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Review of

BERTRAND RUSSELL, *TOWARDS THE "PRINCIPLES OF MATHEMATICS", 1900-02*,

edited by Gregory H. Moore.
Vol. 3 of *The Collected Papers of Bertrand Russell*,
London/New York: Routledge, 1993.
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and

BERTRAND RUSSELL, *FOUNDATIONS OF LOGIC, 1903-05*,

edited by Alasdair Urquhart with the assistance of Albert C. Lewis.
Vol. 4 of *The Collected Papers of Bertrand Russell*,
London/New York: Routledge, 1994.
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These are very big books[1]. Correspondingly, a very lengthy review seems to be required in order to do them full justice. Moreover, an adequate appreciation for the work which Russell undertook in the period 1900-1905 which is the focus of the two books under review requires a comparison with, and consequent recapitulation of, some salient aspects of Russell's work of the years previous. (To that end, it will correspondingly also be necessary to reproduce some of the contents of my previous discussion ([8]) of Russell's earlier work — which I shall however do without the benefit of the apparatus of quotation marks.) I therefore appeal to the readers' indulgence and patience on both counts.

Readers of this review who require a reminder of the editorial and textual apparatus of Russell's *Collected Papers* and of the character, structure, plans, and policies of the Bertrand Russell Editorial Project that is responsible for publishing these *Papers* may refer to Anellis [9]. It is however worth adding here that the *Collected Papers* will *not* include Russell's published books. Thus, for example, Russell's book *A Critical Exposition of the Philosophy of Leibniz*, although published in 1900, will not be found in CP3 despite its importance in the evolution of Russell's thought at this critical juncture in Russell's intellectual career

[1]For the sake of convenience of reference, the first title being reviewed here will be referred to as CP3, the second as CP4.

as a logician and as a philosopher of mathematics. There is in a very important sense a very large gap in the *Collected Papers* because of this policy to exclude Russell's published book-length monographs from the *Collected Papers*. Russell's [1903] *The Principles of Mathematics* is the connecting link between the materials in CP3 and CP4.

1. 1896-1899.

From the time he completed his classroom education and was working on his fellowship thesis on the foundations of geometry in 1895-96 until 1899, Russell worked on what has been called his "Tiergarten Programme" (see Griffin [26]). The program, called so because it was conceived while on a walk in 1895 through the Berlin zoological garden at a time when Russell was working not only on the fellowship thesis but simultaneously on his first political tract, on German social democracy. The program was conceived as two parallel projects, one on philosophy of science which would begin with the most abstract and proceed to the more concrete, from mathematics to physics, and thence to physiology; the other would proceed in the opposite direction, from politics and social questions; the two uniting at last in a grand Hegelian synthesis of the scientific and the practical. The papers of most interest to history and philosophy of logic and history and philosophy of mathematics were presented in volume 2 of the present collection ([62]) and have already been reviewed in this journal (see Anellis [8]). Here it suffices to say first that Russell's efforts in mathematics and foundations of logic in the period from 1896 to 1899 were seriously flawed and frequently distorted, especially as regards his understanding of Cantorian set theory, by Russell's neo-Hegelian philosophy (for a detailed discussion of Russell's views and conception of mathematics at this time, see, *e.g.*, Anellis [1, 2, 3, 4]); second, that his notion of logic and its contents at this time, as he wrote in the *Essay on the Foundations of Geometry* (Russell [46, p. i]), were derived from the neo-Hegelian idealists, most notably Francis Herbert Bradley (1846-1924), and next Bernard Bosanquet (1848-1923) and Christoph Sigwart (1830-1904). One also has the sense that Russell's papers from that period were the jottings of a clever student who had been held back from creative and independent work by the necessities and requirements of student life, and that, once freed of the classroom, said graduate found himself *driven* to write down on paper each and every hitherto pent up thought that crowded his mind.

