongs to $C_0^\infty((r, r+1))$. Thus, by (3.1),

$$\int_r^{r+1} |\psi(\rho)|^N \, d\rho \leq \int_r^{r+1} |\psi'(\rho)|^N \, d\rho,$$

that is,

$$\int_r^{r+1} |\varphi(\rho,\theta)|^N \, d\rho \leq \int_r^{r+1} |\varphi_\rho(\rho,\theta)|^N \, d\rho = \int_r^{r+1} \frac{1}{\rho^{N-1}} |\varphi_\rho(\rho,\theta)|^N \rho^{N-1} \, d\rho,$$

leading to

$$\int_r^{r+1} |\varphi(\rho,\theta)|^N \, d\rho \leq \frac{1}{r^{N-1}} \int_r^{r+1} |\varphi_\rho(\rho,\theta)|^N \rho^{N-1} \, d\rho. \tag{3.3}$$

From (3.2) and (3.3),

$$\int_{\Omega_r} |\varphi(z)|^N \, dz \leq \left(\frac{r+1}{r}\right)^{N-1} \int \int_r^{r+1} |\varphi_\rho(\rho,\theta)|^N \rho^{N-1} \text{sen}(\theta_1, \dots, \theta_{N-1}) \, d\rho \, d\theta.$$

Once that $\varphi_\rho^2 \leq |\nabla\varphi|^2$, the last inequality yields

$$\begin{aligned} &\int_{\Omega_r} |\varphi(z)|^N \, dz \\ &\qquad \leq \left(\frac{r+1}{r}\right)^{N-1} \int \int_r^{r+1} (|\nabla\varphi(\rho,\theta)|^2)^{N/2} \rho^{N-1} \text{sen}(\theta_1, \dots, \theta_{N-1}) \, d\rho \, d\theta. \end{aligned}$$

This way,

$$\int_{\Omega_r} |\varphi(z)|^N \, dz \leq \left(\frac{r+1}{r}\right)^{N-1} \int_{\Omega_r} |\nabla\varphi(z)|^N \, dz,$$

and the result follows by density. □

4. Properties of the levels $J_{k,r}$

LEMMA 4.1. *For each $1 \le k \le \infty$ and $r > 0$, we have $J_{k,r} > 0$.*

PROOF. If k and r are fixed, we claim that there exists $\eta > 0$ such that

$$||u||^N > \eta \quad \text{for all } u \in \mathcal{M}_{k,r}. \tag{4.1}$$

In fact, otherwise, there exists $(u_n) \subset \mathcal{M}_{k,r}$ with $||u_n|| \to 0$ as $n \to \infty$. So, there exists $n_0 \in \mathbb{N}$ such that

$$||u_n||^N < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \quad \text{for all } n \ge n_0,$$

that is,

$$\alpha_0 ||u_n||^{N/(N-1)} < \alpha_N \quad \text{for all } n \ge n_0.$$

Choose $\alpha > \alpha_0$ and $t_1 > 1$ such that $t_1 \alpha ||u_n||^{N/(N-1)} < \alpha_N$, for all $n \ge n_0$. By (H_0) and (H_1), for each $\varepsilon > 0$ and $s > N$, there exists $C_\varepsilon = C(\varepsilon, s) > 0$ such that

$$\begin{aligned} ||u_n||^N &= I'_\lambda(u_n)u_n + \lambda \int_{\Omega_r} f(|x|, u_n)u_n \, dx \\ &\le \varepsilon\lambda \int_{\Omega_r} |u_n|^N \, dx + \lambda C_\varepsilon \int_{\Omega_r} |u_n|^s e^{\alpha |u_n|^{N/(N-1)}} \, dx. \end{aligned}$$

Combining Poincaré's inequality with Hölder's inequality, and choosing ε sufficiently small, we deduce

$$C_1 ||u_n||^N \le C_2 ||u_n||^s \left(\int_{\Omega_r} e^{t_1 \alpha ||u_n||^{N/(N-1)} (|u_n|/||u_n||)^{N/(N-1)}} \, dx \right)^{1/t_1}.$$

The last inequality combined with Applying the Trudinger–Moser leads to

$$C_1 ||u_n||^N \le C_3 ||u_n||^s,$$

or more precisely $||u_n||^{s-N} \ge C_5$, for some positive constant C_5, which is a contradiction, because $||u_n|| \to 0$. Thus, (4.1) is proved.

