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ON CURVED SQUEEZING AND CONLEY INDEX

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Dedicated to Professor Lech Górniewicz on the occasion of his 70-th birthday

Abstract. We consider reaction-diffusion equations on a family of domains depending on a parameter $\varepsilon > 0$. As $\varepsilon \to 0$, the domains degenerate to a lower dimensional manifold. Using some abstract results introduced in the recent paper [2] we show that there is a limit equation as $\varepsilon \to 0$ and obtain various convergence and admissibility results for the corresponding semiflows. As a consequence, we also establish singular Conley index and homology index continuation results. Under an additional dissipativeness assumption, we also prove existence and upper-semicontinuity of global attractors. The results of this paper extend and refine earlier results of [1] and [7].

1. Introduction

Let $\mathcal{M}$ be a smooth k-dimensional submanifold of $\mathbb{R}^\ell$. There is a so called normal neighbourhood $\mathcal{U}$ of $\mathcal{M}$ and for every $\varepsilon \in \,]0,1]$ there is a transformation $\Gamma_\varepsilon : \mathcal{U} \to \mathcal{U}$ which squeezes $\mathcal{U}$ by the factor ε orthogonally towards $\mathcal{M}$. Given a smooth bounded domain Ω with $\operatorname{Cl}\Omega \subset \mathcal{U}$ let $\Omega_\varepsilon = \Gamma_\varepsilon[\Omega]$ be the squeezed domain. Given a function $G: \mathbb{R} \to \mathbb{R}$ satisfying appropriate regularity and growth assumptions to be specified later, we consider, for each $\varepsilon \in \,]0,1]$, the semilinear parabolic Neumann boundary value problem

$$(\mathrm{E}_\varepsilon) \qquad \begin{aligned} \widetilde{u}_t &= \Delta \widetilde{u} + G(\widetilde{u}), && t > 0,\ \widetilde{x} \in \Omega_\varepsilon, \\ \partial_{\nu_\varepsilon} \widetilde{u} &= 0, && t > 0,\ \widetilde{x} \in \partial\Omega_\varepsilon. \end{aligned}$$

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Here, Δ is the Laplace operator in the $\widetilde{x}$-variable and ν_ε is the outer normal vector field on $\partial\Omega_\varepsilon$. If $\mathbf{B}_\varepsilon$ is the linear operator generated on $L^2(\Omega_\varepsilon)$ by the bilinear form

$$\widetilde{a}_\varepsilon\colon H^1(\Omega_\varepsilon)\times H^1(\Omega_\varepsilon)\to\mathbb{R},\quad (\widetilde{u},\widetilde{v})\mapsto\int_{\Omega_\varepsilon}\nabla\widetilde{u}\cdot\nabla\widetilde{v}\,d\widetilde{x}$$

and $\widehat{G}$ is the Nemitski operator of G, then equation (E_ε) can be written abstractly as the semilinear evolution equation

$$(\widetilde{\mathrm{F}}_\varepsilon)\qquad \dot{\widetilde{u}}=-\mathbf{B}_\varepsilon\widetilde{u}+\widehat{G}(\widetilde{u})$$

generating a local semiflow on the space $H^1(\Omega_\varepsilon)$. As $\varepsilon\to 0^+$, the domains degenerate to $\mathcal{M}$. The natural question arises if the family $(\widetilde{\mathrm{F}}_\varepsilon)$ has some limit equation and a corresponding limit local semiflow. This problem was considered in [7], where some previous results from [6] were generalized from the special flat squeezing case to general curved squeezed domains. The idea is to perform the change of variables $\widetilde{u}=u\circ\Gamma_\varepsilon$ in order to transform $(\widetilde{\mathrm{F}}_\varepsilon)$ to the equivalent problem

$$(\mathrm{F}_\varepsilon)\qquad \dot{u}=-\mathbf{A}_\varepsilon u+\widehat{G}(u)$$

with $\mathbf{A}_\varepsilon u=\mathbf{B}_\varepsilon(u\circ\Gamma_\varepsilon^{-1})\circ\Gamma_\varepsilon$, generating a local semiflow π_ε on the fixed space $H^1(\Omega)$. It turns out that the family (F_ε) has a limit equation

$$(\mathrm{F}_0)\qquad \dot{u}=-\mathbf{A}_0 u+\widehat{G}(u)$$

generating a local semiflow π_0 on a closed subspace $H^1_s(\Omega)$ of $H^1(\Omega)$. Some singular linear and nonlinear convergence results together with existence and upper semicontinuity results for attractors are established in [7], extending previous results from [6]. In the paper [1] a singular Conley continuation result for the family $(\pi_\varepsilon)_{\varepsilon\in[0,1]}$ in the flat squeezing case is proved. The essential growth assumption in all these papers is

$$|G'(u)|\le C(|u|^\beta+1),\quad u\in\mathbb{R}$$

for $\ell\ge 3$ where $\beta\le 2/(\ell-2)$. For the important case $\ell=3$, this means that

$$|G'(u)|\le C(|u|^2+1),\quad u\in\mathbb{R}.$$

The purpose of this paper is to extend these result to functions G of higher growth. In fact, we essentially assume, for $\ell=3$, that

$$|G'(u)|\le C(|u|^\beta+1),\quad u\in\mathbb{R}$$

where $\beta<4$. We will then verify the abstract conditions (Spec), (Comp) and (Conv) introduced in the recent paper [2]. The results of [2] then imply several singular linear and nonlinear convergence theorems for the corresponding local semiflows with the resulting singular Conley index and homology index braid continuation principles. We also prove (under some additional dissipativeness hypothesis) existence and upper semicontinuity of attractors. The results of this paper extend and refine results from [1] and [7].

2. Preliminaries

In this paper all linear spaces are over the reals.

Let H be a vector space and V be a linear subspace of H. Let $a\colon V \times V \to \mathbb{R}$ be a bilinear form on V and $b\colon H \times H \to \mathbb{R}$ be a bilinear form on H. If $\lambda \in \mathbb{R}$, $u \in V \setminus \{0\}$ satisfy

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v \in V$$

then we say that λ is *an eigenvalue of the pair* (a,b) and u is *an eigenvector of the pair* (a,b), *corresponding to* λ. The dimension of the span of all eigenvectors of (a,b) corresponding to λ is called *the multiplicity of* λ. If each eigenvalue has finite multiplicity and there is a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the eigenvalues of (a,b) and the number of occurrences of each eigenvalue in this sequence is equal to its multiplicity, then this sequence is uniquely determined and is called *the repeated sequence of the eigenvalues of* (a,b).

Given a and b as above define $R = R(a,b)$ to be the set of all pairs $(u,w) \in V \times H$ such that $a(u,v) = b(w,v)$ for all $v \in V$. We call R the *operator relation generated by the pair* (a,b). If R is the graph of a mapping $B\colon D(B) \to H$, then this map is called the *operator generated by the pair* (a,b).

It follows from the definition that B (if it exists) is linear and (λ,u) is an eigenvalue-eigenvector pair of (a,b) if and only if (λ,u) is an eigenvalue-eigenvector pair of B.

Let us also note that the condition for the existence of a repeated sequence of eigenvalues as given in [6] and [7] is insufficient. This, however, is completely irrelevant for the validity of the results contained in those papers.

The following proposition is well-known:

Proposition 2.1. *Let V, H be two infinite dimensional Hilbert spaces. Suppose $V \subset H$ with compact inclusion, and V is dense in H. Let $b = \langle\,\cdot\,,\,\cdot\,\rangle$ be the inner product of H and $\|\cdot\|$ and $|\cdot|$ denote the Euclidean norms of V and H. Let $a\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. Assume that there are constants d, C, $\alpha \in \mathbb{R}$, $\alpha > 0$, such that, for all u, $v \in V$,*

$$|a(u,v)| \le C\|u\|\|v\|,$$
$$a(u,u) \ge \alpha\|u\|^2 - d|u|^2.$$

Then the operator relation generated by (a,b) is the graph of a linear selfadjoint operator B on (H,b) with compact resolvent. Moreover, each eigenvalue of (a,b) (equivalently, of B) has finite multiplicity and the repeated sequence $(\lambda_n)_n$

of eigenvalues of (a,b) *exists. Furthermore, there exists a* b*-orthonormal and* b*-complete sequence* $(w_n)_n$ *in* V *such that, for each* $n \in \mathbb{N}$, w_n *is an eigenvector of* (a,b) *corresponding to* λ. *Finally, if* $d = 0$, *then* B *is positive,* $D(B^{1/2}) = V$ *and*

$$a(u,v) = b(B^{1/2}u, B^{1/2}v), \quad u, v \in V.$$

Suppose $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ is an infinite dimensional Hilbert space and let $\mathbf{A}\colon D(\mathbf{A}) \subset H \to H$ be a (densely defined) selfadjoint operator on $(H, \langle\,\cdot\,,\,\cdot\,\rangle_H)$ such that, for some $\lambda \in \,]0,\infty[$, the operator $A = \mathbf{A} + \lambda\,\mathrm{Id}_H$ is positive with $A^{-1}\colon H \to H$ compact. Using the notation of [2] the linear space $H_\beta = H_\beta(A) = D(A^{\beta/2})$, $\beta \in [0,\infty[$ is a Hilbert space under the scalar product

$$\langle u, v\rangle_{H_\beta} = \langle A^{\beta/2}u, A^{\beta/2}v\rangle_H, \quad u, v \in H_\beta.$$

For $\beta \in \,]0,\infty[$, let $H_{-\beta} = H'_\beta$ be the dual of H_β. It follows that $H_{-\beta}$ is a Hilbert space under the dual scalar product

$$\langle u, v\rangle_{H_{-\beta}} = \langle F_\beta^{-1}v, F_\beta^{-1}u\rangle_{H_\beta}, \quad u, v \in H_{-\beta},$$

where $F_\beta\colon H_\beta \to H_{-\beta}$, $u \mapsto \langle\,\cdot\,, u\rangle_{H_\beta}$, is the Fréchet–Riesz isomorphism.

For $\beta \in [0,\infty[$ define the map $\psi_\beta\colon H = H_0 \to H_{-\beta}$ by $\psi_\beta(u) = y$, where $y\colon H_\beta \to \mathbb{R}$ is defined by

$$y(v) = \langle v, u\rangle_H, \quad v \in H_\beta.$$

ψ_β is an injection (which we call the *canonical embedding*) so that we can (and will) identify elements $u \in H$ with $\psi_\beta(u) \in H_{-\alpha}$. We thus consider H as a linear subspace of $H_{-\beta}$.

Fix an $\alpha \in [0,\infty[$. Let $\widetilde{A} = \widetilde{A}_\alpha\colon H_{2-\alpha} \to H_{-\alpha}$ be the unique continuous extension of $\psi_\alpha \circ A\colon H_2 \to H_{-\alpha}$. Then $\widetilde{A}$ is a densely defined positive selfadjoint operator on the Hilbert space $H_{-\alpha}$ with $\widetilde{A}^{-1}\colon H \to H$ compact. Moreover, for every $\beta \in \mathbb{R}$, the Hilbert space $H_\beta(\widetilde{A})$ is isometrically isomorphic to the Hilbert space $H_{\beta-\alpha}(A)$. Define $\widetilde{\mathbf{A}} = \widetilde{A} - \lambda\,\mathrm{Id}_{H_{-\alpha}}\colon H_{2-\alpha} \to H_{-\alpha}$. Then $\widetilde{\mathbf{A}}$ is a densely defined selfadjoint operator on the Hilbert space $H_{-\alpha}$. Moreover, $\widetilde{\mathbf{A}}\colon H_{2-\alpha} \to H_{-\alpha}$ is the unique continuous extension of $\psi_\alpha \circ \mathbf{A}\colon H_2 \to H_{-\alpha}$.