Anyone reading the works from the period 1896-99 without reading any of Russell's later writings in mathematics, logic, and their philosophy, would perforce be impelled to ask, as I did (see, *e.g.*, Anellis [8]), how Russell could possibly have earned a reputation as a serious logician. And anyone reading Russell's mathematical and logical work written in the period from 1900 forward, would perforce be impelled to ask, as I did (see Anellis [10]), how Russell so suddenly came to do such excellent work in logic and foundations of mathematics after having shown himself in the previous years to be almost a mathematical buffoon. One explanation offered is that Russell could do good work only under the direction, and with the assistance and guidance, of Whitehead. An explanation more favorable to Russell is that once shaken loose from the distorting prism of neo-Hegelian philosophy and the constriction that demanded theory and reality to be accounted for in all subject-matter in contradictions, *i.e.* as Kantian-Hegelian antinomies, and all knowledge to be found and boxed into Hegelian triads, Russell was able to study seriously and without preconceptions the work of Cantor and Peano, and to analyze logical and mathematical theories in a clear, straightforward, and may we say "logical" way. The true explanation is probably nearer to some combination of these two alternatives. In CP3 and CP4 we are presented with the work that leads Russell from his first tentative steps as a serious technician in logic to his completion of work on his first respectable product as a serious technician and philosopher of logic, namely the *Principles of Mathematics* [51]. As we trace this development, the one question that remains is: How much of what Russell accomplished in the *Principles* — and then in *Principia* — was due exclusively to his own efforts and how much of the perception of Russell the excellent logic technician was a mere façade, deliberate or otherwise, owed, directly or indirectly, to the assistance of others, to Russell's ability to adopt and adapt, even to absorb and copy, the work of a Peano or a Whitehead while making it appear as though it were the product and extension of his own efforts?

CP3, covering materials from the period 1900-1902, is particularly critical for the history of logic insofar as it marks the transition from Russell the mathematical "innocent" or Hegelian-Kantian harlequin that we found in the writings of 1896-1898 to Russell the brilliant technician of logic and proponent, if not founder, of logicism that we were taught by the history of logic and philosophy and foundations of mathematics to respect and admire. It is at the beginning of this period (1900) when the drafts of the *Principles of Mathematics* in the guise in which the *Principles* is familiar to us were first written, when the influence of Peano had already made its mark on Russell, when the

Russell Paradox was fully developed and recognized and recognizable as such, and when the Tiergarten Programme was replaced in Russell's philosophy of mathematics with his logicist program. It is at the end of this period (at the close of 1902) that these efforts culminate in the shipping to the printer of the final version of the manuscript for the *Principles*. It is in the products of the labors of this period 1900-1902 that Russell emerges as a major figure in mathematical logic and in philosophy and foundations of mathematics. It is the Russell of the history textbooks that the labors of this period begin to produce.

Russell's work on the *Principles of Mathematics* grew out of his previous work on the mathematical track of the Tiergarten programme. (These early efforts are presented in Vol. 2 of Russell's *Collected Papers* and discussed in MODERN LOGIC in my review of that volume. I shall not, therefore, reiterate any more than necessary my previous discussion of the earlier efforts.) It is, however, important to note that Grattan-Guinness in his recent sketch of the chronology of the writing of the *Principles* in the context of his analysis of how Russell wrote the *Principles* [25] based upon a close examination of the archival materials, asserts, among other points, that Russell was not yet a logicist at the time he began work on the *Principles*, indeed that even as late as 1900, Russell did not envision the *Principles* as advocating logicism, but that he arrived at his logicist views some time around January 1901. Francisco Rodríguez-Consuegra [43, chap. 2], however, sees marked logicist tendencies in Russell well before his meeting with Peano in July 1900, namely, in the unpublished writings on mathematical philosophy of the period 1898-1900.) In saying "grew out of" here, I am therefore being deliberately vague. The *Principles* is certainly one more effort by Russell to develop a system of mathematics from universal logical principles. But at first, those principles were still Hegelian rather than mathematical, and the earliest work on the *Principles* was just one more attempt at a draft of the grand philosophical scheme as conceived in the Berlin zoo and are hence continuous with the writings created along those lines and to the same purpose that are now found in volume 2 of Russell's *Collected Papers*, just one more in the succession of drafts towards developing the *Hegelian* system. Grattan-Guinness [25] also shows that the received account of the writing of the *Principles* is not substantiated by the documentary remains, that is, that the chapters and parts of the *Principles* were not written in the order in which they appear in the published text nor in the order which Russell retrospectively recollected; *e.g.*, he argues that Parts I and II of the *Principles* as we know it were written no earlier than the summer of 1901, that they did not exist in 1900, except perhaps possibly as preliminary sketches which however are no longer extant. (Grattan-Guinness's study of the chronology builds in large measure upon the chronological and textual studies of the *Principles* of Blackwell [12] and Byrd [14, 15, 16].) Perhaps most importantly, Grattan-Guinness [25] asserts that the famous appendices on Frege's work and on the Russell Paradox to be found therein, and the appendix presenting the theory of types as a means for overcoming the Russell Paradox, were not conceived merely as afterthoughts or ancillaries that arose from Russell's study of Frege, but are central and integral to the conception and focus of Russell's work in the *Principles* and the focal point of the *Principles* itself as a whole.

