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CONSTANT-SIGN AND NODAL SOLUTIONS FOR A NEUMANN PROBLEM WITH p-LAPLACIAN AND EQUI-DIFFUSIVE REACTION TERM

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Abstract. The existence of both constant and sign-changing (namely, nodal) solutions to a Neumann boundary-value problem with p-Laplacian and reaction term depending on a positive parameter is established. Proofs make use of sub- and super-solution techniques as well as critical point theory.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, let $1 < p < \infty$, and let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Given a real parameter $\lambda > 0$, consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{and} \quad \frac{\partial u}{\partial n} = |\nabla u|^{p-2}\nabla u \cdot n,$$

with $n(x)$ being the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.

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In this paper, a smallest positive solution and a biggest negative solution to (1.1) are obtained (see Theorem 3.7) by chiefly assuming that

$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t}=0,\quad \lim_{t\to\infty}\frac{f(x,t)}{|t|^{p-2}t}=\infty\quad\text{uniformly in }\Omega.$$

A third nodal solution exists (cf. Theorem 4.1) as soon as, roughly speaking, λ is not an eigenvalue of the operator $-\Delta_p$ with homogeneous Neumann boundary conditions. The approach taken exploits truncation techniques, sub- and super-solution methods, besides results from critical point theory.

Problem (1.1) has very recently been investigated in [15]. However, that work treats a different situation, i.e. the case when the parameter λ is near resonance. Other papers on related topics are [1], [10], [13]. If $f(x,t):=|t|^{q-2}t$, $(x,t)\in\Omega\times\mathbb{R}$, for some $q\in\,]p,p^*[$, with p^* being the critical Sobolev exponent, then the equation in (1.1) reduces to the so-called equi-diffusive equation

$$-\Delta_p u=\lambda|u|^{p-2}u-|u|^{q-2}u\quad\text{in }\Omega.$$

Under homogeneous Dirichlet boundary conditions, it was thoroughly studied; see for instance [7] (where $N=1$) and [9] (where $N>1$).

2. Basic assumptions and preliminary results

Let $(X,\|\cdot\|)$ be a real Banach space. Given a set $V\subseteq X$, write ∂V for the boundary of V, $\operatorname{int}(V)$ for the interior of V, and $\overline{V}$ for the closure of V. The symbol X^* denotes the dual space of X, while $\langle\cdot\,,\cdot\rangle$ indicates the duality pairing between X and X^*. A function $\Phi\colon X\to\mathbb{R}$ fulfilling

$$\lim_{\|x\|\to\infty}\Phi(x)=\infty$$

is called coercive. Let $\Phi\in C^1(X)$. We say that Φ satisfies the Palais–Smale condition when

$(\mathrm{PS})_\Phi$ *Every sequence $\{x_k\}\subseteq X$ such that $\{\Phi(x_k)\}$ is bounded and*

$$\lim_{k\to\infty}\|\Phi'(x_k)\|_{X^*}=0$$

possesses a convergent subsequence.

If $c\in\mathbb{R}$ then, as usual, $\Phi^c:=\{x\in X:\Phi(x)\le c\}$ while $K_c(\Phi):=K(\Phi)\cap\Phi^{-1}(c)$, with $K(\Phi)$ being the critical set of Φ, i.e. $K(\Phi):=\{x\in X:\Phi'(x)=0\}$.

Let (A,B) be a topological pair fulfilling $B\subset A\subseteq X$. The symbol $H_k(A,B)$, $k\in\mathbb{N}_0$, indicates the k-th-relative singular homology group of (A,B) with integer coefficients. If $x_0\in K_c(\Phi)$ is an isolated point of $K(\Phi)$ then

$$C_k(\Phi,x_0):=H_k(\Phi^c\cap U,\Phi^c\cap U\setminus\{x_0\}),\quad k\in\mathbb{N}_0,$$

are the critical groups of Φ at x_0. Here, U stands for any neighbourhood of x_0 such that $K(\Phi) \cap \Phi^c \cap U = \{x_0\}$. By excision, critical groups turn out to be independent of U. The monographs [3], [5] represent general references on this subject.

Finally, an operator $A: X \to X^*$ is called coercive when

$$\lim_{\|x\|\to\infty} \frac{\langle A(x), x\rangle}{\|x\|} = \infty.$$

We say that A is of type $(\mathrm{S})_+$ if $x_k \rightharpoonup x$ in X and $\limsup_{k\to\infty}\langle A(x_k), x_k - x\rangle \leq 0$ imply $x_k \to x$.

Throughout the paper, Ω denotes a bounded domain of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, with a smooth boundary $\partial\Omega$, $p \in\,]1,\infty[$, $p' := p/(p-1)$, and $\|\cdot\|_p$ is the standard norm of $L^p(\Omega)$. Indicate with p^* the critical exponent for the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = \infty$ otherwise. Moreover, define

$$C^1_n(\overline{\Omega}) := \left\{ u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

If, as usual, $C^1_n(\overline{\Omega})_+ := \{u \in C^1_n(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\}$ then it is known (see e.g. [15, p. 1261]) that

$$\operatorname{int}(C^1_n(\overline{\Omega})_+) = \{u \in C^1_n(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\}.$$

Write $W^{1,p}_n(\Omega)$ for the closure of $C^1_n(\overline{\Omega})$ with respect to the standard norm $\|\cdot\|$ of $W^{1,p}(\Omega)$. When $u, v \in W^{1,p}_n(\Omega)$ and $u(x) \leq v(x)$ almost everywhere in Ω we put

$$[u, v] := \{w \in W^{1,p}_n(\Omega) : u(x) \leq w(x) \leq v(x) \text{ for almost every } x \in \Omega\}.$$

From now on "measurable" always signifies Lebesgue measurable while $m(E)$ indicates the Lebesgue measure of E. To shorten notation, define, for any $u, v: \Omega \to \mathbb{R}$,

$$\Omega(u > v) := \{x \in \Omega : u(x) > v(x)\}, \quad u^+ := \max\{u, 0\}, \quad u^- := \max\{-u, 0\}.$$