Now suppose that $\alpha \in [0,1[$ and let $g\colon H_1 \to H_{-\alpha}$ be a locally Lipschitzian map. Then g can be regarded as a locally Lipschitzian map from $H_{1+\alpha}(\widetilde{A})$ to $H_0(\widetilde{A})$, so that we may consider the abstract semilinear parabolic equation

$$\dot{u} = -\widetilde{\mathbf{A}}u + g(u) \tag{2.1}$$

generating a local semiflow on H_1. By the definition of solution given in [5] and equivalent to the definition of solution of the corresponding integral equation, we see that a function $u\colon [0,t_0[\ \to H_1$, where $t_0 \in\ ]0,\infty]$, is a solution of (2.1) iff u is a solution of

$$\dot u = -\widetilde{A}u + (\lambda u + g(u)). \tag{2.2}$$

Now let $(\underline{H}, \langle\cdot\,,\,\cdot\rangle_{\underline{H}})$ be another Hilbert space and $R\colon H \to \underline{H}$ be a bijective linear isometry. Let $R'\colon \underline{H}' \to H'$ be the dual map. Define the map $\underline{\mathbf{A}}\colon R[D(\mathbf{A})] \to \underline{H}$ by

$$\underline{\mathbf{A}}Ru = R\mathbf{A}u, \quad u \in D(\mathbf{A}).$$

Then $\underline{\mathbf{A}}$ is a densely defined selfadjoint operator on the Hilbert space $\underline{H}$ such that $\underline{A} = \underline{\mathbf{A}} + \lambda\,\mathrm{Id}_{\underline{H}}$ is positive with $\underline{A}^{-1}\colon \underline{H} \to \underline{H}$ compact. Let $\underline{H}_\beta = H_\beta(\underline{A})$, $\beta \in \mathbb{R}$, be the corresponding scale of Hilbert spaces. For $\beta \in [0,\infty[$ let $\underline{\psi}_\beta\colon \underline{H} = \underline{H}_0 \to \underline{H}_{-\beta}$ be the corresponding canonical embedding.

Let $\widetilde{\underline{A}} = \widetilde{\underline{A}}_\alpha\colon \underline{H}_{2-\alpha} \to \underline{H}_{-\alpha}$ be the unique continuous extension of $\psi_\alpha \circ \underline{A}\colon \underline{H}_2 \to \underline{H}_{-\alpha}$. Again $\widetilde{\underline{A}}$ is a densely defined positive selfadjoint operator on the Hilbert space $\underline{H}_{-\alpha}$ with $\widetilde{\underline{A}}^{-1}\colon \underline{H} \to \underline{H}$ compact. Define $\widetilde{\underline{\mathbf{A}}} = \widetilde{\underline{A}} - \lambda\,\mathrm{Id}_{\underline{H}_{-\alpha}}\colon \underline{H}_{2-\alpha} \to \underline{H}_{-\alpha}$. Then again $\widetilde{\underline{\mathbf{A}}}$ is a densely defined selfadjoint operator on the Hilbert space $\underline{H}_{-\alpha}$.

It is easily seen that for each $\beta \in [0,\infty[$ the map R induces, by restriction, an isometry R_β from H_β onto $\underline{H}_\beta$ and so the dual map $R_{-\beta} := R'_\beta$ is an isometry from $\underline{H}_{-\beta}$ onto $H_{-\beta}$. Since, as is immediately seen, $\psi_\beta = R_{-\beta} \circ \underline{\psi}_\beta \circ R_\beta$, it follows that

$$\widetilde{A} = R_{-\alpha} \circ \widetilde{\underline{A}} \circ R_{2-\alpha}.$$

Define the function $\underline{g}\colon \underline{H}_1 \to \underline{H}_{-\alpha}$ by

$$g = R_{-\alpha} \circ \underline{g} \circ R_1.$$

We call $\underline{g}$ *the R-conjugate of g*. It follows that a function $u\colon [0,t_0[\ \to H_1$, where $t_0 \in\ ]0,\infty]$ is a solution of equation (2.1) if and only if the function $\underline{u} = R \circ u$ is a solution of the equation

$$\dot{\underline{u}} = -\widetilde{\underline{\mathbf{A}}}\,\underline{u} + \underline{g}(\underline{u}). \tag{2.3}$$

3. The abstract conditions (Spec) and (Comp).

In this section we will introduce the operators $\mathbf{A}_\varepsilon$ mentioned above and prove some abstract results about them. In particular, we prove the abstract conditions (Spec) and (Comp) introduced in [2]. As a consequence, we obtain, in Conclusion 3.13, two singular convergence theorems for the corresponding families of linear semigroups.

We assume the reader's familiarity with the abstract part of [2] and with the paper [7]. However, for the reader's convenience we collect (with different notation) some relevant technical material from [7] and correct some inaccuracies of that paper. In particular, we give a correct definition of a normal neighbourhood and prove its existence.

We assume throughout that ℓ, k and r are positive integers with $r \geq 2$, $\ell \geq 2$ and $k < \ell$. By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^\ell$.

Let $\mathcal{M} \subset \mathbb{R}^\ell$ be k-dimensional submanifold of $\mathbb{R}^\ell$ of class C^r. For $p \in \mathcal{M}$ we denote by $T_p\mathcal{M}$ the tangent space to $\mathcal{M}$ at the point p. We will identify $T_p\mathcal{M}$ in the usual way with a subspace of $\mathbb{R}^\ell$.

Definition 3.1. An open set $\mathcal{U}$ in $\mathbb{R}^\ell$ is called a *normal neighbouurhood* (or *normal strip*) *of* $\mathcal{M}$ if there is a map $\phi: \mathcal{U} \to \mathcal{M}$ of class C^{r-1}, called an *orthogonal projection of* $\mathcal{U}$ *onto* $\mathcal{M}$ and a continuous function $\delta: \mathcal{M} \to]0, \infty]$, called *the thickness of* $\mathcal{U}$ such that:

(a) whenever $x \in \mathcal{U}$ and $p \in \mathcal{M}$ then $\phi(x) = p$ if and only if the vector $x - p$ is orthogonal to $T_p\mathcal{M}$ (in $\mathbb{R}^\ell$) and $\|x - p\| < \delta(p)$;

(b) $\varepsilon x + (1 - \varepsilon)\phi(x) \in \mathcal{U}$ for all $x \in \mathcal{U}$ and all $\varepsilon \in [0, 1]$.

Examples.

(a) (Flat squeezing case) Let $\mathcal{M} = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^\ell$, $\mathcal{U} = \mathbb{R}^\ell$, $\phi: \mathbb{R}^\ell = \mathbb{R}^k \times \mathbb{R}^{\ell-k} \to \mathcal{M}$, $(x_1, x_2) \mapsto (x_1, 0)$, $\delta \equiv \infty$. With this choice of ϕ and δ, the set $\mathcal{U}$ is a normal neighbourhood of $\mathcal{M}$.

(b) Let $S^{\ell-1} \subset \mathbb{R}^\ell$ be the $\ell - 1$-dimensional unit sphere, $\mathcal{U}$ be the set of all $x \in \mathbb{R}^\ell$ with $0 < \|x\| < 2$, $\phi: \mathcal{U} \to \mathcal{M}$, $x \mapsto x/\|x\|$ and $\delta \equiv 1$. With this choice of ϕ and δ, the set $\mathcal{U}$ is a normal neighbourhood of $\mathcal{M}$.

We will show later that normal neighbourhoods always exist.

Remark 3.2. In the definition of a normal neighbourhood given in [7] the function δ was erroneously omitted. This does not affect the validity of the results of that paper, which hold true under the present, correct Definition 3.1. On the other hand, normal neighbourhoods in the sense of [7] might not exist. This is e.g. the case for $\mathcal{M} = S^{\ell-1}$: for $x \in \mathbb{R}^\ell \setminus \{0\}$ and $p \in \mathcal{M}$, the vector $x - p$ is orthogonal to $T_p(\mathcal{M})$ if and only if $p = x/\|x\|$ or $p = -x/\|x\|$, so there is no map ϕ from a neighbourhood U of $\mathcal{M}$ to $\mathcal{M}$ such that, for each $x \in U$ and each $p \in \mathcal{M}$, $\phi(x) = p$ if and only if $x - p$ is orthogonal to $T_p(\mathcal{M})$.

In the sequel we consider a fixed normal neighbourhood $\mathcal{U}$ of $\mathcal{M}$ with orthogonal projection $\phi: \mathcal{U} \to \mathcal{M}$.

For $\varepsilon \in [0, 1]$ define the *curved squeezing* transformation

$$\Gamma_\varepsilon: \mathcal{U} \to \mathbb{R}^\ell, \quad x \mapsto \varepsilon x + (1 - \varepsilon)\phi(x) = \phi(x) + \varepsilon(x - \phi(x)). \tag{3.1}$$

The following important properties follow from Definition 3.1:

Proposition 3.3. *Let* $\mathcal{U}$ *be a normal neighbourhood of* $\mathcal{M}$. *Then*

(a) $\phi(\mathcal{U}) = \mathcal{M}$ *and* $\phi(x) = x$ *if and only if* $x \in \mathcal{M}$*;*

(b) $D\phi(x)\nu = 0$ *for all* $x \in \mathcal{U}$ *and all vectors* ν *orthogonal to* $T_p\mathcal{M}$*, where* $p := \phi(x)$.

Proof. Part (a) is obvious. If $x \in \mathcal{U}$ and $p = \phi(x)$, then $x - p$ is orthogonal to $T_p\mathcal{M}$ and $\|x - p\| < \delta(p)$. Therefore, for all $t \in \mathbb{R}$ with $|t|$ sufficiently small, $x + t\nu \in \mathcal{U}$, $\|(x + t\nu) - p\| < \delta(p)$ and $(x + t\nu) - p = (x - p) + t\nu$ is orthogonal to $T_p\mathcal{M}$. Thus $\phi(x + t\nu) \equiv p$ for all $|t|$ small enough. In particular, $D\phi(x)\nu = 0$. This proves part (b). □

Proposition 3.4. *If* $\mathcal{M}$ *as above, then there exists a normal neighbourhood of* $\mathcal{M}$ *in the sense of Definition* 3.1.

Proof. Define, as usual, the normal bundle $N(\mathcal{M})$ as the subset of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ consisting of all the ordered pairs (p, w) where $p \in \mathcal{M}$ and w is orthogonal (in $\mathbb{R}^\ell$) to $T_p(\mathcal{M})$. $N(\mathcal{M})$ is a C^{r-1}-submanifold of $\mathbb{R}^\ell \times \mathbb{R}^\ell$. Let $\widetilde{\mathbf{S}}: \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the map $(p, w) \mapsto p + w$. This is a C^∞-map so its restriction $\mathbf{S} = \widetilde{\mathbf{S}}_{|N(\mathcal{M})}$ to $N(\mathcal{M})$ is a C^{r-1}-map. For $\alpha \in \,]0, \infty[$ and $p \in \mathcal{M}$ let $B_\alpha(p)$ be the set of all $(q, w) \in N(\mathcal{M})$ such that $\|q - p\| < \alpha$ and $\|w\| < \alpha$. Clearly,

$$B_\alpha(p') \subset B_{\alpha + \|p' - p\|}(p), \quad p, p' \in \mathcal{M}. \tag{3.2}$$

For each $p \in \mathcal{M}$ the tangent map $T_{(p,0)}\mathbf{S}: T_{(p,0)}N(\mathcal{M}) \to T_p\mathbb{R}^\ell \approx \mathbb{R}^\ell$ is an isomorphism. Thus there is an $\alpha = \alpha(p) > 0$ such that $\mathbf{S}_{|B_\alpha(p)}$ is a C^{r-1}-diffeomorphism onto its image, this image being open in $\mathbb{R}^\ell$. We define $\widetilde{\delta}(p)$ to be the supremum of all numbers $\rho > 0$ such that $\mathbf{S}[B_\rho(p)]$ is open in $\mathbb{R}^\ell$ and $\mathbf{S}_{|B_\rho(p)}$ is a C^{r-1}diffeomorphism of $B_\rho(p)$ onto $\mathbf{S}[B_\rho(p)]$. Thus $\widetilde{\delta}(p) \in \,]0, \infty]$ is defined. If $\widetilde{\delta}(p) = \infty$ then, by (3.2), $\widetilde{\delta}(p') = \infty$ for every $p' \in \mathcal{M}$. Thus either $\widetilde{\delta}(p) \equiv \infty$, so $\widetilde{\delta}$ is continuous or else $\widetilde{\delta}(p) < \infty$ for every $p \in \mathcal{M}$. We also have

$$\rho \le \widetilde{\delta}(p') + \|p' - p\|, \quad p, p' \in \mathcal{M}, \; \rho \in \,]0, \widetilde{\delta}(p)[\,. \tag{3.3}$$