Russell's reputedly dramatic shift from Hegelianism towards logical atomism, and concomitantly logicism in philosophy of mathematics, occurred in large measure through the influence of Russell's friend and fellow philosophy student George Edward Moore. This reversal is well documented in the history of philosophy. From the perspective of the interests of history of logic, it is already evident in Russell's study of Leibniz. In *A Critical Exposition of the Philosophy of Leibniz* (1900), prepared from notes for a course he taught at Cambridge the previous year, Russell already sees relations as the most fundamental of logic, and he argues there that for Leibniz the subject-predicate relation of propositions is the basis upon which Leibniz sought to erect the edifice of his metaphysics. One effort to elaborate this view that is included in the *Collected Papers* is "Leibniz's Doctrine of Substance as Deduced from his Logic", dating from *circa* 1899-1900 (Paper 20, CP3, 514-534), which Russell read to the Aristotelian Society on 5 February 1900 and which comprises much of Chapters III-IV of the *Philosophy of Leibniz*. It is consequently no coincidence that it was while completing the preface for the *Philosophy of Leibniz* that Russell wrote his first major article on the logic of relations.

By 1901, Russell was clearly and indubitably a logicist. The paper "Recent Work on the Principles of Mathematics" (Paper 10, CP3, pp. 366-379) first appeared in the popular journal *International Monthly* [Russell 1901a]. Cantor here comes in for praise as providing with his set theory a solid means for the first time in the history of mathematics for dealing with the likes of Zeno's paradoxes in particular and for problems of the continuity of the infinite in general. Cantor's set theory also finds favor therefore because it provides a foundation for real analysis as presented by Weierstrass, which enables rigorous treatment of problems of the continuity of the infinite. This paper marks the death knell, in case one would be still required at this juncture, for Russell's Kantianism and is the manifesto of his logicism. "The proof that all pure mathematics, including Geometry," Russell writes (CP3, p. 379), "is nothing but formal logic, is a fatal blow to the Kantian philosophy." It was his study of the work of Peano and Peano's school, and in particular of their work in the axiomatization of geometry, that brought forth from Russell and enabled this declaration. It is possible to go even one step further: for Russell, certainly during the period when Peano's work exercised its greatest influence upon him in late 1901 through mid-1902, without geometry in general, and in particular without this work of Peano and his school on the axiomatization of geometry, mathematical logic would not have been conceivable. It is, I think, not by accident that Edwin Bidwell Wilson viewed *The Principles of Mathematics* and *An Essay on the Foundations of Geometry* as part of a consistent and sustained effort by Russell to present the foundations of mathematics. In his double review [69] of the *Principles* and the French translation of the *Essay*, Wilson writes ([69, p. 76]) that Russell's enterprise in these works "is probably the first attempt to give a complete definition of mathematics solely in terms of the laws of thought . . ." through the systematic axiomatic presentation of geometry and arithmetic as [comprised of] propositions of the form "p implies q".

To gain a fulcrum for our continued discussion and a general perspective on the development and trends of Russell's thought from the 1890s to the end of 1902 when Russell completed work on the *Principles* and set his first significant mark on the development of mathematical logic, let us turn to retracing in broad outline the evolution of Russell's work from his early studies in geometry and the significance which they had on the shaping of his views on logic.

Despite their Hegelian philosophical spirit, Russell's studies of non-Euclidean geometry in the 1890s were, I had noted [11], a very important prelude to his later work in logic and foundations of mathematics, leading up to his work with Whitehead on the *Principia Mathematica*. This early work in geometry, I likewise noted (Anellis [11]), exercised a strong influence on the development of Russell's logicism; hence I am most strongly inclined to agree with Grattan-Guinness [25, p. 105] that Russell's logicism had "come to him" as a generalization of his conception of geometries. This influence is evident even despite the fact that the fundamental thrust and principal purpose of Russell's early work on foundations of geometry was to defend a Kantian philosophy of geometry specifically and a Kantian philosophy of mathematics in general. The pivotal piece here was Russell's fellowship dissertation *An Essay* *on the Foundations of Geometry* (1895), which was then published as a book in 1897 ([46])[2].