The result below represents a $W^{1,p}_n(\Omega)$-version of the famous H^1 versus C^1 local minimizers theorem by Brézis and Nirenberg [4]. For its proof we refer the reader to [15, Proposition 2.5]. Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéody function such that

$$|g(x, t)| \leq a_1(1 + |t|^{q-1}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

where $a_1 > 0$ while $q \in\,]1, p^*[$, and let $G(x, \xi) := \int_0^\xi g(x, t)\, dt$, $(x, \xi) \in \Omega \times \mathbb{R}$. Define, for every $u \in W^{1,p}_n(\Omega)$,

$$\varphi(u) := \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G(x, u(x))\, dx.$$

Obviously, $\varphi \in C^1(W_n^{1,p}(\Omega))$. Moreover, one has

Proposition 2.1. *If there exist $u_0 \in W_n^{1,p}(\Omega)$, $\delta_0 > 0$ such that $\varphi(u_0) \leq \varphi(u_0 + v)$ for all $v \in C_n^1(\overline{\Omega})$ satisfying $\|v\|_{C^1(\overline{\Omega})} \leq \delta_0$ then $u_0 \in C_n^1(\overline{\Omega})$ and u_0 turns out to be a $W_n^{1,p}(\Omega)$-local minimizer of φ.*

Let $A: W_n^{1,p}(\Omega) \to (W_n^{1,p}(\Omega))^*$ be the nonlinear operator, arising from the p-Laplacian, defined by

$$\langle A(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx \quad \text{for all } u, v \in W_n^{1,p}(\Omega),$$

and let $\sigma(-\Delta_p)$ the family of eigenvalues of the Neumann problem

$$-\Delta_p u = \lambda|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Recall (vide e.g. [11]) that

(p_1) $\sigma(-\Delta_p)$ contains a strictly increasing sequence $\{\lambda_k\}$ obtained through the Ljusternik–Schnirelman principle.

(p_2) $\lambda_1 = 0$ and $\lim_{k\to\infty}\lambda_k = \infty$.

(p_3) Eigenfunctions corresponding to positive eigenvalues are nodal.

(p_4) The operator A is continuous and of type $(S)_+$.

From now on, to avoid unnecessary technicalities, "for every $x \in \Omega$" will take the place of "for almost every $x \in \Omega$". Moreover, to avoid cumbersome formulae, the variable x will be omitted when no confusion can arise.

Let $f: \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that $f(x,0) = 0$ in Ω and the conditions below hold true.

(f_1) *There exist $a_1 > 0$, $q \in \,]p, p^*[$ such that*

$$|f(x,t)| \leq a_1(1+|t|^{q-1}) \quad \textit{for all } (x,t) \in \Omega\times\mathbb{R}.$$

(f_2) $\displaystyle\lim_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t} = 0$ *uniformly in $x \in \Omega$.*

(f_3) $\displaystyle\lim_{|t|\to\infty}\frac{f(x,t)}{|t|^{p-2}t} = \infty$ *uniformly in $x \in \Omega$.*

(f_4) *To every $\rho > 0$ and every bounded interval $\Lambda \subseteq [\lambda,\infty[$ there correspond constants $r > p$, $\theta > 0$ such that the function*

$$t \mapsto \eta|t|^{p-2}t - f(x,t) + \theta|t|^{r-2}t$$

turns out increasing in $[-\rho,\rho]$ for all $\eta \in \Lambda$, $x \in \Omega$.

A function $\underline{u} \in W^{1,p}(\Omega)$ is called a sub-solution to (1.1) if

$$\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\cdot\nabla v\,dx + \int_\Omega (f(x,\underline{u}) - \lambda|\underline{u}|^{p-2}\underline{u})v\,dx \leq 0 \quad \text{for all } v \in C_n^1(\overline{\Omega})_+.$$

Likewise, we say that $\overline{u} \in W^{1,p}(\Omega)$ is a super-solution of (1.1) when

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla v \, dx + \int_\Omega (f(x,\overline{u}) - \lambda |\overline{u}|^{p-2}\overline{u}) v \, dx \geq 0 \quad \text{for all } v \in C^1_n(\overline{\Omega})_+.$$

Lemma 2.2. *Let* (f_1)–(f_3) *be satisfied. Then* (1.1) *possesses a sub-solution* $\underline{u}_\lambda$ *and a super-solution* $\overline{u}_\lambda$ *such that* $\underline{u}_\lambda \leq \overline{u}_\lambda$ *in* Ω *and* $\underline{u}_\lambda, \overline{u}_\lambda \in \operatorname{int}(C^1_n(\overline{\Omega})_+)$.

Proof. Pick $\lambda_0 > \lambda$, $\mu > 0$, $\eta > \lambda_0 + \mu$. Owing to (f_3) and (f_1) there exists a $c_\eta > 0$ such that

$$f(x,t) > \eta t^{p-1} - c_\eta \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}^+_0 . \tag{2.2}$$

Since $\eta > \lambda_0 + \mu$, the functional $\psi \colon W^{1,p}_n(\Omega) \to \mathbb{R}$ given by

$$\psi(u) := \frac{1}{p} \|\nabla u\|_p^p + \frac{\mu}{p} \|u\|_p^p + \frac{\eta - \lambda_0 - \mu}{p} \|u^+\|_p^p - c_\eta \int_\Omega u \, dx \quad \text{for all } u \in W^{1,p}_n(\Omega)$$

is coercive. A simple argument, based on the compact embedding of $W^{1,p}_n(\Omega)$ in $L^p(\Omega)$, ensures that it is weakly sequentially lower semi-continuous. Therefore,

$$\psi(\overline{u}_\lambda) = \inf_{u \in W^{1,p}_n(\Omega)} \psi(u)$$

for some $\overline{u}_\lambda \in W^{1,p}_n(\Omega)$. This implies $\psi'(\overline{u}_\lambda) = 0$, i.e.

$$A(\overline{u}_\lambda) + \mu |\overline{u}_\lambda|^{p-2} \overline{u}_\lambda = (\lambda_0 + \mu - \eta)(\overline{u}_\lambda^+)^{p-1} + c_\eta \quad \text{in } (W^{1,p}_n(\Omega))^*. \tag{2.3}$$