By (H_2), for each $u \in \mathcal{M}_{k,r}$,

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{\nu} I'_\lambda(u)u \ge \left(\frac{1}{N} - \frac{1}{\nu}\right) ||u||^N > \left(\frac{1}{N} - \frac{1}{\nu}\right)\eta.$$

Therefore,

$$J_{k,r} \ge \left(\frac{1}{N} - \frac{1}{\nu}\right)\eta > 0, \quad \text{for all } 1 \le k \le \infty \text{ and all } r > 0. \qquad \square$$

Lemma 4.2. *For any $1 \leq k < \infty$, there exists $\lambda_0 = \lambda_0(k) > 0$, which is independent of r, such that*

$$J_{k,r} < \frac{1}{2}\bigg(\frac{1}{N} - \frac{1}{\nu}\bigg)\bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{N-1}, \quad \textit{for all } \lambda \geq \lambda_0.$$

Proof. Fix $1 \leq k < \infty$. Notice that we can choose $\delta = \delta(k) > 0$ such that the ball $B_{\delta,r} := B_\delta(((2r+1)/2, 0, , \ldots, 0)) \subset \Omega_r$ satisfies

$$g^i B_{\delta,r} \cap g^j B_{\delta,r} = \emptyset, \quad \text{for all } g^i \in G_k,\ i \neq j,\ i,j = 0,1,\ldots,k-1.$$

Consider $v_r \in W_0^{1,N}(B_{\delta,r}) \setminus \{0\}$, in such a way that

$$S_{p,k} := \inf_{v \in W_0^{1,N}(B_{\delta,r})\setminus\{0\}} \frac{||v||}{|v|_p} = \frac{||v_r||}{|v_r|_p},$$

where $p > N$ is given by (H_3). A direct computation shows that $S_{p,k}$ depends only on p and δ.

Define

$$v := \sum_{g \in G_k} g v_r \in W_{0,G_k}^{1,N}(\Omega_r) \setminus \{0\}.$$

Since

$$I'_\lambda(tv)tv \to -\infty \quad \text{as } t \to \infty \quad \text{and} \quad I'_\lambda(tv)tv > 0, \quad \text{for } t \approx 0,$$

there exists $t_v > 0$ such that $t_v v \in \mathcal{M}_{k,r}$. Observe that

$$J_{k,r} \leq I_\lambda(t_v v) = k I_\lambda(t_v v_r) = k \max_{t \geq 0} I_\lambda(t v_r),$$

and so,

$$J_{k,r} \leq k \max_{t \geq 0}\bigg\{\frac{t^N}{N}||v_r||^N - \lambda \int_{B_{\delta,r}} F(|x|, t v_r)\,dx\bigg\}.$$

From (H_3),

$$J_{k,r} \leq k \max_{t \geq 0}\bigg\{\frac{t^N}{N}||v_r||^N - \lambda \frac{C_p}{p} t^p |v_r|_p^p\bigg\},$$

leading to,

$$\frac{J_{k,r}}{|v_r|_p^N} \leq k \max_{t \geq 0}\bigg\{\frac{t^N}{N} S_{k,p}^N - \lambda \frac{C_p}{p} t^p |v_r|_p^{p-N}\bigg\}.$$

Since the function

$$h(t) = \frac{t^N}{N} S_{k,p}^N - \lambda \frac{C_p}{p} t^p |v_r|_p^{p-N},$$

attains its maximum at

$$t_0 = \bigg[\frac{S_{k,p}^N}{\lambda C_p}\bigg]^{1/(p-N)} \frac{1}{|v_r|_p},$$

a straightforward computation yields

$$J_{k,r} \leq k\bigg(\frac{1}{N} - \frac{1}{p}\bigg) S_{k,p}^{Np/(p-N)} C_p^{N/(N-p)} \lambda^{N/(N-p)}.$$

Choosing

$$\lambda_0 = \frac{S_{k,p}^p}{C_p}\left[\frac{\frac{1}{2}\left(\frac{1}{N}-\frac{1}{\nu}\right)\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}}{k\left(\frac{1}{N}-\frac{1}{p}\right)}\right]^{(N-p)/N},$$

we get

$$J_{k,r} < \frac{1}{2}\left(\frac{1}{N}-\frac{1}{\nu}\right)\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \quad \text{for all } \lambda \geq \lambda_0,$$

and the proof is complete. □

Lemma 4.3. *If $\lambda \geq \lambda_0$ and $1 \leq k < \infty$, then $J_{k,r}$ is attained.*