In his autobiography [58, p. 36], Russell wrote that he first began his study of Euclid at age eleven, under the tutelage of his brother Frank. (The exact date of this first lesson in Euclid was 9 August 1883). He wrote that "This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world." But he was also troubled by the need to accept the axioms of geometry without proof. Thus he formed a need to search for absolute certainty in mathematics, and his doubts, expanded by age fifteen (in 1888) to include questions about differential calculus, led him into the realm of foundations of mathematics. In *My Philosophical Development*, he wrote [60, p. 2] that ". . . I had doubts. Some of Euclid's proofs, especially those that used the method of superposition, appeared to me very shaky. One of my tutors spoke to me of non-Euclidean geometry. Although I knew nothing of it, I found the knowledge that there was such a subject very exciting, intellectually delightful, but a source of disquieting geometrical doubt."

His efforts led him in 1890 to enter the mathematics program at Cambridge, where he became the student of Andrew Russell Forsyth (1858-1942), and thus indirectly of Forsyth's teacher Edward John Routh (1831-1907), and then of Whitehead. Here his youthful doubts about geometry were compounded and aggravated by new doubts raised by his study of infinitesimal analysis, and in 1896 by his discovery of real analysis (see Russell [60, p. 27]; for a description and evaluation of

[2] I had in the past erroneously given 1896 as the year for the fellowship thesis, evidentally confusing it either with the draft of the several series of lectures based upon it which Russell gave in 1896 or with the revised manuscript presented to Cambridge University Press. The correct chronology is as follows. The dissertation was submitted to Trinity College, Cambridge in manuscript form in August 1895. The readers were Whitehead for mathematics and James Ward (1843-1925) for philosophy. Russell won the fellowship for which the thesis was written on 10 October 1895. No copy of the manuscript of the thesis is known to be extant. Russell worked on a revision of the thesis thereafter and until perhaps about the end of September 1897, signing a book publishing contract for the revision of the dissertation on 28 September 1896. The revisions responded in detail to questions which Ward and Whitehead raised during the defense of the thesis. The book manuscript was submitted to Cambridge University Press just before leaving for the US on 3 October 1897. The *Essay* was published by Cambridge University Press on 20 May 1897. (See, *e.g.*, Slater [63, pp. 3-4], as well as the chronologies included in volumes 1 and 2 of Russell's *Collected Papers* for the specifics of the dates).

Russell's difficulties with calculus, see, *e.g.*, Anellis [3]) and by his discovery of non-Euclidean geometries.

First introduced by one of his tutors to non-Euclidean geometry, probably in 1894, Russell began to read Lobachevskii's *Theorie der Parallellinien* in February 1895, after having already read some of the geometrical works by Riemann, Frischauf, Killing and Klein during the previous year, in preparation for writing his fellowship dissertation (1895), *An Essay on the Foundations of Geometry.* (In his log book *What Shall I Read?*, Russell recorded his reading Erdmann's *Die Axiome der Geometrie* in December 1893 and again in May 1896, Riemann's *Hypothesen welche der Geometrie zu Grunde liegen* in February 1894, Frischauf's *Absolute Geometrie* and Killing's *Die Nicht-Euklidischen Raumformen* (*erste Hälfte*) in July 1894, Klein's *Nicht-Euklid* in August 1894, in June 1895 and again in September 1895, Lobachevskii's *Theorie der Parallellinien* in February 1895, Helmholtz's *Sämtliche Schriften über Geometrie* in March 1895, Cayley's *Sixth Memoir upon Quantics*, Stumpf's *Ursprung der Raumvorstellung*, Gauss's *Disquisitiones circa superficias curvas* (the work that founded differential geometry), Beltrami's *Saggio di interpretazione della geometria non-euclidea* and *Teoria fondamentale degli spazii di curvature constante* in May 1895, and Sophus Lie's 1890 *Grundlagen der Geometrie* in June 1895, Lechalas's *L'Espace et le Temps* in November 1895, Grassmann's *Ausdehnungslehre von 1844* in May 1896, and Mansion's *Principes de la Métagéométrie* and Bonnel's *Hypothèses dans la Géométrie* in February 1897.) In his [54, p. 143], Russell wrote of this time in his life that "I discovered that, in addition to Euclidean geometry there were various non-Euclidean varieties, and that no one knew which was right." In 1894 and 1895, he wrote several papers on epistemological questions in geometry in the context of the debate between William Ward and James Fitzjames Stephen on the nature of mathematical truth. These include "The Logic of Geometry" ([44]) and "The *a priori* in Geometry" ([45]). The most significant of these pieces which survive is the "Observation on Space and Geometry", written in the first half of 1895 (published in [59]). Volume 2 of the *Collected Papers of Bertrand Russell* includes a collection of Russell's writings in geometry from the period 1898-1899, some of it previously unpublished, but also in particular, English translations of his two papers on the philosophy of geometry, "Are Euclid's Axioms Empirical?" ([47]) and "The Axioms of Geometry" ([48]), first published in French in the *Revue de métaphysique et de morale*, that grew out of his reply respectively to criticisms of Louis Couturat ([19]) and Henri Poincaré ([40]) in response to their consideration of his book *An Essay on the*