In fact, this is clear if $\rho \le \|p' - p\|$. If $\rho > \|p' - p\|$, then, by (3.2),

$$B_{\rho - \|p' - p\|}(p') \subset B_\rho(p), \quad p, p' \in \mathcal{M}$$

so $\rho - \|p' - p\| \le \widetilde{\delta}(p')$. This proves (3.3) and thus

$$\widetilde{\delta}(p) \le \widetilde{\delta}(p') + \|p' - p\|, \quad p, p' \in \mathcal{M}.$$

By exchanging p with p' we finally obtain

$$|\widetilde{\delta}(p) - \widetilde{\delta}(p')| \le \|p - p'\|, \quad p, p' \in \mathcal{M}.$$

In particular, $\widetilde{\delta}$ again is continuous. Let $\delta = \widetilde{\delta}/2$. Let V be the set of all $(p, w) \in N(\mathcal{M})$ with $\|w\| < \delta(p)$. Continuity of δ implies that V is open in $N(\mathcal{M})$. Since, by construction, $\mathbf{S}$ is a local diffeomorphism on V, it follows that $\mathcal{U} := \mathbf{S}[V]$ is open in $\mathbb{R}^\ell$. We claim that $\mathbf{S}_{|V}$ is injective. In fact, let (p, w) and

$(p', w') \in V$ be such that $p + w = p' + w'$. We may assume that $\delta(p') \le \delta(p)$. It follows that $\|p' - p\| \le \|w\| + \|w'\| < \widetilde{\delta}(p)$. It follows that there is a $\rho \in]0, \widetilde{\delta}(p)[$ such that (p, w), $(p', w') \in B_\rho(p)$. Since $\mathbf{S}_{|B_\rho(p)}$ is injective, $(p, w) = (p', w')$. It follows that $\mathbf{S}_{|V}: V \to \mathcal{U}$ is a C^{r-1}-diffeomorphism. Let $\phi: \mathcal{U} \to \mathcal{M}$ be the map $\pi_1 \circ (\mathbf{S}_{|V})^{-1}$, where $\pi_1: N(\mathcal{M}) \to \mathcal{M}$ is the projection onto the first component. It follows that ϕ is a C^{r-1}-map. It is immediate that the set $\mathcal{U}$ and the functions ϕ and δ satisfy the conditions of Definition 3.1. $\square$

For $x \in \mathcal{U}$ we denote by $Q(x): \mathbb{R}^\ell \to \mathbb{R}^\ell$ the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto $T_p\mathcal{M}$, where $p := \phi(x)$. Let $P(x) = I - Q(x)$. Note that $P(x)$ is the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto the orthogonal complement of $T_p\mathcal{M}$ in $T_p\mathbb{R}^\ell \cong \mathbb{R}^\ell$.

The following properties are an immediate consequence of the definition:

Proposition 3.5. *The map* $[0,1] \times \mathcal{U} \to \mathbb{R}^\ell$, $(\varepsilon, x) \mapsto \Gamma_\varepsilon(x)$, *is continuous. Let* $\varepsilon \in]0,1]$ *be arbitrary. Then*

(a) $\Gamma_\varepsilon[\mathcal{U}] = \{\, y \in \mathcal{U} \mid \phi(y) + (1/\varepsilon)(y - \phi(y)) \in \mathcal{U} \,\}$, $\Gamma_\varepsilon[\mathcal{U}]$ *is open in* $\mathbb{R}^\ell$ *and* $\Gamma_\varepsilon: \mathcal{U} \to \Gamma_\varepsilon[\mathcal{U}]$ *is a diffeomorphism of class* C^{r-1} *with*

$$\Gamma_\varepsilon^{-1}(y) = \phi(y) + (1/\varepsilon)(y - \phi(y)), \quad y \in \Gamma_\varepsilon(\mathcal{U});$$

(b) $\phi(\Gamma_\varepsilon(x)) = \phi(x)$ *for* $x \in \mathcal{U}$.

Theorem 3.6. ([7]) *For* $x \in \mathcal{U}$ *and* $\varepsilon \in [0,1]$ *define*

$$J_\varepsilon(x) := \begin{cases} \varepsilon^{-(\ell-k)/2} |\det D\Gamma_\varepsilon(x)| & \text{if } \varepsilon > 0, \\ |\det(D\phi(x)|_{T_{\phi(x)}\mathcal{M}})| & \text{otherwise.} \end{cases}$$

Then

$$J_\varepsilon(x) > 0 \quad \text{for all } \varepsilon \in [0,1] \text{ and } x \in \mathcal{U}.$$

Moreover, the function $[0,1] \times \mathcal{U} \to \mathbb{R}$, $(\varepsilon, x) \mapsto J_\varepsilon(x)$, *is continuous. For every* $\varepsilon \in [0,1]$ *and for every* $x \in \mathcal{U}$ *there exists a linear map* $S_\varepsilon(x): \mathbb{R}^\ell \to \mathbb{R}^\ell$ *such that, for* $\varepsilon \in]0,1]$,

$$D\Gamma_\varepsilon^{-1}(\Gamma_\varepsilon(x)) = S_\varepsilon(x) + (1/\varepsilon)P(x) \quad \text{for all } x \in \mathcal{U}.$$

Accordingly,

$$(D\Gamma_\varepsilon^{-1}(\Gamma_\varepsilon(x)))^T = S_\varepsilon(x)^T + (1/\varepsilon)P(x) \quad \text{for all } x \in \mathcal{U}.$$

The following properties are satisfied:

(a) *the maps* $[0,1] \times \mathcal{U} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$,

$$(\varepsilon, x) \mapsto S_\varepsilon(x) \quad \text{and} \quad (\varepsilon, x) \mapsto S_\varepsilon(x)^T$$

are continuous;

(b) *for every* $\varepsilon \in [0,1]$, *for every* $x \in \mathcal{U}$ *and for every* ν *orthogonal to* $T_{\phi(x)}\mathcal{M}$

$$S_\varepsilon(x)\nu = S_\varepsilon(x)^T\nu = 0;$$

(c) *for every* $\varepsilon \in [0,1]$ *and for every* $x \in \mathcal{U}$ *the maps*

$$S_\varepsilon(x)|_{T_{\phi(x)}\mathcal{M}}: T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}, \quad S_\varepsilon(x)^T|_{T_{\phi(x)}\mathcal{M}}: T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

are well-defined and bijective. Furthermore,

$$(S_0(x)|_{T_{\phi(x)}\mathcal{M}})^{-1} = D\phi(x)|_{T_{\phi(x)}\mathcal{M}}$$

and

$$(S_0(x)^T|_{T_{\phi(x)}\mathcal{M}})^{-1} = D\phi(x)^T|_{T_{\phi(x)}\mathcal{M}}.$$

Finally, $\phi: \mathcal{U} \to \mathcal{M}$ *is an open map.*

Note that the maps S_ε, $\varepsilon \in [0,1]$ in Theorem 3.6 are uniquely determined.

Let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^\ell$ with Lipschitz boundary and such that $\operatorname{Cl}\Omega \subset \mathcal{U}$.

For $\varepsilon \in \,]0,1]$, define the *curved squeezed domain* $\Omega_\varepsilon := \Gamma_\varepsilon[\Omega]$. For $\varepsilon \in \,]0,1]$ define the following bilinear forms:

$$\widetilde{a}_\varepsilon: H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}, \quad (\widetilde{u},\widetilde{v}) \mapsto \int_{\Omega_\varepsilon} \nabla\widetilde{u}(x)\cdot\nabla\widetilde{v}(x)\,dx;$$

$$\check{a}_\varepsilon: H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}, \quad (\widetilde{u},\widetilde{v}) \mapsto \varepsilon^{-(\ell-k)/2}\int_{\Omega_\varepsilon} \nabla\widetilde{u}(x)\cdot\nabla\widetilde{v}(x)\,dx;$$

$$\begin{aligned} a_\varepsilon: H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \quad (u,v) \mapsto &\int_\Omega J_\varepsilon(x)\langle S_\varepsilon(x)^T\nabla u(x), S_\varepsilon(x)^T\nabla v(x)\rangle\,dx \\ &+ \frac{1}{\varepsilon^2}\int_\Omega J_\varepsilon(x)\langle P(x)\nabla u(x), P(x)\nabla v(x)\rangle\,dx; \end{aligned}$$

$$\widetilde{b}_\varepsilon: L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \to \mathbb{R}, \quad (\widetilde{u},\widetilde{v}) \mapsto \int_{\Omega_\varepsilon} \widetilde{u}(x)\widetilde{v}(x)\,dx.$$

$$\check{b}_\varepsilon: L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \to \mathbb{R}, \quad (\widetilde{u},\widetilde{v}) \mapsto \varepsilon^{-(\ell-k)/2}\int_{\Omega_\varepsilon} \widetilde{u}(x)\widetilde{v}(x)\,dx.$$

For $\varepsilon \in [0,1]$ define the bilinear form

$$b_\varepsilon: L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}, \quad (u,v) \mapsto \int_\Omega J_\varepsilon(x)u(x)v(x)\,dx.$$

We have

$$\widetilde{a}_\varepsilon(u,u) + \widetilde{b}_\varepsilon(u,u) = |u|^2_{H^1(\Omega_\varepsilon)}, \quad \varepsilon \in \,]0,1]\,,\ u \in H^1(\Omega_\varepsilon). \tag{3.4}$$

Let $\varepsilon \in \,]0,1]$ be arbitrary. Then Proposition 2.1 and (3.4) imply that the pair $(\widetilde{a}_\varepsilon, \widetilde{b}_\varepsilon)$ generates a densely defined selfadjoint operator $\mathbf{B}_\varepsilon$ in $(L^2(\Omega_\varepsilon), \widetilde{b}_\varepsilon)$, which we interpret, as usual, as the operator $-\Delta$ on Ω_ε with Neumann boundary condition on $\partial\Omega_\varepsilon$. Since $\check{a}_\varepsilon = \varepsilon^{-(\ell-k)/2}\widetilde{a}_\varepsilon$ and $\check{b}_\varepsilon = \varepsilon^{-(\ell-k)/2}\widetilde{b}_\varepsilon$, we see that

(3.5) the pair $(\check{a}_\varepsilon, \check{b}_\varepsilon)$ generates $\mathbf{B}_\varepsilon$ and both $\mathbf{B}_\varepsilon$ and $B_\varepsilon := \mathbf{B}_\varepsilon + \mathrm{Id}_{L^2(\Omega_\varepsilon)}$ are densely defined selfadjoint operators in $(L^2(\Omega_\varepsilon), \check{b}_\varepsilon)$ with B_ε positive and $B_\varepsilon^{-1}\colon L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ compact.