Proof. Let $(v_n) \subset \mathcal{M}_{k,r}$ be a minimizing sequence for $J_{k,r}$, that is, $(v_n) \subset W^{1,N}_{0,G_k}(\Omega_r) \setminus \{0\}$, $I'_\lambda(v_n)v_n = 0$ and $I_\lambda(v_n) \to J_{k,r}$. We claim that

$$I'_\lambda(v_n) \to 0 \quad \text{in } (W^{1,N}_{0,G_k}(\Omega_r))'.$$

In fact, using Ekeland Variational Principle (see [14]), there exists a sequence $(w_n) \subset \mathcal{M}_{k,r}$ verifying

$$w_n = v_n + o_n(1), \quad I_\lambda(w_n) \to J_{k,r}$$

and

$$I'_\lambda(w_n) - \ell_n E'_\lambda(w_n) = o_n(1), \tag{4.2}$$

where $(\ell_n) \subset \mathbb{R}$ and $E_\lambda(w) = I'_\lambda(w)w$, for $w \in W^{1,N}_{0,G_k}(\Omega_r)$. The below equality

$$\begin{aligned} E'_\lambda(w_n)w_n &= N||w_n||^N - \lambda \int_{\Omega_r}\left[\frac{\partial f}{\partial u}(|x|, w_n)w_n^2 + f(|x|, w_n)w_n\right]dx \\ &= -\lambda \int_{\Omega_r}\left[\frac{\partial f}{\partial u}(|x|, w_n)w_n - (N-1)f(|x|, w_n)\right]w_n\,dx. \end{aligned}$$

together with (H_4) implies that there exist $\sigma \geq N$ and $C_\sigma > 0$ such that

$$-E'_\lambda(w_n)w_n \geq C_\sigma \int_{\Omega_r} w_n^{\sigma+1}\,dx. \tag{4.3}$$

Using the last expression, we can prove that there exists $\delta > 0$ such that $|E'_\lambda(w_n)w_n| \geq \delta$ for all $n \in \mathbb{N}$. Indeed, suppose by contradiction that there exists a subsequence, still denoted by (w_n), such that

$$E'_\lambda(w_n)w_n = o_n(1).$$

By (4.3),

$$\int_{\Omega_r} w_n^{\sigma+1}\,dx = o_n(1),$$

then by interpolation

$$\int_{\Omega_r} w_n^\tau \, dx = o_n(1), \quad \text{for all } \tau \geq \sigma + 1. \tag{4.4}$$

From definition of (w_n), it is easy to show that (w_n) is bounded and satisfies

$$\limsup_{n\to\infty} ||w_n||^N < \frac{J_{k,r}}{\left(\dfrac{1}{N} - \dfrac{1}{\nu}\right)}.$$

Consequently, by Lemma 4.2

$$\limsup_{n\to\infty} ||w_n||^N < \frac{1}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \quad \text{for all } \lambda \geq \lambda_0.$$

Since $(w_n) \subset W_0^{1,N}(\Omega_r)$, by Corollary 2.5, there exist $\alpha > \alpha_0$, $t > 1$ $(t \approx 1)$ and $C > 0$, independent of n, such that

$$\int_{\Omega_r} e^{t\alpha|w_n|^{N/(N-1)}} \, dx \leq C, \quad \text{for all } n \geq n_0. \tag{4.5}$$

From (H_0) and (H_1), for each $\varepsilon > 0$ and $s \geq 1$, there exists $C > 0$ such that

$$||w_n||^N = \lambda \int_{\Omega_r} f(|x|, w_n) w_n \, dx \leq \lambda\varepsilon |u_n|_N^N + C \int_{\Omega_r} |w_n|^s e^{\alpha|w_n|^{N/(N-1)}} \, dx.$$

Choosing ε small enough and using Hölder's inequality together with (4.5), we have

$$||w_n||^N \leq \frac{1}{2}||w_n||_N^N + C|w_n|_{st_1}^s, \tag{4.6}$$

where $t_1 = t/(t-1)$. Therefore, from (4.4), $||w_n||^N = o_n(1)$, showing that $w_n \to 0$ in $W_0^{1,N}(\Omega_r)$. However, using (4.6) $||w_n||^{s-N} \geq C_2 > 0$, for some $C_2 > 0$, which is an absurd. This contradiction yields there exists $\delta > 0$ such that

$$|E'_\lambda(w_n)w_n| \geq \delta, \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

Now, from (4.2)

$$\ell_n E'_\lambda(w_n)w_n = o_n(1),$$

and so, $\ell_n = o_n(1)$. Since (w_n) is bounded, it is not difficult to prove that $(E'_\lambda(w_n))$ is bounded. Using again (4.2),

$$I'_\lambda(w_n) \to 0 \quad \text{in } (W_{0,G_k}^{1,N}(\Omega_r))'.$$

Thus, without loss generality,

$$I_\lambda(v_n) \to J_{k,r} \quad \text{and} \quad I'_\lambda(v_n) \to 0.$$

Since (v_n) is bounded, there exists $v \in W^{1,N}_{0,G_k}(\Omega_r)$ such that, for a subsequence we have

$$\begin{cases} v_n \rightharpoonup v & \text{in } W^{1,N}_{0,G_k}(\Omega_r), \\ v_n(x) \to v(x) & \text{a.e. in } \Omega_r, \\ v_n \to v & \text{in } L^t(\Omega_r) \text{ for } t \geq 1. \end{cases}$$