Foundations of Geometry [46]. The focus of discussion between Russell and Couturat and Poincaré is whether geometrical axioms are *a priori* (necessary) or *a posteriori* (empirical) and whether Euclidean geometry is mere convention or either true or false. In the paper "The Teaching of Euclid" published in the May 1902 issue of the *Mathematical Gazette* (2, pp.165-167) and reprinted in the first volume being here reviewed (pp. 467-469), Russell takes Euclid seriously to task for the lack of "logical excellence" which Euclid was reputed to have presented in his book. The point also recurs in the *Principles* [51, p. 5] where Russell points out the need for rules or "principles" of deduction and proceeds to offer ten such principles [51, pp. 4-5, 10-16], including in particular "formal implication" or the rule of detachment. We may summarize Russell's strong criticisms of Euclid by reminding ourselves of the difference between an axiomatic system and a formal deductive system and by reporting that Russell in essence accuses Euclid of not possessing a formal deductive system. (Richards [42] examines in depth the epistemological context and significance of Russell's *Essay.* The changing attitudes towards non-Euclidean geometries in England in the second half of the nineteenth century, as chronicled and described by Richards [41, pp. 201-229] are, devoted to "Bertrand Russell and the Cambridge Tradition".)

Russell's early articles "The Logic of Geometry" ([44]) and "The *a priori* in Geometry" ([45]) attempt both to give an axiomatic foundation to geometry and to provide a justification for the axioms of geometry. He argues that three axioms, namely the axiom of congruence (also called the axiom of free mobility), the axiom of dimensions, and the axiom of distance (axiom of the straight line), are required by every version of metrical geometry. Russell defends each of the axioms which he introduces, although these "proofs" of the axioms are really just philosophical justifications for their being necessary for the possibility of geometry. Nevertheless, Russell already recognized the argument that these three axioms are required by any metrical geometry, and this led him to conclude that they are *a priori.* The apriority of the axioms becomes a central thesis of the *Essay on the Foundations of Geometry.*

The existence of non-Euclidean geometries side by side with Euclidean geometry deepened Russell's feelings of uncertainty about his resolve to search for the ultimate truth of mathematics. This need to find certainty in mathematics and his dissatisfaction with the education he received in mathematics led him in his fourth year at Cambridge to switch from mathematics to philosophy. In the 1902 essay "The Study of Mathematics", Russell explains his defection from mathematics to philosophy by quoting Plato, writing that Plato had said that "there is in mathematics something which is necessary and cannot be set aside . . . and, if I mistake not, of divine necessity" (see Russell [61, p. 86]). Then Russell adds his own sentiment ([Russell [61, p. 86]): "But the mathematicians do not read Plato, while those who read him know no mathematics, and regard his opinion upon this question as merely a curious aberration." With as much training in mathematics as his preparation for the Mathematical Tripos at Cambridge provided him, Russell then turned to philosophy in search of the necessity of mathematics which he craved and which Plato told him was there. The first major effort to fulfill the quest for mathematical certainty led Russell to write *An Essay on the Foundations of Geometry*.

Whereas Pasch, Peano, and Hilbert and their colleagues were led to develop their axiomatizations of geometry starting from the technical consideration of providing a unified minimal system capable of accommodating Euclidean geometry as well as the various non-Euclidean geometries, Russell during his neo-Hegelian and neo-Kantian period sought to develop "metageometry" starting from the influence of the philosophical question of the epistemological character of geometrical propositions. Thus, the *Essay* is to be understood as an attempt to rescue the Kantian claim of the *a priori* synthetic nature of geometry from the existence of non-Euclidean geometries by finding the common truths of both Euclidean and non-Euclidean geometries and showing that these form the system of "metageometry" and are *a priori* synthetic (see Anellis [6], especially p. 97). The rescue hinged on arguing that non-Euclidean geometries are just special cases of projective geometry.