Let $\alpha \in \,]0,1[$ and $\psi_\alpha\colon H_0(B_\varepsilon) = L^2(\Omega) \to H_{-\alpha}(B_\varepsilon)$ be the canonical embedding and $\widetilde{\mathbf{B}}_\varepsilon\colon H_{2-\alpha}(B_\varepsilon) \to H_{-\alpha}(B_\varepsilon)$ be the unique continuous extension of $\psi_\alpha \circ \mathbf{B}_\varepsilon$. Then, for $u \in D(B_\varepsilon) = H_2(B_\varepsilon)$ and $v \in H_1(B_\varepsilon)$,

$$\widetilde{\mathbf{B}}_\varepsilon u(v) = \psi_\alpha(\mathbf{B}_\varepsilon u)(v) = \check{b}_\varepsilon(\mathbf{B}_\varepsilon u, v) = \check{a}_\varepsilon(u,v).$$

Thus a simple density and continuity argument shows that

$$\widetilde{\mathbf{B}}_\varepsilon u(v) = \check{a}_\varepsilon(u,v), \quad u \in H_{2-\alpha}(B_\varepsilon),\ v \in H_1(B_\varepsilon). \tag{3.6}$$

Note that, by Theorem 3.6 there are constants C, $c \in \,]0,\infty[$ such that

$$cb_\varepsilon(u,u) \le |u|^2_{L^2(\Omega)} \le Cb_\varepsilon(u,u), \quad \text{for } \varepsilon \in [0,1] \text{ and } u \in L^2(\Omega). \tag{3.7}$$

Let us now define the space

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid P(x)\nabla u(x) = 0 \ \text{ a.e.}\,\}.$$

Note that

(3.8) $H^1_s(\Omega)$ is a closed linear subspace of the Hilbert space $H^1(\Omega)$.

We have the following

Proposition 3.7. ([7]) *The space $H^1_s(\Omega)$ is infinite dimensional.*

Now define the "limit" bilinear form

$$a_0\colon H^1_s(\Omega) \times H^1_s(\Omega) \to \mathbb{R}, \quad (u,v) \mapsto \int_\Omega J_0(x)\langle S_0(x)^T\nabla u(x), S_0(x)^T\nabla v(x)\rangle\,dx.$$

Finally, let $L^2_s(\Omega)$ be the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$. Note that

(3.9) $L^2_s(\Omega)$ is a closed linear subspace of the Hilbert space $L^2(\Omega)$.

For $\varepsilon \in \,]0,1]$ and u, $v \in L^2(\Omega)$ set

$$\langle u,v\rangle_\varepsilon := b_\varepsilon(u,v).$$

For $\varepsilon \in \,]0,1]$ and u, $v \in H^1(\Omega)$ set

$$\langle\!\langle u,v\rangle\!\rangle_\varepsilon := a_\varepsilon(u,v) + b_\varepsilon(u,v).$$

By (3.7), $\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_\varepsilon$) is a scalar product on $L^2(\Omega)$ (resp. $H^1(\Omega)$). Let $|\cdot|_\varepsilon$ (resp. $\|\cdot\|_\varepsilon$) be the Euclidean norm on $L^2(\Omega)$ (resp. $H^1(\Omega)$) induced by $\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_\varepsilon$). Furthermore, for u, $v \in L^2_s(\Omega)$ set

$$\langle u, v\rangle_0 := b_0(u, v).$$

Finally, for u, $v \in H^1_s(\Omega)$ set

$$\langle\!\langle u, v\rangle\!\rangle_0 := a_0(u, v) + b_0(u, v).$$

Again by (3.7), $\langle\,\cdot\,,\,\cdot\,\rangle_0$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_0$) is a scalar product on $L^2_s(\Omega)$ (resp. $H^1_s(\Omega)$).

Let $|\cdot|_0$ (resp. $\|\cdot\|_0$) be the Euclidean norm on $L^2_s(\Omega)$ (resp. $H^1_s(\Omega)$) induced by $\langle\,\cdot\,,\,\cdot\,\rangle_0$ (resp. $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_0$).

Proposition 3.8 ([7]). *The following statements hold:*

(a) *For every $\delta \in\,]0,1[$ there exists an $\overline{\varepsilon} \in\,]0,1]$ such that*

$$(1-\delta)b_0(u,u) \le b_\varepsilon(u,u) \le (1+\delta)b_0(u,u) \tag{3.10}$$

for all $u \in L^2(\Omega)$ and $\varepsilon \in\,]0,\overline{\varepsilon}]$, and

$$(1-\delta)a_0(u,u) \le a_\varepsilon(u,u) \le (1+\delta)a_0(u,u) \tag{3.11}$$

for all $u \in H^1_s(\Omega)$ and $\varepsilon \in\,]0,\overline{\varepsilon}[$.

(b) *Whenever u and $v \in L^2(\Omega)$, then*

$$b_\varepsilon(u,v) \to b_0(u,v) \quad \text{as } \varepsilon \to 0. \tag{3.12}$$

(c) *There is a constant $C \in\,]1,\infty[$ such that*

$$\|u\|_\varepsilon \le C\|u\|_0 \ \text{and}\ \|u\|_0 \le C\|u\|_\varepsilon$$

for all $u \in H^1_s(\Omega)$ and all $\varepsilon \in\,]0,1]$.

(d)

$$a_\varepsilon(u,u) \to a_0(u,u) \quad \text{as } \varepsilon \to 0, \tag{3.13}$$

for all $u \in H^1_s(\Omega)$.

(e) *There exists a $\gamma \in\,]0,\infty[$ such that*

$$\gamma|u|_{H^1(\Omega)} \le \|u\|_\varepsilon \quad \text{for all } \varepsilon \in\,]0,1] \text{ and } u \in H^1(\Omega). \tag{3.14}$$

By (3.7) the norms $|\cdot|_\varepsilon$, $\varepsilon \in [0,1]$, are all equivalent to the usual norm on $L^2(\Omega)$, with equivalence constants independent of ε. Writing $H^\varepsilon = L^2(\Omega)$ for $\varepsilon \in\,]0,1]$ and $H^0 = L^2_s(\Omega)$ we thus see, using Proposition 3.7, that

$$\text{for } \varepsilon \in [0,1],\ (H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon) \text{ is an infinite-dimensional Hilbert space.} \tag{3.15}$$

By (3.14) and a trivial estimate the norm $\|\cdot\|_\varepsilon$ is, for each $\varepsilon \in\,]0,1]$, equivalent to the usual norm $|\cdot|_{H^1(\Omega)}$, one of the constants depending on ε this time. It follows that, for $\varepsilon \in\,]0,1]$, $(H^1(\Omega), \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_\varepsilon)$ is a Hilbert space which is densely and compactly embedded in $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$.

Now let $(u_k)_k$ in $H^1_s(\Omega)$ be a $\|\cdot\|_0$-Cauchy sequence. By Proposition 3.8 part (c) we have that, for any given $\varepsilon \in\,]0,1]$, $(u_k)_k$ is a $\|\cdot\|_\varepsilon$-Cauchy sequence and consequently also a $|\cdot|_{H^1(\Omega)}$-Cauchy sequence. Thus, for some $u \in H^1(\Omega)$, the sequence $(u_k)_k$ converges to u in the $|\cdot|_{H^1(\Omega)}$-norm. Hence $u \in H^1_s(\Omega)$ as $H^1_s(\Omega)$ is closed in $H^1(\Omega)$ in the $|\cdot|_{H^1(\Omega)}$-norm. It follows that $(u_k)_k$ converges to u in the $|\cdot|_\varepsilon$-norm and thus, since $u_k - u \in H^1_s(\Omega)$ for $k \in \mathbb{N}$, $(u_k)_k$ converges to u in the $|\cdot|_0$-norm. It follows that $(H^1_s(\Omega), \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_0)$ is a Hilbert space. By definition, $H^1_s(\Omega)$ is dense in $(H^0, \langle\cdot\,,\,\cdot\rangle_0)$.

Now let $(u_k)_k$ in $H^1_s(\Omega)$ be $\|\cdot\|_0$-bounded. By Proposition 3.8 part (c) we have that, for any given $\varepsilon \in\,]0,1]$, $(u_k)_k$ is $\|\cdot\|_\varepsilon$-bounded and so $|\cdot|_{H^1(\Omega)}$-bounded. Thus, for some $u \in L^2(\Omega)$, a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, converges to u in the $|\cdot|_{L^2(\Omega)}$-norm. Hence $u \in L^2_s(\Omega)$. From (3.7) we obtain that $(u_k)_k$ converges to u in $(H^0, \langle\cdot\,,\,\cdot\rangle_0)$.

Altogether we obtain, using Proposition 3.7, that

(3.16) for $\varepsilon \in [0,1]$, $(H^1(\Omega), \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_\varepsilon)$ is an infinite dimensional Hilbert space which is densely and compactly embedded in $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$.

Now, using Proposition 3.8, part (e) and then part (a), we obtain the estimates

$$a_\varepsilon(u,u) \geq \gamma^2 |u|^2_{H^1(\Omega)} - |u|^2_\varepsilon, \qquad \varepsilon \in\,]0,1]\,,\ u \in H^1(\Omega), \tag{3.17}$$

$$a_0(u,u) \geq (1+\delta)^{-1}\gamma^2 |u|^2_{H^1(\Omega)} - |u|^2_0, \quad \varepsilon \in\,]0,1]\,,\ u \in H^1_s(\Omega). \tag{3.18}$$

Proposition 2.1 implies that, for $\varepsilon \in [0,1]$, the pair $(a_\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$ generates a densely defined selfadjoint operator $\mathbf{A}_\varepsilon$ on $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$ with compact resolvent. Moreover, there exists the repeated sequence $(\mu_{\varepsilon,j})_j$ of eigenvalues of $\mathbf{A}_\varepsilon$. For each $\varepsilon \in\,]0,1]$ we also choose a corresponding $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$-orthonormal and complete sequence $(w_{\varepsilon,j})_j$ of eigenfunctions.

Theorem 3.9 ([7]). *The following properties hold:*

(a) *For every $j \in \mathbb{N}$, $\mu_{\varepsilon,j} \to \mu_{0,j}$ as $\varepsilon \to 0$.*
(b) *Let $(\varepsilon_n)_n$ be an arbitrary sequence in $]0,1]$ converging to 0. Then there is a subsequence of $(\varepsilon_n)_n$, again denoted by $(\varepsilon_n)_n$, and there exists an $(H^0, \langle\cdot\,,\,\cdot\rangle_0)$-orthonormal and complete sequence $(w_{0,j})_j$ of eigenvectors of $(a_0, \langle\cdot\,,\,\cdot\rangle_0)$ corresponding to $(\mu_{0,j})_j$ such that, for every j,*

$$\|w_{\varepsilon_n,j} - w_{0,j}\|_{\varepsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Now define, for $\varepsilon \in [0,1]$, $A_\varepsilon = \mathbf{A}_\varepsilon + I_\varepsilon$, where I_ε is the identity operator on H^ε. Then A_ε is the operator generated by the pair $(\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_\varepsilon, \langle\cdot\,,\,\cdot\rangle_\varepsilon)$.

By Proposition 2.1,

(3.19) $A_\varepsilon\colon D(A_\varepsilon) = D(\mathbf{A}_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined positive selfadjoint operator in $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon})$ with ${A_\varepsilon}^{-1}\colon H^\varepsilon \to H^\varepsilon$ compact. For $\alpha \in \mathbb{R}$ write $H^\varepsilon_\alpha := H_\alpha(A_\varepsilon) := D(A_\varepsilon^{1/2})$. In particular, $H^\varepsilon_0 = H^\varepsilon$. We have $H^\varepsilon_1 = H^1(\Omega)$ if $\varepsilon > 0$ and $H^\varepsilon_1 = H^1_s(\Omega)$ if $\varepsilon = 0$. Finally, $\langle\!\langle u, v\rangle\!\rangle_\varepsilon = \langle A_\varepsilon^{1/2} u, A_\varepsilon^{1/2} v\rangle_\varepsilon$ for u, $v \in H^\varepsilon_1$.

We also have that the sequence $(\lambda_{\varepsilon,j})_j$, with $\lambda_{\varepsilon,j} = \mu_{\varepsilon,j} + 1$ for each $j \in \mathbb{N}$ is the repeated sequence of eigenvalues of the operator A_ε with $(w_{\varepsilon,j})_j$ as a corresponding sequence of eigenfunctions. Using (3.19) and Theorem 3.9 we thus obtain that

(3.20) whenever $(\varepsilon_n)_n$ is a sequence in $]0,1]$ with $\varepsilon_n \to 0$ then $\lambda_{\varepsilon_n,j} \to \lambda_{0,j}$ as $n \to \infty$, for all $j \in \mathbb{N}$. Moreover, there is a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ and there is an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ of A_0 corresponding to $(\lambda_{0,j})_j$ such that $|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$, for all $j \in \mathbb{N}$.