The above limits imply that

$$\int_{\Omega_r} \Big(f(|x|, v_n)v_n - f(|x|, v_n)v \Big)\, dx = o_n(1). \tag{4.8}$$

In fact, by (H_0)–(H_1),

$$|f(|x|, v_n)v_n| \leq |v_n|^N + C|v_n|e^{\alpha |v_n|^{N/(N-1)}}. \tag{4.9}$$

Consider α and t given by Corollary 2.5 and define

$$Q_n := e^{\alpha|v_n|^{N/(N-1)}} \quad \text{and} \quad Q := e^{\alpha|v|^{N/(N-1)}}.$$

From Corollary 2.5, $Q_n \in L^t(\Omega_r)$ and (Q_n) is bounded in $L^t(\Omega_r)$. Moreover, $Q_n(x) \to Q(x)$ almost everywhere in Ω_r. Using a result due to Brezis–Lieb Lemma (see [18]), we derive

$$Q_n \rightharpoonup Q \quad \text{in } L^t(\Omega_r). \tag{4.10}$$

Since $v_n \to v$ strongly in $L^q(\Omega_r)$ for every $q \geq 1$, we have

$$|v_n| \to |v| \quad \text{in } L^{t'}(\Omega_r), \tag{4.11}$$

where $t' = t/(t-1)$. Hence, from (4.10)–(4.11),

$$\int_{\Omega_r} |v_n| Q_n \, dx \to \int_{\Omega_r} |v| Q \, dx. \tag{4.12}$$

Then (4.9)–(4.12) combined with Lebesgue's Dominated Convergence Theorem give

$$\int_{\Omega_r} f(|x|, v_n)v_n \, dx \to \int_{\Omega_r} f(|x|, v)v \, dx.$$

A similar argument shows that

$$\int_{\Omega_r} f(|x|, v_n)v \, dx \to \int_{\Omega_r} f(|x|, v)v \, dx,$$

which proves (4.8).

Now, we will prove that $v_n \to v$ in $W^{1,N}_{0,G_k}(\Omega_r)$. To this end, we begin recalling that there exists $C > 0$ such that

$$\langle |x|^{N-2}x - |y|^{N-2}y, x - y\rangle \geq C|x-y|^N \quad \text{(see [17])},$$

for every $x,y\in\mathbb{R}^N$ $(N\geq 2)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in $\mathbb{R}^N$. The above inequality leads to

$$\begin{aligned}C\int_{\Omega_r}|\nabla v_n-\nabla v|^N\,dx &\leq \int_{\Omega_r}\langle|\nabla v_n|^{N-2}\nabla v_n-|\nabla v|^{N-2}\nabla v,\nabla v_n-\nabla v\rangle\,dx\\ &=\int_{\Omega_r}|\nabla v_n|^N\,dx-\int_{\Omega_r}|\nabla v_n|^{N-2}\nabla v_n\nabla v\,dx\\ &\quad-\int_{\Omega_r}|\nabla v|^{N-2}\langle\nabla v,\nabla v_n-\nabla v\rangle\,dx.\end{aligned}$$

On the other hand, since (v_n) is bounded, the limit $I'_\lambda(v_n)\to 0$ gives

$$\int_{\Omega_r}|\nabla v_n|^{N-2}\nabla v_n\nabla v\,dx-\lambda\int_{\Omega_r}f(|x|,v_n)v\,dx=o_n(1),$$

and

$$\int_{\Omega_r}|\nabla v_n|^N\,dx-\lambda\int_{\Omega_r}f(|x|,v_n)v_n\,dx=o_n(1).$$

Consequently

$$\begin{aligned}C\int_{\Omega_r}|\nabla v_n-\nabla v|^N\,dx\leq{}&\lambda\int_{\Omega_r}f(|x|,v_n)v_n\,dx-\lambda\int_{\Omega}f(|x|,v_n)v\,dx\\ &-\int_{\Omega_r}|\nabla v|^{N-2}\langle\nabla v,\nabla v_n-\nabla v\rangle\,dx+o_n(1).\end{aligned}$$