In the last two or three years of the nineteenth century, Russell abandoned his neo-Hegelian idealism and simultaneously began shifting the focus of his attention away from geometry to the study of real analysis and its foundations, especially set theory and logic, in the very last year of the nineteenth century. Thus, he wrote in [53, p. 15] that, in mid-1898, he began work on a book about the principles of mathematics, the writing of which he described as his "chief ambition ever since age of eleven," that is, since he first came in contact with Euclid's *Elements*. Russell's new studies of the work of Peano, Frege, Cantor, Weierstrass and others carried him in the direction of developing logic as the axiomatic foundation of all of mathematics. After two false starts, when the book finally appeared, the result of the effort was *The Principles of Mathematics*. This new direction in Russell's work was clearly reinforced by the axiomatic concerns begun under the influence of his work in "metageometry". This direction was already hinted at in the *Essay on the Foundations of Geometry*, when Russell wrote in the *Essay* ([46, p. 14]) that in the second period of the history of metageometry, "which was largely philosophical in its aims and constructive in its methods. It aimed at no less than a logical analysis of all the essential axioms of Geometry, and regarded space as a particular case of the more general conception of a *manifold.* Taking its stand on the methods of analytical metrical Geometry, it established two non-Euclidean systems, the first that of Lobatchewskii, the second — in which the axiom of the straight line, in Euclid's form, was also denied — a new variety, by analogy called spherical." Thus we see that the work of Lobachevskii and others on non-Euclidean geometry exerted an important early influence on the later direction of Russell's contributions to logic and to his logicist philosophy. During the period when he was at work on the *Principles*, Russell continued to write on geometry and on the philosophy of space and time. These writings are found in CP3 as "Part IV. Geometry" (pp. 455-504) and "Part II. Absolute Space and Time" (pp. 215-282) respectively, and among the materials included in the appendices of the volume at hand. We see from the writings included in Part II that Russell argued that space and time are absolute, not relative, thereby adopting a Newtonian (and Kantian), rather than Leibnizian, position. Moreover, the issues raised are dealt with in sections of the drafts of the *Principles* found in this volume, and, of course, recur in the published version of the *Principles* in Parts VI and VII. Russell never lost the need for the mathematical certainty that he was taught through his first encounter with Euclid, and his work on foundations of mathematics is a manifestation of the search to underwrite that certainty with logical rigor and, at least at the outset, with Kantian apodicticity. However, in the writings on geometry dating from 1902 and included in Part IV of CP3, Russell has come to reject the Kantian view which he previously had held, that the axioms of geometry are *a priori.*

During the period from 1896 to 1899 Russell produced various miscellaneous writings on philosophy of mathematics and philosophy of science, much of it intended for a project developing the "dialectic of the sciences", written with the intent of presenting a Hegelian system of science that began with general concepts of mathematics and, working from the general to the concrete, to a consideration of physics. Many of these writings were collected in a notebook entitled "Various Notes on Mathematical Philosophy." Some of these writings were incorporated in revised form into Chapter 4 of Russell's *My Philosophical Development.* In 1896-97, Russell learned about Cantorian set theory. Now the questions about calculus which he entertained as a student at Cambridge combined with a misunderstanding of Cantorian set theory (see, *e.g.*, Anellis [3]), the old worries about uncritical acceptance of the axioms of Euclidean geometry and newer doubts about the truth of the varieties of non-Euclidean geometries, and led Russell to undertake a project on the foundations of mathematics. Kolesnikov [35, pp. 23-39] adopts a broader, philosophical, view of Russell's work in this period, especially the *Essay* and "Various Notes on Mathematical Philosophy", so that this work is seen as "prolegomena" to Russell's work on the theory of knowledge, rather than as a more restrictive and specific preparation for Russell's logicist program. This led Kolesnikov to assert that it was through his work on his *Critical Exposition of the Philosophy of Leibniz* (1900) that Russell first formulated the task of applying the tools of mathematical logic to the analysis and elaboration of all science, including epistemology, while I take a mathematical perspective to claim that it was already in the geometrical studies of this period that Russell gradually formulated his logicist program and took his first steps towards its elaboration, and in the *Philosophy of Leibniz*, Russell formulated the next decisive step towards the logicist program of developing a metaphysics on the basis of the logico-grammatical structure of language, which in turn pointed towards the next step in the succession, of the reduction of linguistic and mathematical structures into logic.)