Acting on (2.3) with $v := -\overline{u}_\lambda^-$ we obtain

$$\frac{1}{p} \|\nabla \overline{u}_\lambda^-\|_p^p + \mu \|\overline{u}_\lambda^-\|_p^p = -c_\eta \int_\Omega \overline{u}_\lambda^- \, dx \leq 0.$$

Consequently, $\overline{u}_\lambda^- = 0$, which, on account of (2.3), means $\overline{u}_\lambda \geq 0$ in Ω and $\overline{u}_\lambda \neq 0$. Since, by (2.3) again,

$$-\Delta_p \overline{u}_\lambda + \eta \overline{u}_\lambda^{p-1} = \lambda_0 \overline{u}_\lambda^{p-1} + c_\eta \quad \text{in } \Omega, \qquad \frac{\partial \overline{u}_\lambda}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

standard results from nonlinear regularity theory (see e.g. [10]) ensure that

$$\overline{u}_\lambda \in C^1_n(\overline{\Omega})_+ \setminus \{0\}.$$

Thanks to [16, Theorem 5], the obvious inequality $\Delta_p \overline{u}_\lambda \leq \eta \overline{u}_\lambda^{p-1}$ yields

$$\overline{u}_\lambda \in \operatorname{int}(C^1_n(\overline{\Omega})_+).$$

Finally, gathering (2.4) and (2.2) together we have

$$-\Delta_p \overline{u}_\lambda = \lambda_0 \overline{u}_\lambda^{p-1} - (\eta \overline{u}_\lambda^{p-1} - c_\eta) \geq \lambda_0 \overline{u}_\lambda^{p-1} - f(x, \overline{u}_\lambda). \tag{2.5}$$

Hence, $\overline{u}_\lambda$ turns out to be a super-solution of (1.1).

Now, pick $\varepsilon \in \,]0, \lambda[$. Due to ($f_2$), there exists a $\delta > 0$ such that

$$f(x,t) \leq \varepsilon t^{p-1} \quad \text{for all } (x,t) \in \Omega \times [0, \delta]. \tag{2.6}$$

Suppose, as we allow,

$$\delta \le \min_{x\in\overline{\Omega}} \overline{u}_\lambda(x), \tag{2.7}$$

fix any $\xi \in]0,\delta]$, and define

$$\underline{u}_\lambda(x) := \xi \quad \text{for all } x \in \Omega.$$

Obviously, $\underline{u}_\lambda \in \mathrm{int}(C^1_n(\overline{\Omega})_+)$. Using (2.6) we then see that $\underline{u}_\lambda$ is a sub-solution of (1.1). Finally, (2.7) evidently gives $\underline{u}_\lambda \le \overline{u}_\lambda$ in Ω, which completes the proof.□

Likewise, one has

Lemma 2.3. *Let* (f_1)–(f_3) *be satisfied. Then,* (1.1) *possesses a sub-solution* $\underline{v}_\lambda$ *and a super-solution* $\overline{v}_\lambda$ *such that* $\underline{v}_\lambda \le \overline{v}_\lambda$ *in* Ω *and* $\underline{v}_\lambda, \overline{v}_\lambda \in -\mathrm{int}(C^1_n(\overline{\Omega})_+)$.

3. Constant-sign solutions

Let $\eta > 0$, let $f^+_\eta\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by setting, for every $(x,t) \in \Omega \times \mathbb{R}$,

$$f^+_\eta(x,t) := \begin{cases} 0 & \text{if } t \le 0, \\ \eta t^{p-1} - f(x,t) & \text{otherwise,} \end{cases}$$

and let $F^+_\eta(x,\xi) := \int_0^\xi f^+_\eta(x,t)\,dt$. For $\mu > 0$, write

$$\varphi^+_\mu(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{p}\|u\|_p^p - \int_\Omega F^+_{\lambda+\mu}(x,u(x))\,dx \quad \text{for all } u \in W^{1,p}_n(\Omega).$$

Since $f^+_{\lambda+\mu}$ is of Carathéodory's type, one has $\varphi^+_\mu \in C^1(W^{1,p}_n(\Omega))$.

Theorem 3.1. *Suppose* (f_1)–(f_4) *hold true. Then problem* (1.1) *possesses a solution* $u_0 \in \mathrm{int}(C^1_n(\overline{\Omega})_+) \cap [\underline{u}_\lambda, \overline{u}_\lambda]$*, which is a local minimizer of* φ^+_μ.

Proof. Put, for every $(x,t) \in \Omega \times \mathbb{R}$,

$$f_{\lambda+\mu}(x,t) := \begin{cases} (\lambda+\mu)\underline{u}_\lambda(x)^{p-1} - f(x,\underline{u}_\lambda(x)) & \text{if } t < \underline{u}_\lambda(x), \\ (\lambda+\mu)t^{p-1} - f(x,t) & \text{if } \underline{u}_\lambda(x) \le t \le \overline{u}_\lambda(x), \\ (\lambda+\mu)\overline{u}_\lambda(x)^{p-1} - f(x,\overline{u}_\lambda(x)) & \text{if } t > \overline{u}_\lambda(x), \end{cases} \tag{3.1}$$

where $\underline{u}_\lambda$ and $\overline{u}_\lambda$ are as in Lemma 2.2. Since the functional

$$\varphi_\mu(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{p}\|u\|_p^p - \int_\Omega F_{\lambda+\mu}(x,u(x))\,dx, \quad u \in W^{1,p}_n(\Omega),$$

with $F(x,\xi) := \int_0^\xi f_{\lambda+\mu}(x,t)\,dt$, is weakly sequentially lower semi-continuous and coercive, one has

$$\varphi_\mu(u_0) = \inf_{u\in W^{1,p}_n(\Omega)} \varphi_\mu(u) \tag{3.2}$$

for some $u_0 \in W_n^{1,p}(\Omega)$. This implies $\varphi'_\mu(u_0) = 0$, namely

$$A(u_0) + \mu|u_0|^{p-2}u_0 = f_{\lambda+\mu}(\cdot, u_0) \quad \text{in } (W_n^{1,p}(\Omega))^*. \tag{3.3}$$