Applying (4.8) and using the fact that $v_n\rightharpoonup v$ in $W^{1,N}_{0,G_k}(\Omega_r)$, the last inequality implies that

$$\lim_{n\to\infty}\int_{\Omega_r}|\nabla v_n-\nabla v|^N\,dx=0$$

or equivalently,

$$v_n\to v\quad\text{in } W^{1,N}_{0,G_k}(\Omega_r).$$

From this,

$$I_\lambda(v_n)\to I_\lambda(v)=J_{k,r}>0\quad\text{and}\quad I'_\lambda(v_n)\to I'_\lambda(v)=0.$$

Therefore, $v\in\mathcal{M}_{k,r}$ and $I_\lambda(v)=J_{k,r}$. □

Lemma 4.4. *There exists $r_0=r_0(\lambda)>0$ such that*

$$J_{\infty,r}\geq\frac12\bigg(\frac1N-\frac1\nu\bigg)\bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{N-1},\quad\textit{for all } r>r_0.$$

Proof. Arguing by contradiction, we assume that there exists a sequence (r_n), with $r_n\to+\infty$ satisfying

$$J_{\infty,r_n}<\frac12\bigg(\frac1N-\frac1\nu\bigg)\bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{N-1},\quad\text{for all } n\in\mathbb{N}. \tag{4.13}$$

Now, we claim that J_{∞,r_n} is attained, for all $n \in \mathbb{N}$. In fact, fixed n, let $(v_k) \subset \mathcal{M}_{\infty,r_n}$ be a minimizing sequence for J_{∞,r_n}, that is, $(v_k) \subset W^{1,N}_{0,G_\infty}(\Omega_{r_n}) \setminus \{0\}$ and satisfies

$$I'_\lambda(v_k)v_k = 0 \quad \text{and} \quad I_\lambda(v_k) \to J_{\infty,r_n}, \quad \text{as } k \to \infty.$$

Note that

$$o_k(1) + J_{\infty,r_n} = I_\lambda(v_k) - \frac{1}{\nu} I'_\lambda(v_k)v_k \geq \left(\frac{1}{N} - \frac{1}{\nu}\right) ||v_k||^N. \tag{4.14}$$

From (4.13) and (4.14),

$$\limsup_{k\to\infty} ||v_k||^N < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Now, we can repeat the same arguments employed in the proof of Lemma 4.3 to conclude that

$$I'_\lambda(v_k) \to 0 \quad \text{in } (W^{1,N}_{0,G_\infty}(\Omega_{r_n}))' \qquad \text{and} \qquad v_k \to v \quad \text{in } W^{1,N}_{0,G_\infty}(\Omega_{r_n}),$$

where $v \in W^{1,N}_{0,G_\infty}(\Omega_{r_n})$ is the weak limit of (v_k) in $W^{1,N}_{0,G_\infty}(\Omega_{r_n})$. Then,

$$I_\lambda(v_k) \to I_\lambda(v) = J_{\infty,r_n} > 0 \quad \text{and} \quad I'_\lambda(v_k) \to I'_\lambda(v) = 0,$$

from where it follows that $v \in \mathcal{M}_{\infty,r_n}$ and $I_\lambda(v) = J_{\infty,r_n}$, proving that J_{∞,r_n} is attained.

Since J_{∞,r_n} is attained, for each $n \in \mathbb{N}$, we can choose a sequence $(u_n) \subset W^{1,N}_{0,G_\infty}(\Omega_{r_n}) \setminus \{0\}$ satisfying

$$I'_\lambda(u_n)u_n = 0 \quad \text{and} \quad I_\lambda(u_n) = J_{\infty,r_n}.$$

Consequently,

$$\frac{1}{2}\left(\frac{1}{N} - \frac{1}{\nu}\right)\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} > J_{\infty,r_n} = I_\lambda(u_n) - \frac{1}{\nu} I'_\lambda(u_n)u_n \geq \left(\frac{1}{N} - \frac{1}{\nu}\right) ||u_n||^N,$$

which implies

$$\limsup_{n\to\infty} ||u_n||^N < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.15}$$

Let $(\widetilde{u}_n)$ be a sequence given by

$$\widetilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega_{r_n}, \\ 0 & \text{if } x \notin \Omega_{r_n}. \end{cases}$$

Observe that the following properties occur:

(1) $(\widetilde{u}_n) \subset W^{1,N}_{G_\infty}(\mathbb{R}^N)$;
(2) $||\widetilde{u}_n||_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} = ||u_n||_{W^{1,N}_{0,G_\infty}(\Omega_{r_n})}$;
(3) $\widetilde{u}_n \rightharpoonup 0$ in $W^{1,N}_{G_\infty}(\mathbb{R}^N)$, because $\widetilde{u}_n(x) \to 0$ a.e. in $\mathbb{R}^N$.