We also claim that

(3.21) Under the notation of (3.20), $\langle u, w_{\varepsilon_{n_k},j}\rangle_{\varepsilon_{n_k}} \to \langle u, w_{0,j}\rangle_0$ as $k \to \infty$, for all $u \in H^0$ and all $j \in \mathbb{N}$.

Indeed,

$$\begin{aligned}
|\langle u, w_{\varepsilon_{n_k},j}\rangle_{\varepsilon_{n_k}} - \langle u, w_{0,j}\rangle_0| &= \bigg| \int_\Omega (J_{\varepsilon_{n_k}} u w_{\varepsilon_{n_k},j} - J_0 u w_{0,j})\,dx \bigg| \\
&\le |J_{\varepsilon_{n_k}} - J_0|_{L^\infty(\Omega)} \cdot |u|_{L^2(\Omega)} \cdot (|w_{\varepsilon_{n_k},j} - w_{0,j}|_{L^2(\Omega)} + |w_{0,j}|_{L^2(\Omega)}) \\
&\quad + |J_0|_{L^\infty(\Omega)} \cdot |u|_{L^2(\Omega)} \cdot |w_{\varepsilon_{n_k},j} - w_{0,j}|_{L^2(\Omega)}.
\end{aligned}$$

By (3.7),

$$\begin{aligned}
|w_{\varepsilon_{n_k},j} - w_{0,j}|^2_{L^2(\Omega)} &\le C\langle w_{\varepsilon_{n_k},j} - w_{0,j}, w_{\varepsilon_{n_k},j} - w_{0,j}\rangle_{\varepsilon_{n_k}} \\
&\le C\langle\!\langle w_{\varepsilon_{n_k},j} - w_{0,j}, w_{\varepsilon_{n_k},j} - w_{0,j}\rangle\!\rangle_{\varepsilon_{n_k}} \\
&= C|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Since $|J_{\varepsilon_{n_k}} - J_0|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$, the claim follows.

The statements (3.15), (3.19), (3.8), (3.9), (3.20), (3.21) and Proposition 3.8 now imply the following result.

Proposition 3.10. *The family $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,1]}$ satisfies hypothesis* [2, (Spec)].

Now we claim that

(3.22) whenever $(\varepsilon_n)_n$ is a sequence in $]0,1]$ with $\varepsilon_n \to 0$ and $(\xi_n)_n$ is a sequence with $\xi_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} |\xi_n|_{H_1^{\varepsilon_n}} < \infty$, then there exist a $v \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that $|\xi_{n_k} - v|_{H^{\varepsilon_{n_k}}} \to 0$ $k \to \infty$.

Indeed, by Proposition 3.8 part (e) the sequence $(\xi_n)_n$ is bounded in $H^1(\Omega)$. Thus there are a $v \in H^1(\Omega)$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that $(\xi_{n_k})_k$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to v. In view of (3.7) we only need to prove that $v \in H_s^1(\Omega)$. Now the operator $\nabla\colon H^1(\Omega) \to L^2(\Omega)$ is linear and (strongly) continuous, so ∇ is weakly continuous. In particular,

(3.23) $(\nabla \xi_{n_k})_k$ converges weakly in $L^2(\Omega, \mathbb{R}^\ell)$ to ∇v.

Since the map $P\colon \mathcal{U} \to L(\mathbb{R}^\ell, \mathbb{R}^\ell)$ is of class C^{r-1}, it is bounded on Ω, so for each $\mathbf{u} \in L^2(\Omega, \mathbb{R}^\ell)$, the function $P(\,\cdot\,)\mathbf{u}$, $x \mapsto P(x)\mathbf{u}(x)$, lies in $L^2(\Omega, \mathbb{R}^\ell)$. Thus by (3.23)

$$\begin{aligned}\int_\Omega \langle P(x)\nabla\xi_{n_k}(x), \mathbf{u}(x)\rangle_{\mathbb{R}^\ell}\,dx &= \int_\Omega \langle \nabla\xi_{n_k}(x), P(x)\mathbf{u}(x)\rangle_{\mathbb{R}^\ell}\,dx \\ &\underset{k\to\infty}{\longrightarrow} \int_\Omega \langle \nabla v(x), P(x)\mathbf{u}(x)\rangle_{\mathbb{R}^\ell}\,dx = \int_\Omega \langle P(x)\nabla v(x), \mathbf{u}(x)\rangle_{\mathbb{R}^\ell}\,dx,\end{aligned}$$

so

(3.24) $(P(\,\cdot\,)\nabla\xi_{n_k})_k$ converges weakly in $L^2(\Omega, \mathbb{R}^\ell)$ to $P(\,\cdot\,)\nabla v$.

Now for each $n \in \mathbb{N}$

$$\frac{1}{\varepsilon_n^2}\int_\Omega \langle P(x)\nabla\xi_n(x), P(x)\nabla\xi_n(x)\rangle_{\mathbb{R}^\ell}\,dx \le |\xi_n|^2_{H_1^{\varepsilon_n}} \le \sup_{n\in\mathbb{N}} |\xi_n|^2_{H_1^{\varepsilon_n}<\infty}.$$

Therefore, since $\varepsilon_n \to 0$ for $n \to \infty$ and $\inf_{n\in\mathbb{N}} \inf_{x\in\Omega} J_{\varepsilon_n}(x) > 0$, we conclude that $(P(\,\cdot\,)\nabla\xi_n)_n$ converges strongly in $L^2(\Omega, \mathbb{R}^\ell)$ to 0, so

(3.25) $(P(\,\cdot\,)\nabla\xi_n)_n$ converges weakly in $L^2(\Omega, \mathbb{R}^\ell)$ to 0.

From (3.24) and (3.25) we conclude that $P(\,\cdot\,)\nabla v = 0$ almost everywhere so $v \in H_s^1(\Omega)$. The claim is proved. This proves the following result.

Proposition 3.11. *The family $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,1]}$ satisfies assumption* [2, (Comp)].

Let us now relate the operators $\mathbf{A}_\varepsilon$ and $\mathbf{B}_\varepsilon$, to each other.

It is clear that, for $\varepsilon \in]0,1]$, the assignment

$$u \mapsto u \circ \Gamma_\varepsilon$$

restricts to linear isomorphisms $L^2(\Omega_\varepsilon) \to L^2(\Omega)$ and $H^1(\Omega_\varepsilon) \to H^1(\Omega)$. Using the change-of-variables formula and Theorem 3.6 we see that, for $\varepsilon \in]0,1]$,

$$a_\varepsilon(u \circ \Gamma_\varepsilon, v \circ \Gamma_\varepsilon) = \check{a}_\varepsilon(u,v) \tag{3.26}$$

for all $u, v \in H^1(\Omega_\varepsilon)$. Moreover,

$$b_\varepsilon(u \circ \Gamma_\varepsilon, v \circ \Gamma_\varepsilon) = \check{b}_\varepsilon(u,v) \tag{3.27}$$

for all $u, v \in L^2(\Omega_\varepsilon)$.

Using formulas (3.26) and (3.27) we obtain the following

Proposition 3.12. *The (linear) operators $\mathbf{B}_\varepsilon$ (resp. $\mathbf{A}_\varepsilon$) defined by $(\check{a}_\varepsilon, \check{b}_\varepsilon)$ (resp. $(a_\varepsilon, b_\varepsilon)$) satisfy the following properties:*

(a) *$u \in D(\mathbf{B}_\varepsilon)$ if and only if $u \circ \Gamma_\varepsilon \in D(\mathbf{A}_\varepsilon)$;*
(b) *$\mathbf{A}_\varepsilon(u \circ \Gamma_\varepsilon) = (\mathbf{B}_\varepsilon u) \circ \Gamma_\varepsilon$ for $u \in D(\mathbf{B}_\varepsilon)$.*

Given $\varepsilon \in [0,1]$ and $\alpha \in]0,1[$ let $\psi_\alpha = \psi^\varepsilon_\alpha \colon H^\varepsilon_0 \to H^\varepsilon_{-\alpha}$ be the canonical embedding and $\widetilde{A}_\varepsilon \colon H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$ be the unique continuous extension of $\psi_\alpha \circ A_\varepsilon$. We set $\widetilde{\mathbf{A}}_\varepsilon = \widetilde{A}_\varepsilon - \mathrm{Id}_{H^\varepsilon_{-\alpha}}$. Proceding as in the proof of formula (3.6) we see that

$$\widetilde{\mathbf{A}}_\varepsilon w(v) = a_\varepsilon(w,v), \quad w \in H^\varepsilon_{2-\alpha},\ v \in H^\varepsilon_1. \tag{3.28}$$

Using Proposition 3.10 we now obtain the following

Conclusion 3.13. *The linear singular convergence results* [2, Theorems 4.6 and 4.7] *hold in the present case.*

4. Nonlinear semiflows on squeezed domains

In this section, we consider semilinear parabolic equations on the curved squeezed domains Ω_ε. We first transform these equations to equivalent equations on the fixed domain Ω generating a family π_ε, $\varepsilon \in [0,1]$, of local semiflows. Then, under appropriate hypotheses on the nonlinearities we establish the validity of the abstract condition (Conv) introduced in [2]. As a consequence we obtain various convergence and compactness results for the family π_ε, $\varepsilon \in [0,1]$ with the resulting singular Conley index and homology index braid continuation principles.

As usual, set

$$2^* = \begin{cases} \dfrac{2\ell}{\ell-2} & \text{if } \ell \geq 3, \\ \text{an arbitrary } p^* \in]2,\infty[& \text{if } \ell = 2. \end{cases}$$

For $\theta \in [0,1]$ let

$$p_{(\theta)} = \left(\theta \frac{1}{2^*_\Omega} + (1-\theta)\frac{1}{2}\right)^{-1}.$$

Let $q \in \,](1-(1/2^*))^{-1}, 2^*[$ be arbitrary. Then there is an $\alpha \in \,]0,1[$ such that $p := q/(q-1) < p_{(\alpha)}$.

By interpolation theory, given an $\varepsilon \in \,]0,1]$ there is a continuous embedding from $H_\alpha(B_\varepsilon)$ to $L^p(\Omega_\varepsilon)$ (with embedding constant depending on ε and α). Let $Z_\varepsilon \colon H^1(\Omega_\varepsilon) \to L^q(\Omega_\varepsilon)$ be a locally Lipschitzian map.

Example 4.1. Let $\zeta_\varepsilon \colon \Omega_\varepsilon \times \mathbb{R} \to \mathbb{R}$, $(x,s) \mapsto \zeta_\varepsilon(x,s)$, be a function such that

(a) there is a null set N_ε in Ω_ε with $\zeta_\varepsilon(x,\,\cdot\,) \in C^1(\mathbb{R},\mathbb{R})$ for all $x \in \Omega_\varepsilon \setminus N_\varepsilon$;
(b) for all $s \in \mathbb{R}$, $\zeta_\varepsilon(\,\cdot\,,s)$ and $\partial_s\zeta_\varepsilon(\,\cdot\,,s)$ are measurable on Ω_ε;

Moreover, with

$$r = \frac{2^* q}{2^* - q}, \quad \beta = \frac{2^*}{q} - 1$$

there is a constant $C_\varepsilon \in \,]0,\infty[$ and functions $\overline{a}_\varepsilon \in L^r(\Omega_\varepsilon)$ and $\overline{b}_\varepsilon \in L^q(\Omega_\varepsilon)$ such that

$$\begin{aligned} |\partial_s\zeta_\varepsilon(x,s)| &\le \widetilde{C}_\varepsilon(\overline{a}_\varepsilon(x) + |s|^\beta), && \text{for } (x,s) \in (\Omega_\varepsilon \setminus N_\varepsilon) \times \mathbb{R}, \\ |\zeta_\varepsilon(x,0)| &\le \overline{b}_\varepsilon(x), && \text{for } x \in \Omega_\varepsilon \setminus N_\varepsilon, \end{aligned}$$

For $u \in H^1(\Omega_\varepsilon)$ and $x \in \Omega_\varepsilon$ set $Z_\varepsilon(u)(x) = \zeta_\varepsilon(x,u(x))$. Then, by standard results (cf. the proof of [3, Theorem 2.6]), $Z_\varepsilon(u) \in L^q(\Omega_\varepsilon)$ and the map $Z_\varepsilon \colon H^1(\Omega_\varepsilon) \to L^q(\Omega_\varepsilon)$ is Lipschitzian on bounded subsets of $H^1(\Omega_\varepsilon)$.