Acting on (3.3) with $v := (\underline{u}_\lambda - u_0)^+$ and using (2.6) we obtain

$$\begin{aligned}
\langle A(u_0), (\underline{u}_\lambda - u_0)^+\rangle &+ \mu \int_\Omega |u_0|^{p-2}u_0(\underline{u}_\lambda - u_0)^+\,dx \\
&= (\lambda+\mu)\int_\Omega \underline{u}_\lambda^{p-1}(\underline{u}_\lambda - u_0)^+\,dx - \int_\Omega f(x,\underline{u}_\lambda)(\underline{u}_\lambda - u_0)^+\,dx \\
&\ge (\lambda+\mu-\varepsilon)\int_\Omega \underline{u}_\lambda^{p-1}(\underline{u}_\lambda - u_0)^+\,dx.
\end{aligned}$$

Observe that $A(\underline{u}_\lambda) = 0$. The choice of ε forces

$$\langle A(u_0) - A(\underline{u}_\lambda), (\underline{u}_\lambda - u_0)^+\rangle + \mu\int_\Omega (|u_0|^{p-2}u_0 - \underline{u}_\lambda^{p-1})(\underline{u}_\lambda - u_0)^+\,dx \ge 0.$$

By monotonicity we thus have $m(\Omega(\underline{u}_\lambda > u_0)) = 0$, that is $\underline{u}_\lambda \le u_0$ in Ω. A similar reasoning then provides $u_0 \le \overline{u}_\lambda$. Therefore, on account of (3.1) and (3.3), the function u_0 turns out to be a solution of (1.1). Through standard results from nonlinear regularity theory we finally get $u_0 \in \operatorname{int}(C_n^1(\overline{\Omega})_+)$.

Next, pick $\sigma \in\,]0,\xi[$ and define $u_\sigma := u_0 - \sigma$. Obviously, $u_\sigma \in \operatorname{int}(C_n^1(\overline{\Omega})_+)$ because

$$u_\sigma(x) \ge \underline{u}_\lambda(x) - \sigma = \xi - \sigma > 0 \quad \text{for all } x \in \Omega.$$

Moreover,

$$\begin{aligned}
-\Delta_p u_\sigma(x) + \theta u_\sigma(x)^{r-1} &= -\Delta_p u_0(x) + \theta u_0(x)^{r-1} - h(\sigma) \\
&= \lambda u_0(x)^{p-1} - f(x,u_0(x)) + \theta u_0(x)^{r-1} - h(\sigma),
\end{aligned} \tag{3.4}$$

where θ, r come from (f_4) written for $\rho := \|\overline{u}_\lambda\|_\infty$ and $\Lambda := [\lambda, \lambda+1]$, while $h(\sigma) \to 0$ as $\sigma \to 0^+$. Combining (3.4) with (f_4) and (2.6) we achieve

$$\begin{aligned}
-\Delta_p u_\sigma(x) + \theta u_\sigma(x)^{r-1} &\ge \lambda\xi^{p-1} - f(x,\xi) + \theta\xi^{r-1} - h(\sigma) \\
&\ge (\lambda-\varepsilon)\xi^{p-1} + \theta\xi^{r-1} - h(\sigma).
\end{aligned} \tag{3.5}$$

Choose $\sigma > 0$ so small that $h(\sigma) < (\lambda-\varepsilon)\xi^{p-1}$. Then (3.5) leads to

$$-\Delta_p u_\sigma(x) + \theta u_\sigma(x)^{r-1} \ge \theta\xi^{r-1} = -\Delta_p \underline{u}_\lambda(x) + \theta\underline{u}_\lambda(x)^{r-1}$$

which implies $u_\sigma \ge \underline{u}_\lambda$ in Ω. So, a fortiori,

$$u_0 - \underline{u}_\lambda \in \operatorname{int}(C_n^1(\overline{\Omega})_+). \tag{3.6}$$

Likewise, if $\eta > 0$ and $u_\eta := u_0 + \eta$ then, by (2.5),

$$\begin{aligned}(3.7)\quad -\Delta_p u_\eta(x) + \theta u_\eta(x)^{r-1} &= -\Delta_p u_0(x) + \theta u_0(x)^{r-1} + h(\eta)\\ &= \lambda u_0(x)^{p-1} - f(x, u_0(x)) + \theta u_0(x)^{r-1} + h(\eta)\\ &\le \lambda \overline{u}_\lambda(x)^{p-1} - f(x, \overline{u}_\lambda(x)) + \theta \overline{u}_\lambda(x)^{r-1} + h(\eta)\\ &\le (\lambda - \lambda_0)\overline{m}_\lambda + \lambda_0 \overline{u}_\lambda(x)^{p-1} - f(x, \overline{u}_\lambda(x)) + \theta \overline{u}_\lambda(x)^{r-1} + h(\eta)\\ &\le (\lambda - \lambda_0)\overline{m}_\lambda - \Delta_p \overline{u}_\lambda(x) + \theta \overline{u}_\lambda(x)^{r-1} + h(\eta)\end{aligned}$$

where $\lim_{\eta\to 0^+} h(\eta) = 0$ while $\overline{m}_\lambda = \min_{x\in\overline{\Omega}} \overline{u}_\lambda(x)$. Choosing η in (3.7) so small that $h(\eta) < (\lambda_0 - \lambda)\overline{m}_\lambda$ and arguing as before provides

$$\overline{u}_\lambda - u_0 \in \operatorname{int}(C^1_n(\overline{\Omega})_+). \tag{3.8}$$

Write, for $\widehat{\delta} > 0$, $\widehat{u} \in C^1_n(\overline{\Omega})$,

$$B_{\widehat{\delta}}(\widehat{u}) := \{u \in C^1_n(\overline{\Omega}) : \|u - \widehat{u}\|_{C^1(\overline{\Omega})} \le \widehat{\delta}\}.$$

Due to (3.6), (3.8) we can find a $\delta_0 > 0$ such that $B_{\delta_0}(u_0) \subseteq [\underline{u}_\lambda, \overline{u}_\lambda]$. Fix any $v \in B_{\delta_0}(0)$. By the above inclusion and (3.1) one has

$$(\varphi^+_\mu)'(u_0 + tv) = \varphi'_\mu(u_0 + tv) \quad \text{for all } t \in [0, 1].$$