Using the compact embedding $W^{1,N}_{G_\infty}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$, $N < t < \infty$, we derive that

$$\widetilde{u}_n \to 0 \quad \text{in } L^t(\mathbb{R}^N), \text{ for } N < t < \infty. \tag{4.16}$$

Now, observe that

$$||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} = I'_\lambda(u_n)u_n + \lambda \int_{\Omega_{r_n}} f(|x|, u_n)u_n \, dx = \lambda \int_{\mathbb{R}^N} f(|x|, \widetilde{u}_n)\widetilde{u}_n \, dx.$$

From $(H_0) - (H_1)$, given $\varepsilon > 0$, $q > N$ and $\alpha > \alpha_0$, there exists $C_\varepsilon > 0$ such that

$$||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \leq \varepsilon\lambda \int_{\Omega_{r_n}} |u_n|^N \, dx + C_\varepsilon\lambda \int_{\mathbb{R}^N} |\widetilde{u}_n|^q \Big(e^{\alpha|\widetilde{u}_n|^{N/(N-1)}} - S(\alpha, \widetilde{u}_n)\Big) \, dx,$$

hence by Poincaré's inequality,

$$\begin{aligned} ||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \leq \varepsilon\lambda \bigg(\frac{r_n+1}{r_n}\bigg)^{N-1} &\int_{\Omega_{r_n}} |\nabla u_n|^N \, dx \\ &+ C_\varepsilon\lambda \int_{\mathbb{R}^N} |\widetilde{u}_n|^q \Big(e^{\alpha|\widetilde{u}_n|^{N/(N-1)}} - S(\alpha, \widetilde{u}_n)\Big) \, dx. \end{aligned}$$

Choosing ε sufficiently small, there are positive constants C_1, C_2 such that

$$C_1||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \leq C_2\lambda \int_{\mathbb{R}^N} |\widetilde{u}_n|^q \Big(e^{\alpha|\widetilde{u}_n|^{N/(N-1)}} - S(\alpha, \widetilde{u}_n)\Big) \, dx.$$

Applying Hölder's inequality,

$$C_1||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \leq C_2\lambda|\widetilde{u}_n|^q_{qt_1} \bigg[\int_{\mathbb{R}^N} \Big(e^{\alpha|\widetilde{u}_n|^{N/(N-1)}} - S(\alpha, \widetilde{u}_n)\Big)^t \, dx\bigg]^{1/t},$$

where t is given by Lemma 2.4.

Now, the last inequality combined with Lemma 2.4 and (4.15) leads to

$$C_1||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \leq C_3\lambda|\widetilde{u}_n|^q_{qt_1}. \tag{4.17}$$

Then, by (4.16) and (4.17)

$$\widetilde{u}_n \to 0 \quad \text{in } W^{1,N}_{G_\infty}(\mathbb{R}^N). \tag{4.18}$$

On the other hand, from (4.17), there exist constants $C_1, C_2 > 0$ independent of r, such that

$$C_1||\widetilde{u}_n||^N_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \leq C_2||\widetilde{u}_n||^q_{W^{1,N}_{G_\infty}(\mathbb{R}^N)}$$

and so,

$$||u_n||_{W^{1,N}_{G_\infty}(\mathbb{R}^N)} \geq C_4 > 0,$$

where C_4 is independent of r, obtaining this way, a contradiction with (4.18). $\square$

LEMMA 4.5. *Suppose that $J_{km,r}$ is attained for some $1 \le k < \infty$ and some $2 \le m < \infty$. Suppose also that $J_{km,r} < J_{\infty,r}$. Then, $J_{k,r} < J_{km,r}$.*

PROOF. Consider $u \in \mathcal{M}_{km,r}$ such that $I_\lambda(u) = J_{km,r}$. Let $x = (\theta,\rho)$ be the polar coordinates of $x \in \mathbb{R}^2$. Then, $u = u(\theta,\rho,|y|)$, $y \in \mathbb{R}^{N-2}$. It is easy to derive that

$$|\nabla u|^N = \left(\frac{1}{\rho^2}u_\theta^2 + u_\rho^2 + |\nabla_y u|^2\right)^{N/2}.$$

Thus,

$$\int_{\Omega_r} |\nabla u|^N \,dx\,dy = \int\int_r^{r+1}\int_0^{2\pi}\left(\frac{1}{\rho^2}u_\theta^2 + u_\rho^2 + |\nabla_y u|^2\right)^{N/2}\rho\,d\theta\,d\rho\,dy.$$

Define

$$v(\theta,\rho,|y|) := u\left(\frac{\theta}{m},\rho,|y|\right).$$

It is possible to show the following properties:

(i) $v \in W^{1,N}_{0,G_k}(\Omega_r)$;