During this period from 1896 to 1899, we find Russell's starts at the writing of a book, which never was completed, whose aim was to provide Russell's fulfillment of his so-called "Tiergarten programme", formulated in 1895 and described [58, p. 125] as the writing of "one series of books on the philosophy of the sciences from pure mathematics to psychology" and "largely inspired by Hegelian ideas" (see, *e.g.*, Anellis [5, especially pp. 162-163] for geometry; for more detail on the Tiergarten program, see Griffin [26, 27]). Most of the material on calculus, set theory, and number theory included in "Various Notes on Mathematical Philosophy" was written in 1896 and 1897, whereas "An Analysis of Mathematical Reasoning" was written in 1898, and "The Philosophy of Mathematics" was written in 1898-1899. Already in "An Analysis of Mathematical Reasoning," if one looks carefully, there is a hint here of the transition of Russell's interest away from providing a Hegelian synopsis of mathematics and science towards a concern for the relationship between logic and language, if not for presenting a fully developed logicism. (The origin and course of Russell's emerging logicism is traced in detail by Rodríguez-Consuegra [43].) The first glimmerings of this shift arose precisely out of Russell's Hegelian concern in his work on the philosophical foundations of geometry to settle the questions of the analyticity *versus* syntheticity of geometrical propositions and whether Euclid's axioms are necessarily true or empirically contingent. Thus, the first chapter of the first "Book", on manifolds, of "An Analysis of Mathematical Reasoning" opens with a discussion of "The Elements of Judgment", followed by a chapter on "Subject and Predicate" before taking up the topic of manifold. Here, a manifold is defined ([62, p. 179]) as a "collection of terms having that kind of unity and relation which is found associated with a common predicate"; therefore, a manifold is precisely Cantor's *Mannigfaltigkeit*. At this point Russell introduces the term *class* as a synonym for *manifold*. He concludes ([62, pp. 184-185]) that though "a manifold ... may, for purely numerical purposes, be a mere assemblage", it must, however, be "coextensive with the terms of which some predicate can be asserted," so that a manifold is the extension of a concept found in traditional logic. Having once defined manifolds, Russell was prepared in the next chapter to consider "the true topic of Symbolic Logic" — the mutual relations of addition and synthesis, after which he defines this "Logical Calculus" as the science which deals "with manifolds as such" and he leans heavily on Whitehead's *Treatise on Universal Algebra* to discuss the relations of symbolic logic, including especially equality and equivalence. The remainder of the book deals with increasingly concrete applications of the symbolic logic, moving from number to quantity. Much of the material from these chapters is missing and it is difficult to judge how much progress, if any, Russell was making towards understanding Cantorian set theory or the work in real analysis of Weierstrass and his colleagues, but the surviving paragraph of the chapter on infinity ([62, p. 234]) does not offer any suggestion of greater comprehension.

There were, we noted, two separate attempts to write a synoptic work on the foundations of mathematics between the publication of the *Essay on the Foundations of Geometry* in 1897 and the appearance in 1903 of *The Principles of Mathematics*. The first attempt contains parts of two chapters, the first on cardinal numbers, the second on ordinal numbers. In this treatment, cardinals are viewed as adjectives which apply to manifolds taken as a whole, not necessarily as the "*Anzahl*" (understood in the colloquial sense of the cardinality of a *Zahl*) or power of the manifold. In the next chapter, on ordinals, Russell distinguishes between cardinals and ordinals by noting ([62, p. 251]) that ordinals involve the notion of *order* and that "the ordinal numbers involve and presuppose the cardinal numbers, but the converse does not seem to be the case." Now Russell rescues his treatment of cardinals from the previous chapter by stating ([62, p. 251]) that cardinals express the sum of terms of manifolds. At last, then, Russell has apparently begun to admit the difference between a *Zahl* and its cardinality. But it is still unclear that he now understands the difference between natural numbers and cardinals. Nor is he prepared to talk about the real numbers. He still speaks ([62, p. 255]) of the conception of "the number infinity, as the last term of a series which has no last term" in which all the terms are natural numbers. He has not yet come to recognize Cantor's transfinite; he still clings to his belief that the concept of the number infinity involves a contradiction, and he revives the old arguments which he used in the writings from 1896-1897, in which no distinction is made between the actual infinite and the potential infinite.