Thus, on account of (3.2),

$$\begin{gathered}\varphi^+_\mu(u_0 + v) - \varphi^+_\mu(u_0) = \int_0^1 \frac{d}{dt}\varphi^+_\mu(u_0 + tv)\, dt = \int_0^1 \langle (\varphi^+_\mu)'(u_0 + tv), v\rangle\, dt,\\ \int_0^1 \langle \varphi'_\mu(u_0 + tv), v\rangle\, dt = \int_0^1 \frac{d}{dt}\varphi_\mu(u_0 + tv)\, dt = \varphi_\mu(u_0 + v) - \varphi_\mu(u_0) \ge 0.\end{gathered}$$

As $v \in B_{\delta_0}(0)$ was arbitrary, the function u_0 turns out to be a $C^1_n(\overline{\Omega})$-local minimizer of φ^+_μ. Bearing in mind Proposition 2.1, the conclusion follows. □

Now, let $\eta > 0$, let $f^-_\eta \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by setting, for every $(x, t) \in \Omega \times \mathbb{R}$,

$$f^-_\eta(x, t) := \begin{cases} \eta|t|^{p-2}t - f(x, t) & \text{if } t \le 0,\\ 0 & \text{otherwise,}\end{cases}$$

and let $F^-_\eta(x, \xi) := \int_0^\xi f^-_\eta(x, t)\, dt$. If $\mu > 0$, put

$$\varphi^-_\mu(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{p}\|u\|_p^p - \int_\Omega F^-_{\lambda+\mu}(x, u(x))\, dx \quad \text{for all } u \in W^{1,p}_n(\Omega).$$

Since $f^-_{\lambda+\mu}$ is of Carathéodory's type, one has $\varphi^-_\mu \in C^1(W^{1,p}_n(\Omega))$. The next result can be established through arguments analogous to those adopted in proving Theorem 3.1.

Theorem 3.2. *Suppose* (f_1)–(f_4) *hold true. Then problem* (1.1) *possesses a solution* $v_0 \in -\mathrm{int}(C^1_n(\overline{\Omega})_+) \cap [\underline{v}_\lambda, \overline{v}_\lambda]$*, which is a local minimizer of* φ^-_μ.

Theorem 3.3. *If assumptions* (f_1)–(f_4) *are satisfied then* (1.1) *has the smallest solution* $u_* \in \mathrm{int}(C^1_n(\overline{\Omega})_+)$ *in the order interval* $[\underline{u}_\lambda, \overline{u}_\lambda]$.

Proof. Define $S^+_\lambda := \{u \in [\underline{u}_\lambda, \overline{u}_\lambda] : u \text{ is a solution to (1.1)}\}$. Theorem 3.1 yields $S^+_\lambda \neq \emptyset$ while standard results from nonlinear regularity theory (cf. e.g. [10]) combined with Lemma 2.2 give $S^+_\lambda \subseteq \mathrm{int}(C^1_n(\overline{\Omega})_+)$. We claim that S^+_λ turns out to be downward directed. Indeed, pick $u_1, u_2 \in S^+_\lambda$ and put $\overline{u} := \min\{u_1, u_2\}$. The same reasoning exploited in the proof of [1, Lemma 1] ensures here that $\overline{u}$ is a super-solution to (1.1). Hence, as before (see Theorem 3.1), one can find a solution $u_3 \in \mathrm{int}(C^1_n(\overline{\Omega})_+) \cap [\underline{u}_\lambda, \overline{u}]$ of problem (1.1). Since $u_3 \in S^+_\lambda$, $u_3 \le u_1$, and $u_3 \le u_2$, the assertion follows.

Our next goal is to show that S^+_λ possesses a minimal element. So, let $C \subseteq S^+_\lambda$ be a chain. By [6, p. 336] we have

$$\inf C = \inf\{u_k : k \in \mathbb{N}\} \tag{3.9}$$

for some $\{u_k\} \subseteq C$, while Lemma 1.1.5 of [8] allows this sequence to be decreasing. Moreover, $\{u_k\}$ is bounded in $W^{1,p}_n(\Omega)$, because

$$u_k \in [\underline{u}_\lambda, \overline{u}_\lambda] \text{ and } A(u_k) = \lambda u_k^{p-1} - f(\,\cdot\,, u_k) \text{ in } (W^{1,p}_n(\Omega))^* \text{ for all } k \in \mathbb{N}. \tag{3.10}$$

Passing to a subsequence when necessary, we may thus suppose $u_k \rightharpoonup u$ in $W^{1,p}_n(\Omega)$ as well as $u_k \to u$ in $L^q(\Omega)$, with

$$u = \inf\{u_k : k \in \mathbb{N}\}. \tag{3.11}$$

Hypothesis (f_1) forces

$$\lim_{k\to\infty} \int_\Omega f(x, u_k(x))(u_k(x) - u(x))\, dx = 0.$$

Therefore, on account of (3.10),

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0.$$

Property (p_4) ensures that $u_k \to u$ in $W^{1,p}_n(\Omega)$. From (3.10) it follows, letting $k \to \infty$,

$$u \in [\underline{u}_\lambda, \overline{u}_\lambda], \quad A(u) = \lambda u^{p-1} - f(\,\cdot\,, u) \quad \text{in } (W^{1,p}_n(\Omega))^*,$$

i.e. $u \in S^+_\lambda$. Now, (3.9) and (3.11) lead to $\inf C \in S^+_\lambda$, as desired.

By Zorn's lemma the set S^+_λ possesses a minimal element, say u_*. If $u \in S^+_\lambda$ then there exists $\widetilde{u} \in S^+_\lambda$ such that $\widetilde{u} \le \min\{u_*, u\}$, because S^+_λ is downward directed. The minimality of u_* gives $u_* = \widetilde{u}$. Hence, $u_* \le u$ in Ω, and the proof is complete. $\square$

Using Lemma 2.3 instead of Lemma 2.2 and arguing as before provides the next result.

Theorem 3.4. *If assumptions* (f$_1$)–(f$_4$) *are satisfied then problem* (1.1) *has the greatest solution* $v_* \in -\mathrm{int}(C^1_n(\overline{\Omega})_+)$ *in the order interval* $[\underline{v}_\lambda, \overline{v}_\lambda]$.

Theorems 3.3 and 3.4 lead to the existence of extremal constant sign solutions.