(ii) $|\nabla v|^N = \left(\dfrac{1}{\rho^2 m^2}u_\theta^2 + u_\rho^2 + |\nabla_y u|^2\right)^{N/2}$;

(iii) $\displaystyle\int_{\Omega_r} F(v)\,dx\,dy = \int_{\Omega_r} F(u)\,dx\,dy.$

We know that, there exists $t_0 > 0$ such that $t_0 v \in \mathcal{M}_{k,r}$. For simplicity, we denote $v := t_0 v$. Now, since $v \in \mathcal{M}_{k,r}$,

$$J_{k,r} \le I_\lambda(v) = \frac{1}{N}\int_{\Omega_r}|\nabla v|^N\,dx\,dy - \lambda\int_{\Omega_r}F(v)\,dx\,dy.$$

Using (ii)–(iii),

$$J_{k,r} \le \frac{1}{N}\int\int\int_0^{2\pi}\left(\frac{1}{m^2\rho^2}u_\theta^2 + u_\rho^2 + |\nabla_y u|^2\right)^{N/2}\rho\,d\theta\,d\rho\,dy - \lambda\int_{\Omega_r}F(u)\,dx\,dy. \tag{4.19}$$

Once that $I_\lambda(u) = J_{km,r} < J_{\infty,r}$, we have $u \notin W^{1,N}_{0,G_\infty}(\Omega_r)$ and therefore, u_θ^2 is not identically zero. Then, using that $m > 1$, we obtain

$$\int\int_r^{r+1}\int_0^{2\pi}\frac{1}{m^2\rho^2}u_\theta^2\rho\,d\theta\,d\rho\,dy < \int\int_r^{r+1}\int_0^{2\pi}\frac{1}{\rho^2}u_\theta^2\rho\,d\theta\,d\rho\,dy,$$

which together with (4.19) implies $J_{k,r} < I_\lambda(u) = J_{km,r}$ and the proof is complete. $\square$

5. Proof of Theorema 1.1

In this section, we establish the proof of Theorem 1.1. First, notice that by Lemma 4.2, for each $n \in \mathbb{N}$, there exists $\lambda_0 = \lambda_0(n) > 0$ satisfying

$$J_{2^n,r} < \frac{1}{2}\left(\frac{1}{N} - \frac{1}{\nu}\right)\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \quad \text{for all } \lambda > \lambda(n).$$

On the other hand, by Lemma 4.4, there exists $r_0 = r_0(\lambda_0(n)) > 0$ such that

$$J_{\infty,r} \geq \frac{1}{2}\left(\frac{1}{N} - \frac{1}{\nu}\right)\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \quad \text{for all } r > r_0.$$

Thus,

$$0 < J_{2^n,r} = J_{2\cdot 2^{n-1},r} < \frac{1}{2}\left(\frac{1}{N} - \frac{1}{\nu}\right)\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \leq J_{\infty,r},$$

for all $\lambda > \lambda_0$ and for all $r > r_0$. Once that $J_{2^n,r}$ is attained, we can apply Lemma 4.5 to obtain

$$J_{2^{n-1},r} < J_{2^n,r} \quad \text{for all } \lambda > \lambda_0 \text{ and for all } r > r_0.$$

Since $J_{2^{n-2}2,r}$ is attained also and satisfies

$$J_{2^{n-2}2,r} = J_{2^{n-1},r} < J_{2^n,r} < J_{\infty,r},$$

by Lemma 4.5 $J_{2^{n-2},r} < J_{2^{n-1},r}$. Inductively,

$$0 < J_{2,r} < J_{2^2,r} < \ldots < J_{2^n,r} < J_{\infty,r},$$

for all $\lambda > \lambda_0$ and all $r > r_0$.

By Lemma 4.3, we have that the minimizers of $J_{k,m}$ are critical points of I_λ in $W^{1,N}_{0,G_k}(\Omega_r)$. Applying the Principle of symmetric criticality (see [23]), it follows that they are critical points of I_λ in $W^{1,N}_0(\Omega_r)$ and therefore are solutions of (P). This way, all minimizers of $J_{2^m,r}, m = 1, \ldots, n$ are nonradial, rotationally non-equivalent and non-negative solutions of (P). Now, invoking the Harnack's inequality [29], we have that the solutions are strictly positive. $\square$

References

[1] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problems with critical growth for the N-Laplacian*, Ann. Scuola Norm. Sup. Pisa **17** (1990), 393–413.

[2] C.O. Alves and G.M. Figueiredo, *On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$*, J. Differential Equations **246** (2009), 1288–1311.

[3] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math **36** (1983), 437–477.

[4] J. Byeon, *Existence of many nonequivalent nonradial positive solutions of semilinear elliptic equations on three-dimensional annuli*, J. Differential Equations **136** (1997), 136–165.