Consider the equation

$$\dot{u} = -\mathbf{B}_\varepsilon u + Z_\varepsilon(u). \tag{4.1}$$

Intuitively, if $t_0 \in \,]0,\infty]$ and $u \colon [0,t_0[\, \to H^1(\Omega_\varepsilon)$, we say that *$u$ is a 'solution' of* (4.1) if and only if (a) u is continuous into $H^1(\Omega_\varepsilon)$ and (b) $u_{|]0,t_0[}$ is differentiable into $L^2(\Omega_\varepsilon)$ and for every $t \in \,]0,t_0[$ and every $h \in H^1(\Omega_\varepsilon)$

$$\int_{\Omega_\varepsilon} \dot{u}(t)(x)h(x)\,dx = -\int_{\Omega_\varepsilon} \nabla u(t)(x)\nabla h(x)\,dx + \int_{\Omega_\varepsilon} Z_\varepsilon(u(t))(x)h(x)\,dx. \tag{4.2}$$

Since $H^1(\Omega) = H_1(B_\varepsilon) \subset H_\alpha(B_\varepsilon)$, the integrals in (4.2) make sense.

Now (4.2) is clearly equivalent to

$$\check{b}_\varepsilon(\dot{u}(t),h) = -\check{a}_\varepsilon(u(t),h) + \varepsilon^{-(\ell-k)/2}\int_{\Omega_\varepsilon} Z_\varepsilon(u(t))(x)h(x)\,dx.$$

Define the map $g_\varepsilon \colon H^1(\Omega_\varepsilon) \to H_{-\alpha}(B_\varepsilon)$ by

$$g_\varepsilon(u)(h) = \varepsilon^{-(\ell-k)/2}\int_{\Omega_\varepsilon} Z_\varepsilon(u(t))(x)h(x)\;dx, \quad u \in H^1(\Omega_\varepsilon),\ h \in H_\alpha(B_\varepsilon).$$

We see that g_ε is defined and locally Lipschitzian.

Thus by formula (3.6) and the density of $H^1(\Omega_\varepsilon)$ in $H_\alpha(B_\varepsilon)$ we see that u is a 'solution' of (4.1) if and only if (a) u is continuous into $H^1(\Omega_\varepsilon)$ and (b) $u_{|]0,t_0[}$ is differentiable into $L^2(\Omega_\varepsilon)$ and for every $t \in]0, t_0[$

$$\dot{u}(t) = -\widetilde{\mathbf{B}}_\varepsilon u(t) + g_\varepsilon(u(t)).$$

Thus u is a "solution" of (4.1) iff u is a solution of

$$\dot{u} = -\widetilde{\mathbf{B}}_\varepsilon u + g_\varepsilon(u). \tag{4.3}$$

Here we have used the smoothing property of solutions of (4.3), cf. [4]. Define $R = R^{(\varepsilon)} \colon L^2(\Omega_\varepsilon) \to L^2(\Omega)$ by $u \mapsto u \circ \Gamma_\varepsilon$. Formula (3.27) implies that R is a linear isometry from $(L^2(\Omega_\varepsilon), \check{b}_\varepsilon)$ onto $(L^2(\Omega), b_\varepsilon)$. By the definition of J_ε and the change-of-variables formula we have

$$\begin{aligned} g_\varepsilon(u)(h) &= \varepsilon^{-(\ell-k)/2} \int_{\Omega_\varepsilon} Z_\varepsilon(u)(x) h(x)\, dx \\ &= \int_\Omega J_\varepsilon(x) Z_\varepsilon(u)(\Gamma_\varepsilon(x)) h(\Gamma_\varepsilon(x))\, dx, \quad u \in H^1(\Omega_\varepsilon),\ h \in H_\alpha(B_\varepsilon). \end{aligned}$$

Thus the R-conjugate of g_ε is the map $\underline{g}_\varepsilon \colon H^1(\Omega) \to H_{-\alpha}(A_\varepsilon)$ given by

$$\underline{g}_\varepsilon(\underline{u})(\underline{h}) = \int_\Omega J_\varepsilon(x) Z_\varepsilon(\underline{u} \circ \Gamma_\varepsilon^{-1})(\Gamma_\varepsilon(x)) \underline{h}(x)\, dx, \quad \underline{u} \in H^1(\Omega),\ \underline{h} \in H_\alpha(A_\varepsilon).$$

Thus, dropping the underscores and using the results of Section 2 (cf. (2.3) and (2.2)) we see that the local semiflow $\widetilde{\pi}_\varepsilon$ generated on $H^1(\Omega_\varepsilon)$ by equation (4.3) is conjugated, via $R^{(\varepsilon)}$, to the local semiflow on the fixed space $H^1(\Omega)$ generated by equation

$$\dot{u} = -\widetilde{A}_\varepsilon u + \check{f}_\varepsilon(u), \tag{4.4}$$

where $\widetilde{A}_\varepsilon$ is defined after the statement of Proposition 3.12 and the map $\check{f}_\varepsilon \colon H_1^\varepsilon = H^1(\Omega) \to H_{-\alpha}^\varepsilon$ is given by

$$\check{f}_\varepsilon(u)(h) = \int_\Omega J_\varepsilon(x)(u(x) + Z_\varepsilon(u \circ \Gamma_\varepsilon^{-1})(\Gamma_\varepsilon(x)) h(x)\, dx, \tag{4.5}$$

$u \in H^1(\Omega)$, $h \in H_{-\alpha}$. Therefore, from now on, we will analyze, for $\varepsilon \in]0, 1]$, equation (4.4) on the fixed domain Ω in place of the original equation (4.3) on the variable curved squeezed domain Ω_ε.

We will now provide a condition ensuring that the family of equations (4.4) has a limit equation as $\varepsilon \to 0^+$.

PROPOSITION 4.2. *For $\varepsilon \in [0, 1]$, let $\Phi_\varepsilon \colon H^1(\Omega) \to L^q(\Omega)$ be a map satisfying the following assumptions:*

(a) *For all $M \in [0, \infty[$ there is an $L = L_M \in [0, \infty[$ such that*

$$|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)|_{L^q(\Omega)} \le L|u - v|_{H^1(\Omega)}$$

for all $\varepsilon \in [0,1]$ *and all* u, $v \in H^1(\Omega)$ *with* $|u|_{H^1(\Omega)}$, $|v|_{H^1(\Omega)} \leq M$.

(b) *For every* $u \in H^1_s(\Omega)$,

$$|\Phi_\varepsilon(u) - \Phi_0(u)|_{L^q(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

For $\varepsilon \in [0,1]$ *and* $u \in H^\varepsilon_1$ *define, for* $h \in H^\varepsilon_\alpha$,

$$f_\varepsilon(u)(h) = \int_\Omega J_\varepsilon(x)\Phi_\varepsilon(u)(x)h(x)\,dx.$$

Then $f_\varepsilon(u) \in H^\varepsilon_{-\alpha}$ *and the family* $(f_\varepsilon)_{\varepsilon\in[0,1]}$ *of maps satisfies condition* [2, (Conv)].

Proof. By the definition of the bilinear forms a_ε and b_ε, for every $\varepsilon \in [0,1]$ there are continuous embeddings from H^ε_0 to $H^0(\Omega) = L^2(\Omega)$ and from H^ε_1 to $H^1(\Omega)$ with embedding constants independent of $\varepsilon \in [0,1]$. Thus by interpolation theory (cf. [10]) for every $\theta \in [0,1]$ and every $\varepsilon \in [0,1]$ there is a continuous embedding from H^ε_θ to $H^\theta(\Omega)$ with embedding constant $C_{1,\theta} \in \,]0,\infty[$ independent of $\varepsilon \in [0,1]$. Furthermore, there is a continuous embedding from $H^\theta(\Omega)$ into $L^{p(\theta)}(\Omega)$ with embedding constant $C_{2,\theta} \in \,]0,\infty[$. Moreover, there is a continuous embedding from $L^{p(\theta)}(\Omega)$ to $L^p(\Omega)$ with embedding constant C_3. Finally, $C_4 := \sup_{(x,\varepsilon)\in\Omega\times[0,1]} |J_\varepsilon(x)| < \infty$. It follows that $f_\varepsilon(u)(h)$ is defined and $|f_\varepsilon(u)(h)| \leq C_{1,\alpha}C_{2,\alpha}C_3C_4|\Phi_\varepsilon(u)|_{L^q(\Omega)}|h|_{H^\varepsilon_\alpha}$, so f_ε maps H^ε_1 into $H^\varepsilon_{-\alpha}$. Thus [2, (Conv), part (a)] is satisfied.

Let $M \in [0,\infty[$ be arbitrary and $L = L_M$ be as in assumption (a). If $\varepsilon \in [0,1]$ and u, $v \in H^\varepsilon_1$ with $|u|_{H^\varepsilon_1}$, $|v|_{H^\varepsilon_1} \leq M/C_{1,1}$ then u, $v \in H^1(\Omega)$ with $|u|_{H^1(\Omega)}$, $|v|_{H^1(\Omega)} \leq M$ so

$$\begin{aligned}|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon_{-\alpha}} &\leq C_{1,\alpha}C_{2,\alpha}C_3C_4|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)|_{L^q(\Omega)}\\ &\leq C_{1,\alpha}C_{2,\alpha}C_3C_4L|u-v|_{H^1(\Omega)} \leq C_{1,\alpha}C_{2,\alpha}C_3C_4LC_{1,1}|u-v|_{H^\varepsilon_1}.\end{aligned}$$

This together with assumption (a) implies [2, (Conv), part (c)]. If $u \in H^0_1$ then, for all $\varepsilon \in [0,1]$

$$|f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} \leq C_{1,\alpha}C_{2,\alpha}C_3C_4|\Phi_\varepsilon(u)|_{L^q(\Omega)}.$$

This together with assumption (b) easily implies [2, (Conv), part (d)].

Now let $w \in H^0_1$ be arbitrary and $(\varepsilon_n)_n$ be a sequence in $]0,1]$ with $\varepsilon_n \to 0$. Let $t \in \,]0,\infty[$ be arbitrary. We will show that

$$\lim_{n\to\infty} |e^{-t\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(w) - e^{-t\widetilde{A}_0} f_0(w)|_{H^{\varepsilon_n}_1} = 0, \tag{4.6}$$

proving [2, (Conv), part (b)].

To this end, we will use [2, Theorem 4.6], which holds in the present case in view of Conclusion 3.13, and so we have to verify the assumptions of that result. For $n \in \mathbb{N}$ set $u_n = f_{\varepsilon_n}(w)$ and define $v_n \in H^{\varepsilon_n}_{-\alpha}$ by

$$v_n(h) = \int_\Omega J_{\varepsilon_n}(x)\Phi_0(w)(x)h(x)\,dx, \quad h \in H^{\varepsilon_n}_\alpha.$$

Finally, set $u = f_0(w)$. Then

$$|u_n - v_n|_{H^{\varepsilon_n}_{-\alpha}} \le C_{1,\alpha}C_{2,\alpha}C_3C_4|\Phi_{\varepsilon_n}(w) - \Phi_0(w)|_{L^q(\Omega)}.$$

Notice that, by assumption (b), the right hand side of this estimate goes to zero as $n \to \infty$. Thus [2, Theorem 4.6, assumption (a)] is satisfied.