Theorem 3.5. *Suppose* (f$_1$)–(f$_4$) *hold true. Then* (1.1) *possesses a smallest positive solution* $u_+ \in \mathrm{int}(C^1_n(\overline{\Omega})_+)$.

Proof. Pick $\{t_k\} \subseteq]0,1[$ fulfilling $t_k \to 0$ and define $\underline{u}_{\lambda,k} := t_k \underline{u}_\lambda$. For each $k \in \mathbb{N}$, Theorem 3.3 provides a function $u_{*,k} \in \mathrm{int}(C^1_n(\overline{\Omega})_+) \cap [\underline{u}_{\lambda,k}, \overline{u}_\lambda]$ such that

$$A(u_{*,k}) = \lambda (u_{*,k})^{p-1} - f(\,\cdot\,, u_{*,k}) \quad \text{in } (W^{1,p}_n(\Omega))^*. \tag{3.12}$$

Through the same arguments exploited in the proof of this result we then obtain a solution $u_+ \in C^1_n(\overline{\Omega})_+$ to (1.1) enjoying the property

$$u_{*,k} \to u_+ \quad \text{in } W^{1,p}_n(\Omega). \tag{3.13}$$

One has $u_+ \neq 0$. Indeed, if $w_k := u_{*,k}/\|u_{*,k}\|$ then

$$w_k \rightharpoonup w \quad \text{in } W^{1,p}_n(\Omega) \quad \text{and} \quad w_k \to w \quad \text{in } L^p(\Omega) \tag{3.14}$$

for some $w \in W^{1,p}_n(\Omega)$. Moreover, on account of (3.12),

$$A(w_k) = \lambda w_k^{p-1} - \frac{f(\,\cdot\,, u_{*,k})}{\|u_{*,k}\|^{p-1}} \quad \text{in } (W^{1,p}_n(\Omega))^* \text{ for all } k \in \mathbb{N}. \tag{3.15}$$

Suppose, contrary to our claim, that $u_+ = 0$. Acting on (3.15) with $v := w_k - w$ and using (3.14), besides (f$_2$), it results in

$$\lim_{k\to\infty} \langle A(w_k), w_k - w\rangle = 0.$$

Hence, by (p$_4$),

$$w_k \to w \text{ in } W^{1,p}_n(\Omega), \text{ which forces } \|w\| = 1. \tag{3.16}$$

Due to (3.13) we get

$$\frac{f(\,\cdot\,, u_{*,k})}{\|u_{*,k}\|^{p-1}} \rightharpoonup 0 \quad \text{in } L^p(\Omega). \tag{3.17}$$

Gathering (3.15)–(3.17) together directly yields $A(w) = \lambda w^{p-1}$, namely w turns out to be an eigenfunction of (2.1) corresponding to the eigenvalue λ. Since $w(x) > 0$ for all $x \in \Omega$, Property (p$_3$) forces $\lambda = 0$, against the choice of λ. Therefore, $u_+ \in C^1_n(\overline{\Omega})_+ \setminus \{0\}$.

Next, pick any $\rho \geq \max_{x\in\overline{\Omega}} u_+(x)$. Assumption (f_4) provides $r > p$, $\theta > 0$ such that

$$-\Delta_p u_+(x) + \theta u_+(x)^{r-1} = \lambda u_+(x)^{p-1} - f(x, u_+(x)) + \theta u_+(x)^{r-1} \geq 0$$

almost everywhere in Ω. Thus, $\Delta_p u_+ \leq \theta u_+^{r-1}$ and so, on account of [16, Theorem 5], $u_+ \in \mathrm{int}(C^1_n(\overline{\Omega})_+)$. It remains to verify that u_+ is the smallest solution of (1.1) inside $\mathrm{int}(C^1_n(\overline{\Omega})_+)$. If u belongs to $\mathrm{int}(C^1_n(\overline{\Omega})_+)$ and solves (1.1) then $\underline{u}_{\lambda,k} \leq u$ for any k large enough. By the minimality of $u_{*,k}$ we get $u_{*,k} \leq u$. Via (3.13), letting $k \to \infty$ yields $u_+ \leq u$. □

A similar argument, with Theorem 3.4, $\underline{v}_\lambda$, and $\overline{v}_{\lambda,k} := t_k\overline{v}_\lambda$ in place of Theorem 3.3, $\underline{u}_{\lambda,k}$, and $\overline{u}_\lambda$, respectively, produces the next result.

Theorem 3.6. *If* (f_1)–(f_4) *are satisfied then problem* (1.1) *has a biggest negative solution* $v_- \in -\mathrm{int}(C^1_n(\overline{\Omega})_+)$.

Gathering Theorems 3.5 and 3.6 together we obtain

Theorem 3.7. *Suppose* (f_1)–(f_4) *hold true. Then* (1.1) *possesses a biggest negative solution* $v_- \in -\mathrm{int}(C^1_n(\overline{\Omega})_+)$ *and a smallest positive solution* $u_+ \in \mathrm{int}(C^1_n(\overline{\Omega})_+)$.

Finally, when $f(x,t) := |t|^{q-2}t$, $(x,t) \in \Omega \times \mathbb{R}$, the positive solution given by Theorem 3.5 is unique, as the next result shows.

Theorem 3.8. *Let* $q \in \,]p, p^*[$ *and let* $\lambda > 0$. *Then the Neumann problem*

$$-\Delta_p u = \lambda u^{p-1} - u^{q-1} \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{3.18}$$

has only one solution in $\mathrm{int}(C^1_n(\overline{\Omega})_+)$.