[5] D.M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407–435.

[6] A. Castro and B.M. Finan, *Existence of many positive nonradial solutions for a superlinear Dirichlet problem on thin annuli*, Nonlinear Differential Equations **5** (2000), 21–31.

[7] F. Catrina and Z.-Q. Wang, *Nonlinear elliptic equations on expanding symmetric domains*, J. Differential Equations **156** (1999), 153–181.

[8] C. Coffman, *A non-linear boundary value problem with many positive solutions*, J. Differential Equations **54** (1984), 429–437.

[9] D.G. de Figueiredo and O.H. Miyagaki, *Multiplicity of non-radial solutions of critical elliptic problems in an annulus*, Proc. Roy. Soc. Edinburgh Sect. A **135** (2005), 25–37.

[10] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), 139–153.

[11] J.M.B. do Ó, *N-Laplacian equations in* $\mathbb{R}^N$ *with critical growth*, Abstr. Appl. Anal. **2** (1997), 301–315.

[12] ______, *Semilinear Dirichlet problems for the N-Laplacian in* $\mathbb{R}^N$ *with nonlinearities in critical growth range*, Differential Integral Equations **5** (1996), 967–979.

[13] J.M.B. do Ó, E. Medeiros and U. Severo, *On a quasilinear nonhomogeneos elliptic equation with critical growth in* $\mathbb{R}^N$, J. Differential Equations **246** (2009), 1363–1386.

[14] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.

[15] B. Gidas, W.N. Ni and L. Nirenberg, *Symmetric and related proprieties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[16] N. Hirano and N. Mizoguchi, *Nonradial solutions of semilinear elliptic equations on annuli*, J. Math. Soc. Japan **46** (1994), 111–117.

[17] Y. Jianfu, *Positive solutions of quasilinear elliptic obstacle problems with critical exponents*, Nonlinear Anal. **25** (1995), 1283–1306.

[18] O. Kavian, *Introduction à la Théorie des Points Critiques et Applications aux Problèms Elliptiques*, Springer–Verlag, 1993.

[19] Y.Y. Li, *Existence of many positive solutions of semilinear elliptic equations on annulus*, J. Differential Equations **83** (1990), 348–367.

[20] S.S. Lin, *Existence of many positive nonradial solutions for nonlinear elliptic equations on an annulus*, J. Differential Equations **103** (1993), 338–349.

[21] N. Mizoguchi and T. Suzuki, *Semilinear elliptic equations on a annuli in three and higher dimensions*, Houston J. Math. **22** (1996), 199–215.

[22] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[23] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), 19–30.

[24] R. Panda, *On semilinear Neumann problems with critical growth for the N-Laplacian*, Nonlinear Anal. **26** (1996), 1347–1366.

[25] E.A.B. Silva and S.H.M. Soares, *Liouville–Gelfand type problems for the N-Laplacian on bounded domains of* $\mathbb{R}^N$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1999), 1–30.

[26] T. Suzuki, *Positive solutions for semilinear elliptic equations on expanding annului: mountain pass approach*, Funkcial. Ekvac. **39** (1996), 143–164.

[27] E. Tonkes, *Solutions to a pertubed critical semilinear equation concerning the N-Laplacian in* $\mathbb{R}^N$, Comm. Math. Univ. Carolinae **40** (1999), 679–699.

[28] N. Trudinger, *On imbedding into Orlicz space and some applications*, J. Math. Mech. **17** (1967), 473–484.

[29] ______, *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. (1967), 721–747.

[30] Z. Wang and M. Willem, *Existence of many positive solutions of semilinear elliptic equations on an annulus*, Proc. Amer. Math. Soc. **127** (1999), 1711–1714.

[31] Y. Wang, J. Yang and Y. Zhang, *Quasilinear elliptic equations involving the N-Laplacian with critical exponential growth in $\mathbb{R}^N$*, Nonlinear Anal. **71** (2009), 6157–6169.

[32] M. Willem, *Minimax Theorems*, Birkhäuser, 1986.

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Claudianor O. Alves
Unidade Acadêmica de Matemática e Estatística
Universidade Federal de Campina Grande – UFCG
58109-970 – Campina Grande – PB, BRAZIL

E-mail address: coalves@dme.ufpb.br

Luciana Roze de Freitas
Departamento de Matematica e Estatística
Universidade Estadual da Paraíba
58429-900 – Campina Grande, PB, BRAZIL
and
Departamento de Matemática, ICMC/USP
Universidade de São Paulo
13560-970 São Carlos, SP, BRAZIL

E-mail address: lucianarfreitas@hotmail.com