Let $C_5 \in\,]0,\infty[$ be a bound for the embedding $H^1(\Omega) \to H^\alpha(\Omega)$. Then, for every $j \in \mathbb{N}$,

$$\begin{aligned} |v_n(w_{\varepsilon_n,j}) - u(w_{0,j})| &\le |J_{\varepsilon_n}|_{L^\infty(\Omega)}|\Phi_0(w)|_{L^q(\Omega)}|w_{\varepsilon_n,j} - w_{0,j}|_{L^p(\Omega)} \\ &\quad + |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)}|\Phi_0(w)|_{L^q(\Omega)}|w_{0,j}|_{L^p(\Omega)} \\ &\le C_4|\Phi_0(w)|_{L^q(\Omega)}C_5C_{1,1}C_{2,\alpha}C_3|w_{\varepsilon_n,j} - w_{0,j}|_{H^{\varepsilon_n}_1} \\ &\quad + |J_{\varepsilon_n} - J_0|_{L^\infty(\Omega)}|\Phi_0(w)|_{L^q(\Omega)}|w_{0,j}|_{L^p(\Omega)}. \end{aligned}$$

Hence $|v_n(w_{\varepsilon_n,j}) - u(w_{0,j})| \to 0$ as $n \to \infty$. Thus [2, Theorem 4.6, assumption (b)] is satisfied.

Now, for all $n \in \mathbb{N}$,

$$|v_n|_{H^{\varepsilon_n}_{-\alpha}} \le C_{1,\alpha}C_{2,\alpha}C_3C_4|\Phi_0(w)|_{L^q(\Omega)}.$$

Thus [2, Theorem 4.6, assumption (c)] is satisfied. Now (4.6) follows from [2, Theorem 4.6]. □

From now on we assume the following hypothesis:

Hypothesis 4.3. *For $\varepsilon \in [0,1]$, $\varphi_\varepsilon\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that*

(a) *there is a null set N in Ω with $\varphi_\varepsilon(x,\,\cdot\,) \in C^1(\mathbb{R},\mathbb{R})$ for all $x \in \Omega \setminus N$;*
(b) *for all $s \in \mathbb{R}$, $\varphi_\varepsilon(\,\cdot\,,s)$ and $\partial_s\varphi_\varepsilon(\,\cdot\,,s)$ are measurable on Ω;*

Moreover, with

$$r = \frac{2^*q}{2^* - q}, \quad \beta = \frac{2^*}{q} - 1$$

there is a constant $C \in\,]0,\infty[$ and functions $a \in L^r(\Omega)$ and $b \in L^q(\Omega)$ such that for all $\varepsilon \in [0,1]$

$$\begin{aligned} &|\partial_s\varphi_\varepsilon(x,s)| \le C(a(x) + |s|^\beta), && \text{for } (x,s) \in (\Omega \setminus N) \times \mathbb{R}, \\ &|\varphi_\varepsilon(x,0)| \le b(x), && \text{for } x \in \Omega \setminus N, \\ &|\varphi_\varepsilon(x,s) - \varphi_0(x,s)| \to 0, && \text{for } (x,s) \in (\Omega \setminus N) \times \mathbb{R}. \end{aligned}$$

REMARK. By redefining φ_ε to be $= 0$ on $N \times \mathbb{R}$, we may and will assume that $N = \emptyset$.

For $\varepsilon \in [0,1]$, $u \in H^1(\Omega)$ and $x \in \Omega$ define

$$\check{\Phi}_\varepsilon(u)(x) = \varphi_\varepsilon(x, u(x)).$$

Again the arguments from the proof of [3, Theorem 2.6] imply that $\check{\Phi}_\varepsilon(u) \in L^q(\Omega)$ and that the family $\check{\Phi}_\varepsilon$, $\varepsilon \in [0,1]$, satisfies the assumptions of Proposition 4.2. Therefore, trivially, the family $\Phi_\varepsilon(u)$, $\varepsilon \in [0,1]$, also satisfies the assumptions of Proposition 4.2, where, for $\varepsilon \in [0,1]$, the map $\Phi_\varepsilon \colon H^1(\Omega) \to L^q(\Omega)$ is given by

$$\Phi_\varepsilon(u) = u + \check{\Phi}_\varepsilon(u), \quad u \in H^1(\Omega).$$

We let the map f_ε, $\varepsilon \in [0,1]$ be defined as in Proposition 4.2 with this choice of Φ_ε. It follows that the family f_ε, $\varepsilon \in [0,1]$, satisfies condition [2, (Conv)]. Let π_ε be the local semiflow on H_1^ε generated by the equation

$$\dot{u} = -\widetilde{A}_\varepsilon u + f_\varepsilon(u). \tag{4.7}$$

REMARK. For $\varepsilon \in]0,1]$ define the function $\zeta_\varepsilon \colon \Omega_\varepsilon \times \mathbb{R} \to \mathbb{R}$ by

$$\zeta_\varepsilon(x, s) = \varphi_\varepsilon(\Gamma_\varepsilon^{-1}(x), s).$$

Then ζ_ε satisfies the assumptions of Example 4.1. Let Z_ε be defined as in that example. Then

$$Z_\varepsilon(u \circ \Gamma_\varepsilon^{-1})(\Gamma_\varepsilon(x)) = \Phi_\varepsilon(u)(x), \quad u \in H^1(\Omega),\ x \in \Omega$$

so f_ε is as in (4.5).

We now obtain the following

CONCLUSION 4.4. *For the above family π_ε, $\varepsilon \in [0,1]$, the following results hold: the convergence results* [2, Theorems 5.4, 5.5, 5.7 and Corollary 5.6]*, the admissibility results* [2, Theorems 6.1, 6.2 and Corollary 6.3]*, the Conley index continuation principle* [2, Theorem 7.3] *and the homology braid continuation principle* [2, Theorem 7.5].

To conclude this section let us note the following results which we will use in Section 5.

PROPOSITION 4.5. *For $\varepsilon \in [0,1]$ define the function $F_\varepsilon \colon \Omega \times \mathbb{R} \to \mathbb{R}$ by*

$$F_\varepsilon(x, t) = \int_0^t \varphi_\varepsilon(x, s)\, dx, \quad x \in \Omega,\ t \in \mathbb{R}.$$

For $u \in L^{2^*}(\Omega)$ and $x \in \Omega$ define $\widehat{F}_\varepsilon(u)(x) = F_\varepsilon(x, u(x))$. Then $\widehat{F}_\varepsilon(u) \in L^{2^*q/(2^*+q)}(\Omega)$ and the map $\widehat{F}_\varepsilon$ is continuously differentiable from $L^{2^*}(\Omega)$ to $L^{2^*q/(2^*+q)}(\Omega)$. Moreover,

$$D\widehat{F}_\varepsilon(u)(v)(x) = \varphi_\varepsilon(x, u(x))v(x), \quad u, v \in L^{2^*}(\Omega),\ x \in \Omega.$$

PROOF. By hypothesis 4.3, integrating and using Young's inequality, we obtain

$$|\varphi_\varepsilon(x,t)| \leq b(x) + Ca(x)|t| + \frac{C}{\beta+1}|t|^{\beta+1} \leq c(x) + \frac{C+1}{\beta+1}|t|^{\beta+1}$$

i.e.

$$|\varphi_\varepsilon(x,t)| \leq c(x) + C'|t|^{\beta+1} \quad x \in \Omega,\ t \in \mathbb{R} \tag{4.8}$$

where

$$c(x) = b(x) + C(\beta/(\beta+1))a(x)^{(\beta+1)/\beta}, \quad x \in \Omega, \quad \text{and} \quad C' = \frac{C+1}{\beta+1}.$$

Using the same arguments with (4.8) we get

$$|F_\varepsilon(x,t)| \leq d(x) + C''|t|^{\beta+2} \quad x \in \Omega,\ t \in \mathbb{R} \tag{4.9}$$

where

$$d(x) = \frac{\beta+1}{\beta+2}c(x)^{(\beta+2)/(\beta+1)}, \quad x \in \Omega, \quad \text{and} \quad C'' = \left(\frac{1}{\beta+2} + \frac{C+1}{(\beta+1)(\beta+2)}\right).$$

Since $r\beta/(\beta+1) = q$, we have $c \in L^q(\Omega)$. Now an application of [3, Theorem 2.6] shows that $\widehat{F}_\varepsilon$ is a C^1-map from $L^{q(\beta+1)}(\Omega)$ to $L^{q(\beta+1)/(\beta+2)}(\Omega)$. Since $q(\beta+1) = 2^*$ and $q(\beta+1)/(\beta+2) = 2^*q/(2^*+q)$, the assertion follows. □

COROLLARY 4.6. *For $\varepsilon \in [0,1]$ the function $V_\varepsilon\colon H_1^\varepsilon \to \mathbb{R}$ given by*

$$V_\varepsilon(u) = (1/2)a_\varepsilon(u,u) - \int_\Omega J_\varepsilon(x)F_\varepsilon(x,u(x))\,dx, \quad u \in H_1^\varepsilon,$$

is defined, continuously differentiable and for each $M \in [0,\infty[$

$$\sup_{\varepsilon\in[0,1]} \sup_{\|u\|_\varepsilon \leq M} |V_\varepsilon(u)| < \infty. \tag{4.10}$$

Moreover,

$$DV_\varepsilon(u)(v) = a_\varepsilon(u,v) - \int_\Omega J_\varepsilon(x)\varphi_\varepsilon(x,u(x))v(x)\,dx, \quad u, v \in H_1^\varepsilon. \tag{4.11}$$

Whenever $I \subset \mathbb{R}$ is an open interval and $u\colon I \to H_1^\varepsilon$ is a solution of the local semiflow π_ε, then u is differentiable into H_1^ε with derivative $\dot{u}$. The function $V_\varepsilon \circ u$ is differentiable and

$$(V_\varepsilon \circ u)'(t) = -|\dot{u}(t)|_\varepsilon^2, \quad t \in I.$$

Proof. By our assumption on q we have $2^*q/(2^*+q) > 1$ so we have a continuous inclusion from $L^{2^*q/(2^*+q)}(\Omega)$ to $L^1(\Omega)$, with embedding constant C_6. We also have a continuous inclusion from H_1^ε to $L^{2^*}(\Omega)$ with embedding constant $C_{1,1}$ (independent of $\varepsilon \in [0,1]$). Finally, the integral is a linear continuous map from $L^1(\Omega)$ to $\mathbb{R}$ with embedding constant 1. Now Proposition 4.5 together with the fact that a_ε is bilinear and bounded from $H_1^\varepsilon \times H_1^\varepsilon$ to $\mathbb{R}$ imply that V_ε is defined, continuously differentiable, and that formula (4.11) holds.

Formula (4.9) implies

$$|V_\varepsilon(u)| \le (1/2)\|u\|_\varepsilon^2 + C_4|d|_{L^1(\Omega)} + C_4C''C_6C_{1,1}^{\beta+2}\|u\|_\varepsilon^{\beta+2}, \tag{4.12}$$

$\varepsilon \in [0,1]$, $u \in H_1^\varepsilon$, with $C_4 := \sup_{(x,\varepsilon)\in\Omega\times[0,1]} |J_\varepsilon(x)|$. Formula (4.12) in turn implies (4.10).

Now the smoothing property for solutions of (4.7) implies that u is differentiable into H_1^ε.