Proof. If $u, v \in W^{1,p}_n(\Omega)$ are two solutions of (3.18) then standard results from nonlinear regularity theory (vide for instance [10]) and [16, Theorem 5] guarantee that $u, v \in \mathrm{int}(C^1_n(\overline{\Omega})_+)$. Thus, thanks to [2, Theorem 1.1],

$$\begin{aligned}
0 &\leq \int_\Omega \left(|\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right) \cdot |\nabla v|^{p-2}\nabla v \right) dx \\
&= \int_\Omega \left(|\nabla u|^p - \frac{u^p}{v^{p-1}}(-\Delta_p v) \right) dx = \int_\Omega (|\nabla u|^p - \lambda u^p + u^p v^{q-p})\, dx \\
&= \int_\Omega (-u^q + u^p v^{q-p})\, dx = \int_\Omega u^p (v^{q-p} - u^{q-p})\, dx.
\end{aligned}$$

Interchanging the role of u and v provides

$$\int_\Omega v^p \left(u^{q-p} - v^{q-p}\right)\, dx \geq 0.$$

Consequently,

$$\int_\Omega (u^p - v^p)\left(u^{q-p} - v^{q-p}\right) dx \le 0. \tag{3.19}$$

Since the function $t \mapsto t^{q/p-1}$, $t \in \mathbb{R}^+$, is strictly monotone because $q > p$, inequality (3.19) forces $u = v$. $\square$

4. Nodal solutions

A third non-zero, sign-changing (i.e. nodal) solution can be obtained provided $\lambda \in]\lambda_2, \infty[\setminus \sigma(-\Delta_p)$, as the result below shows. Let v_- and u_+ the solutions of problem (1.1) given by Theorem 3.7. Define, for every $\eta > 0$, $(x,t) \in \Omega \times \mathbb{R}$,

$$f_\eta(x,t) := \begin{cases} \eta|v_-(x)|^{p-2}v_-(x) - f(x, v_-(x)) & \text{if } t < v_-(x), \\ \eta|t|^{p-2}t - f(x,t) & \text{if } v_-(x) \le t \le u_+(x), \\ \eta u_+(x)^{p-1} - f(x, u_+(x)) & \text{if } t > u_+(x), \end{cases}$$

$$f_\eta^-(x,t) := \begin{cases} \eta|v_-(x)|^{p-2}v_-(x) - f(x, v_-(x)) & \text{if } t < v_-(x), \\ \eta|t|^{p-2}t - f(x,t) & \text{if } v_-(x) \le t \le 0, \\ 0 & \text{if } t > 0, \end{cases}$$

$$f_\eta^+(x,t) := \begin{cases} 0 & \text{if } t < 0, \\ \eta t^{p-1} - f(x,t) & \text{if } 0 \le t \le u_+(x), \\ \eta u_+(x)^{p-1} - f(x, u_+(x)) & \text{if } t > u_+(x). \end{cases} \tag{4.1}$$

Obviously, $f_\eta, f_\eta^-, f_\eta^+ \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Moreover, set

$$F_\eta(x,\xi) := \int_0^\xi f_\eta(x,t)\, dt, \quad F_\eta^\pm(x,\xi) := \int_0^\xi f_\eta^\pm(x,t)\, dt, \quad (x,\xi) \in \Omega \times \mathbb{R}.$$

Theorem 4.1. *If* $\lambda \in]\lambda_2, \infty[\setminus \sigma(-\Delta_p)$ *and* (f$_1$)–(f$_4$) *are satisfied then* (1.1) *possesses a nodal solution* $\overline{u} \in C^1_n(\overline{\Omega})$.

Proof. Write, for $\mu > 0$,

$$\varphi_\mu^+(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{p}\|u\|_p^p - \int_\Omega F_{\lambda+\mu}^+(x, u(x))\, dx \quad \text{for all } u \in W_n^{1,p}(\Omega).$$

By (4.1) the functional φ_μ^+ belongs to $C^1(W_n^{1,p}(\Omega))$, is coercive and sequentially weakly lower semi-continuous. Hence, there exists $\widehat{u} \in W_n^{1,p}(\Omega)$ such that

$$\varphi_\mu^+(\widehat{u}) = \inf_{u \in W_n^{1,p}(\Omega)} \varphi_\mu^+(u). \tag{4.2}$$

One clearly has $\widehat{u} \in K(\varphi_\mu^+)$. Moreover, $\widehat{u} \neq 0$. Indeed, pick $\varepsilon \in]0, \lambda[$. Since

$$\min_{x \in \overline{\Omega}} u_+(x) > 0,$$

on account of (f_2) for any $\xi > 0$ sufficiently small we get

$$\varphi_\mu^+(\xi) = -\frac{\lambda}{p}\xi^p m(\Omega) + \int_\Omega \left(\int_0^\xi f(x,t)\,dt\right) dx \le -\frac{\lambda-\varepsilon}{p}\xi^p m(\Omega) < 0,$$

which forces

$$\varphi_\mu^+(\widehat{u}) < 0 = \varphi_\mu^+(0),$$

and the assertion follows. The next goal is to prove that

$$\widehat{u} = u_+. \tag{4.3}$$

If $u \in K(\varphi_\mu^+) \setminus \{0\}$ then

$$A(u) + \mu|u|^{p-2}u = f_{\lambda+\mu}^+(\,\cdot\,, u) \quad \text{in } (W_n^{1,p}(\Omega))^*. \tag{4.4}$$

Acting on (4.4) with $-u^- \in W_n^{1,p}(\Omega)$ we obtain $\min\{1,\mu\}\|u^-\|^p \le 0$. Therefore, $u \ge 0$. Through (4.4) again it results in

$$\begin{aligned}\langle A(u), (u-u_+)^+\rangle + \mu \int_\Omega u^{p-1}(u-u_+)^+\,dx \\ &= \int_\Omega \Big[(\lambda+\mu)u_+^{p-1} - f(x,u_+)\Big]\,(u-u_+)^+\,dx \\ &= \langle A(u_+), (u-u_+)^+\rangle + \mu\int_\Omega u_+^{p-1}(u-u_+)^+\,dx.\end{aligned}$$

By the strict monotonicity of $t \mapsto |t|^{p-2}t$, $t \in \mathbb{R}$, this implies $u \le u_+$. Consequently, $u \in [0, u_+] \setminus \{0\}$, and (4.4) becomes

$$A(u) = \lambda u^{p-1} - f(\,\cdot\,, u) \quad \text{in } (W_n^{1,p}(\Omega))^*,$$

namely u turns out to be a solution of (1.1), $u \in \operatorname{int}(C_n^1(\overline{\Omega})_+)$, besides $u \le u_+$. Thanks to Theorem 3.7 we thus have $u = u_+$. So, $K(\varphi_\mu^+) \setminus \{0\} = \{u_+\}$. Now, (4.3) comes at once from $\widehat{u} \in K(\varphi_\mu^+) \setminus \{0\}$. Define

$$\varphi_\mu(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{p}\|u\|_p^p - \int_\Omega F_{\lambda+\mu}(x, u(x))\,dx$$

for all $u \in W_n^{1,p}(\Omega)$. Since

$$\varphi_\mu^+|_{C_n^1(\overline{\Omega})_+} = \varphi_\mu|_{C_n^1(\overline{\Omega})_+}$$

and $u_+ \in \operatorname{int}(C_n^1(\overline{\Omega})_+)$, combining (4.2), (4.3) with Proposition 2.1 ensures that u_+ is a $W_n^{1,p}(\Omega)$-local minimizer of φ_μ.

Similarly, the function v_- turns out a $W_n^{1,p}(\Omega)$-local minimizer of φ_μ. This can be verified as before, but with φ_μ^+ replaced by

$$\varphi_\mu^-(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{p}\|u\|_p^p - \int_\Omega F_{\lambda+\mu}^-(x, u(x))\,dx \quad \text{for all } u \in W_n^{1,p}(\Omega).$$

Without loss of generality we may assume that

$$\varphi_\mu(v_-) \le \varphi_\mu(u^+). \tag{4.5}$$

If u_+ is not an isolated critical point of φ_μ then there exists a sequence $\{u_k\} \subseteq W^{1,p}_n(\Omega)$ of pairwise distinct critical points for φ_μ converging to u_+. Since an argument analogous to that involving φ_μ^+ yields here

$$K(\varphi_\mu) \subseteq [v_-, u_+], \tag{4.6}$$

by the properties of v_- and u_+, each u_k turns out a nodal solution of (1.1), and the conclusion follows. Suppose now u_+ is isolated. The same reasoning exploited in the proof of [14, Proposition 6] provides $r > 0$ fulfilling

$$r < \|u_+ - v_-\|, \quad \varphi_\mu(u_+) < \inf_{u \in \partial B_r(u_+)} \varphi_\mu(u). \tag{4.7}$$

Moreover, the functional φ_μ satisfies the Palais–Smale condition, because it evidently is coercive. By (4.5) and (4.7), the classical Mountain pass Theorem can be applied. Thus, there exists $\overline{u} \in W^{1,p}_n(\Omega)$ such that

$$\varphi_\mu'(\overline{u}) = 0, \qquad \inf_{u \in \partial B_r(u_+)} \varphi_\mu(u) \le \varphi_\mu(\overline{u}).$$

On account of (4.5) and (4.7) again, the above inequality forces $\overline{u} \notin \{v_-, u_+\}$. Due to (4.6) we then get

$$A(\overline{u}) = \lambda |\overline{u}|^{p-2}\overline{u} - f(\,\cdot\,, \overline{u}) \quad \text{in } (W^{1,p}_n(\Omega))^*,$$

i.e. the function $\overline{u}$ solves problem (1.1). Standard results from nonlinear regularity theory (cf. [10]) finally give $\overline{u} \in C^1_n(\overline{\Omega})$. Bearing in mind Theorem 3.7, besides (4.6), the conclusion is achieved once we show that $\overline{u} \neq 0$. Define, for every $(t, u) \in [0,1] \times W^{1,p}_n(\Omega)$,

$$h(t,u) := t\varphi_\mu(u) + (1-t)\psi(u), \quad \text{where } \psi(u) := \frac{1}{p}(\|\nabla u\|_p^p - \lambda \|u\|_p^p).$$

Since φ_μ is coercive and $\lambda \notin \sigma(-\Delta_p)$, the function $h(t,\,\cdot\,)$, $t \in [0,1]$, satisfies the Palais–Smale condition. We claim that zero turns out an isolated critical point of $h(t,\,\cdot\,)$ uniformly in $t \in [0,1]$. If, on the contrary, $h_u'(t_k, u_k) = 0$ for some $\{(t_k, u_k)\} \subseteq [0,1] \times W^{1,p}_n(\Omega)$ with $(t_k, u_k) \to (t, 0)$ in $[0,1] \times W^{1,p}_n(\Omega)$, then

$$\begin{aligned} -\Delta_p u_k + t_k \mu |u_k|^{p-2} u_k &= t_k f_{\lambda+\mu}(x, u_k) + (1-t_k)\lambda |u_k|^{p-2} u_k \quad \text{in } \Omega, \\ \frac{\partial u_k}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.8}$$

Theorem 2 of [12] provides $\alpha \in \,]0,1[$, $M > 0$ fulfilling

$$\{u_k\} \subseteq C^{1,\alpha}_n(\overline{\Omega}) \quad \text{and} \quad \|u_k\|_{C^{1,\alpha}_n(\overline{\Omega})} \le M \quad \text{for all } k \in \mathbb{N}.$$

By compactness of the embedding $C_n^{1,\alpha}(\overline{\Omega}) \subseteq C_n^1(\overline{\Omega})$ this forces $u_k \to 0$ in $C_n^1(\overline{\Omega})$, where a subsequence is considered when necessary. Consequently, $u_k \in [v_-, u_+]$ for any sufficiently large k. Due to (4.8) we thus obtain

$$A(u_k) = \lambda |u_k|^{p-2} u_k - t_k f(\,\cdot\,, u_k) \quad \text{in } (W_n^{1,p}(\Omega))^*.$$

Now, arguing exactly as in the proof of Theorem 3.5 yields $\lambda \in \sigma(-\Delta_p)$, which is impossible.

Through the homotopy invariance property of critical groups [5, p. 334] one has

$$C_k(\varphi_\mu, 0) = C_k(h(1, \,\cdot\,), 0) = C_k(h(0, \,\cdot\,), 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

From Proposition 2.6 in [13] it follows

$$C_0(\varphi_\mu, 0) = C_1(\varphi_\mu, 0) = 0. \tag{4.9}$$

Observe that

$$C_1(\varphi_\mu, \overline{u}) \neq 0, \tag{4.10}$$

because $\overline{u}$ is a mountain pass point [5, Corollary 5.2.5]. Comparing (4.9) with (4.10) finally leads to $\overline{u} \neq 0$. □

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