We also have

$$b_\varepsilon(\dot u(t), v) = -\widetilde{\mathbf{A}}_\varepsilon(u(t))(v) + f_\varepsilon(u(t))(v), \quad t \in I,\ v \in H_1^\varepsilon$$

where $\widetilde{\mathbf{A}}_\varepsilon$ is defined after the statement of Proposition 3.12. Using (3.28) we thus obtain, for $t \in I$,

$$\begin{aligned}(V_\varepsilon \circ u)'(t) = DV_\varepsilon(u(t))(\dot u(t)) &= a_\varepsilon(u(t), \dot u(t)) - \int_\Omega J_\varepsilon(x)\varphi_\varepsilon(x, u(t)(x))\dot u(t)(x)\,dx \\ &= \widetilde{\mathbf{A}}_\varepsilon(u(t))(\dot u(t)) - f_\varepsilon(u(t))(\dot u(t)) = -b_\varepsilon(\dot u(t), \dot u(t)) = -|\dot u(t)|_\varepsilon^2.\end{aligned}$$

The corollary is proved. □

5. Global attractors

In this section we will prove that, under a dissipativeness condition on the family $(\varphi_\varepsilon)_{\varepsilon\in[0,1]}$, for each $\varepsilon \in [0,1]$ the local semiflow π_ε is actually a global semiflow and it has a global attractor $\mathcal{A}_\varepsilon$. We will also show that the family of these attractors is upper-semicontinuous at $\varepsilon = 0$. These results extend the corresponding results from [6] and [7].

Assume the following

Hypothesis 5.1. *There is an $\eta \in\ ]0,1[$ and for every $\varepsilon \in [0,1]$ there is a positive function $\mu_\varepsilon \in L^{\beta+2}(\Omega)$ such that $M = \sup_{\varepsilon\in[0,1]} |\mu_\varepsilon|_{L^{\beta+2}(\Omega)} < \infty$ and for every $\varepsilon \in [0,1]$*

$$\varphi_\varepsilon(x,t)/t \le -\eta, \quad x \in \Omega,\ |t| > \mu_\varepsilon(x).$$

Remark 5.2. Hypothesis 5.1 is satisfied if the following uniform dissipativeness condition holds:

$$\limsup_{|t|\to\infty} \sup_{(\varepsilon,x)\in[0,1]\times\Omega} \varphi_\varepsilon(x,t)/t < 0.$$

In this case μ_ε can be chosen to be a constant function independent of $\varepsilon \in [0,1]$.

Hypothesis 5.1 together with (4.8) implies that,

$$\varphi_\varepsilon(x,t)t \le -\eta t^2, \quad \varepsilon \in [0,1]\,,\ x \in \Omega,\ |t| > \mu_\varepsilon(x)$$

and

$$\begin{aligned}\varphi_\varepsilon(x,t)t \le (c(x) + C'|t|^{\beta+1})|t| &\le \frac{\beta+1}{\beta+2}c(x)^{(\beta+2)/(\beta+1)} + \left(\frac{1}{\beta+2} + C'\right)|t|^{\beta+2} \\ &\le \frac{\beta+1}{\beta+2}c(x)^{(\beta+2)/(\beta+1)} + \left(\frac{1}{\beta+2} + C'\right)\mu_\varepsilon(x)^{\beta+2},\end{aligned}$$

$\varepsilon \in [0,1]$, $x \in \Omega$, $|t| \le \mu_\varepsilon(x)$. Thus

$$\varphi_\varepsilon(x,t)t \le -\eta t^2 + \frac{\beta+1}{\beta+2}c(x)^{(\beta+2)/(\beta+1)} + \left(\frac{1}{\beta+2} + C'\right)\mu_\varepsilon(x)^{\beta+2}, \tag{5.1}$$

$(\varepsilon, x, t) \in [0,1] \times \Omega \times \mathbb{R}$. Consequently,

$$f_\varepsilon(u)(u) = \int_\Omega J_\varepsilon(x)\varphi_\varepsilon(x,u(x))u(x)\,dx \le -\eta|u|_\varepsilon^2 + \check{C}, \quad u \in H_1^\varepsilon \tag{5.2}$$

where

$$\check{C} = C_4|d|_{L^1(\Omega)} + \left(\frac{1}{\beta+2} + C'\right)C_4 M^{\beta+2} < \infty.$$

Moreover, by a simple integration

$$F_\varepsilon(x,t) \le F_\varepsilon(x,\mu_\varepsilon(x)) - \frac{1}{2}\eta t^2 + \frac{1}{2}\eta\mu_\varepsilon(x)^2, \quad \varepsilon \in [0,1]\,,\ x \in \Omega,\ t > \mu_\varepsilon(x)$$

and

$$F_\varepsilon(x,t) \le F_\varepsilon(x,-\mu_\varepsilon(x)) - \frac{1}{2}\eta t^2 + \frac{1}{2}\eta\mu_\varepsilon(x)^2, \quad \varepsilon \in [0,1]\,,\ x \in \Omega,\ t < -\mu_\varepsilon(x).$$

Since $\mu_\varepsilon(x)^2 \le (2/(\beta+2))\mu_\varepsilon(x)^{\beta+2} + \beta/(\beta+2)$, it follows from (4.9) that

$$F_\varepsilon(x,t) \le -\frac{1}{2}\eta t^2 + 2d(x) + \left(2C'' + \frac{\eta}{\beta+2}\right)\mu_\varepsilon^{\beta+2} + \frac{\eta\beta}{2(\beta+2)},$$

$(\varepsilon, x, t) \in [0,1] \times \Omega \times \mathbb{R}$, so that

$$\int_\Omega J_\varepsilon(x)F_\varepsilon(x,u(x))\,dx \le -\frac{1}{2}\eta|u|_\varepsilon^2 + \check{\check{C}}, \quad (\varepsilon, x, t) \in [0,1] \times \Omega \times \mathbb{R} \tag{5.3}$$

where

$$\check{\check{C}} = 2C_4|d|_{L^1(\Omega)} + \left(2C'' + \frac{\eta}{\beta+2}\right)C_4 M^{\beta+2} + \frac{\eta\beta}{2(\beta+2)}C_4\lambda(\Omega) < \infty$$

with $\lambda(\Omega)$ denoting the measure of Ω.

Lemma 5.3. *If $\varepsilon \in [0,1]$ and u is an equilibrium of π_ε, then $\|u\|_\varepsilon^2 \le \check{C}/\eta$.*

Proof. By our assumption $u \in D(\widetilde{\mathbf{A}}_\varepsilon) = H^\varepsilon_{2-\alpha}$ and $-\widetilde{\mathbf{A}}_\varepsilon u + f_\varepsilon(u) = 0$. It follows that $\widetilde{\mathbf{A}}_\varepsilon(u)(u) = f_\varepsilon(u)(u)$, so in view of (3.28) and (5.2), $a_\varepsilon(u,u) = f_\varepsilon(u)(u) \le -\eta|u|_\varepsilon^2 + \check{C}$ so $\|u\|_\varepsilon^2 = a_\varepsilon(u,u) + |u|_\varepsilon^2 \le \check{C}/\eta$. □

Lemma 5.4. *If $\varepsilon \in [0,1]$ and $u \in H_1^\varepsilon$, then $\|u\|_\varepsilon^2 \le (2/\eta)(V_\varepsilon(u) + \check{\check{C}})$ and $\|u\|_\varepsilon \le K(V_\varepsilon(u)+1)$, where $K = \max(1/\eta, (2\check{\check{C}} + \eta)/2)$.*

Proof. Indeed, by (5.3), $V_\varepsilon(u) \ge (1/2)a_\varepsilon(u,u) + (\eta/2)|u|_\varepsilon^2 - \check{\check{C}}$, so $\|u\|_\varepsilon^2 \le (2/\eta)(V_\varepsilon(u) + \check{\check{C}})$. Since $2\|u\|_\varepsilon \le \|u\|_\varepsilon^2 + 1$, the result follows. □

For $\varepsilon \in [0,1]$ let $\mathcal{A}_\varepsilon$ be the union of all full bounded orbits of π_ε. Corollary 4.6 and Lemmas 5.3 and 5.4 show that all assumptions of [6, Theorem 5.6] are satisfied. Using that theorem we thus obtain

Theorem 5.5. *For every $\varepsilon \in [0,1]$, π_ε is a global semiflow and $\mathcal{A}_\varepsilon$ is a global attractor of π_ε.*

Lemma 5.6. *Let $M_1 = \sup_{\|v\|_\varepsilon^2 \le (\check{C}/\eta)} V_\varepsilon(v)$. Then $M_1 \in [0,\infty[$ and*

$$\sup_{\varepsilon\in[0,1]} \sup_{u\in\mathcal{A}_\varepsilon} \|u\|_\varepsilon^2 \le \frac{2}{\eta}(M_1 + \check{\check{C}}) =: M_2.$$

Proof. $M_1 < \infty$ by (4.10). If $\varepsilon \in [0,1]$ and $u \in \mathcal{A}_\varepsilon$, then there is a full solution $\sigma\colon \mathbb{R} \to \mathcal{A}_\varepsilon$ of π_ε with $\sigma(0) = u$. $\sigma[\mathbb{R}]$ lies in a compact set so its α- and ω-limit sets are nonempty. Since π_ε is gradient-like with respect to V_ε, we have that $V_\varepsilon(\sigma(t)) \le V(v)$ for each $t \in \mathbb{R}$ and all $v \in \alpha(\sigma)$, and $\alpha(\sigma)$ consists only of equilibria of π_ε. Thus, by Lemma 5.3, $V_\varepsilon(u) \le \sup_{\|v\|_\varepsilon^2 \le (\check{C}/\eta)} V_\varepsilon(v) = M_1$, thus $\|u\|_\varepsilon^2 \le (2/\eta)(M_1 + \check{\check{C}})$ by Lemma 5.4. □

Theorem 5.7. *The family $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,1]}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the family $\|\cdot\|_\varepsilon$ of norms i.e.*

$$\lim_{\varepsilon\to 0^+} \sup_{w\in\mathcal{A}_\varepsilon} \inf_{u\in\mathcal{A}_0} \|w-u\|_\varepsilon = 0.$$

Proof. For every $\varepsilon \in]0,1]$ let $Q_\varepsilon\colon H_1^\varepsilon \to H_1^\varepsilon$ be the $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\varepsilon$-orthogonal projection of H_1^ε onto its closed subspace $H_s^1(\Omega) = H_1^0$. Let C be as in Proposition 3.8, part (c). Define N_0 to be the set of all $u \in H_1^0$ with $\|u\|_0^2 \le C^2M_2 + 1$. For $\varepsilon \in]0,1]$ define N_ε to be the set of all $u \in H_1^\varepsilon$ with $Q_\varepsilon u \in N_0$ and $\|(\mathrm{Id}_{H_1^\varepsilon} - Q_\varepsilon)u\|_\varepsilon^2 \le M_2 + 1$. For $\varepsilon \in [0,1]$ let K_ε be the largest π_ε-invariant set included in N_ε. Since N_ε is bounded, $K_\varepsilon \subset \mathcal{A}_\varepsilon$ by the definition of $\mathcal{A}_\varepsilon$. We prove the reverse inclusion. Let $u \in \mathcal{A}_\varepsilon$ be arbitrary and $\sigma\colon\mathbb{R} \to \mathcal{A}_\varepsilon$ be a full solution of π_ε with $\sigma(0) = u$. Let $t \in \mathbb{R}$ be arbitrary. If $\varepsilon = 0$, then $\|\sigma(t)\|_0^2 \le M_2 < C^2M_2 + 1$, so $\sigma(t) \in N_0 = N_\varepsilon$. If $\varepsilon > 0$, then $\|Q_\varepsilon\sigma(t)\|_0^2 \le C^2\|Q_\varepsilon\sigma(t)\|_\varepsilon^2 \le C^2\|\sigma(t)\|_\varepsilon^2 \le C^2M_2 < C^2M_2 + 1$ and $\|(\mathrm{Id}_{H_1^\varepsilon} - Q_\varepsilon)\sigma(t)\|_\varepsilon^2 \le$

$\|\sigma(t)\|_\varepsilon^2 \leq M_2 < M_2 + 1$, so again $\sigma(t) \in N_\varepsilon$. It follows that $u \in K_\varepsilon$ which proves that $K_\varepsilon = \mathcal{A}_\varepsilon$. The above estimates also prove that N_ε is an isolating neighbourhood of K_ε, $\varepsilon \in [0,1]$.

Now the statement of this theorem follows by an application of [2, Theorem 7.3], which holds in view of Conclusion 4.4. □

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