The material from his next attempt, "The Fundamental Ideas and Axioms of Mathematics", is the last before he embarked on writing *The Principles of Mathematics*. It is clearly somewhat more sophisticated in its recognition and treatment of number theory than any of the previous attempts. But it is still clearly deficient in its understanding of the infinite and lacks the technical acuity and breadth of familiarity with the technical literature found in the *Principles*, and the approach and concerns remain largely philosophical rather than mathematical. At this stage in Russell's thinking (as a vestige of his atomism), the natural numbers are built from the fundamental *unit*, the number one, which is primitive. We find then that "$1+1 \neq 2$" still means for Russell [62, p. 288] that if one adds one unit to one unit, we have two *units*, or that "*two* is a mere abbreviation" or synonym, for "one unit and one unit". In a fragment ([62, p. 298]) for part of the "Fundamental Ideas", he declares that "1 seems to mean exactly the same as *term* or *concept* or *logical subject*" and much of this material is devoted to examining the linguistic conditions under which a proposition of the form "$1 + 1 = 2$" is true or false and what it means ("if A is one and B is one, then A and B are two" and 1 is the logical subject, 2 is a predicate). In a note ([62, p. 296]) for the "Fundamental Ideas", he at last comes to accept the notion of irrationals. In the extant parts of the "Fundamental Idea", the infinite is treated only in the "Synoptic Table of Contents"; there is no surviving textual material in the *nachgelassene* parts of the manuscript that deals with the question of the infinite. However, the "antinomy" of the infinite that is found in Russell's earlier writings, that is in his misconstrued direct attempts from 1896 and 1897 to deal with real and infinitesimal analysis and Cantor's transfinite, still survives in the "Synoptic Table of Contents" ([62, p. 267]). There, Russell says that, where N is the number of numbers, then since N + 1 is also a number, there is no number of numbers.

In [60, p. 30] Russell recites a list of some of the mathematics books which he read after 1896; although the works listed were texts in analysis (Dini), in differential geometry (Gauss, Darboux), in linear algebra (Grassmann) and in universal algebra (Whitehead) which he "afterwards discovered, [were] quite irrelevant to my main purpose," we nevertheless can assume from the context of their presentation that he was looking for texts that would help him specifically with his work on the foundations of geometry rather than works that might help him out of his difficulties with either calculus or Cantorian set theory. Thus it appears that the questions raised by the lack of proofs for the axioms of geometry and the creation of several competing varieties of non-Euclidean geometry, each in competition with Euclidean geometry, provided Russell with the motivation for his work not only in set theory and foundations of analysis, but more broadly in logic and foundations of mathematics generally.

The use of philosophical arguments for proofs in "The Logic of Geometry" [44] and "The *a priori* in Geometry" [45] indicates that Russell was not yet thinking in modern terms about the nature of logical proof. But it does show that Russell already clearly recognized that a logic conception of justification must play a rôle in establishing the "certainty" of mathematical claims. Thus, already in "The *a priori* in Geometry" [45, p. 97], Russell began by declaring that the purpose of the paper "is purely logical, and aims at applying principles of general logic to geometrical reasoning," adding that his inquiry has two parts, "(1) an analysis of the actual reasoning of Geometry, with a view to discovering those essential axioms, and that fundamental postulate, without which this reasoning would become formally impossible; (2) a deduction, from the fundamental nature of a form of externality, of the principles which must be true of any such form, when treated in abstraction as the subject-matter of a special science." Russell says this more succinctly in "The Logic of Geometry" [44, p. 1] which he begins by declaring that his concern is "with Geometry simply as a body of reasoning." The concept of proof is firmly and thoroughly linked with the aims of metageometry in Russell's thought by 1902, when he wrote (in 1902; see Russell [61, p. 92]) that "It is a merit in Euclid that he advances as far as he is able to go without employing the axiom of parallels — not, as is often said, because the axiom is inherently objectionable, but because, in mathematics, every new axiom diminishes the generality of the resulting theorems, and the greatest possible generality is before all things to be sought." Even as late as 1960, Russell could exclaim that "what delighted me about mathematics was that things could be proved" (see Marquand [36, p. 24]).

The long-term impact which geometry had on Russell's development of the logicist program is also evidenced by the plans to write a fourth volume of the *Principia Mathematica* to be devoted entirely to geometry (see Harrell [28] for a description of the plans to write a volume of *Principia* on geometry and of the materials prepared towards that project.)

With this background in mind, let us now finally turn to the details of the development of Russell's work in the period from 1900 to the end of 1902.

2. 1900-1902.

We have within the context of our discussion on the influence which geometry had on the formation of Russell's logicist program spent so much time discussing the material from the period 1896 to 1899 because Russell's "An Analysis of Mathematical Reasoning", completed in July 1898, was the "first substantial attempt" at a draft of what eventually became the *Principles* (see p. xix of Moore's "Introduction" to CP3). Moreover, the influence of Bradley on Russell's thought remained, even after his neo-Hegelian idealism dissipated. We see this in Russell's belief, evident, as already noted, from his *Exposition of the Philosophy of Leibniz*, that relations were the heart of the mathematical enterprise. In *Essay on the Foundations of Geometry*, he accepted the Kantian view of the basis of relations; judgments are both analytic and synthetic, holding between parts and wholes, synthesis combining parts into a whole, analysis analyzing wholes into parts. Diversity, he thought, was the fundamental relation of mathematics (except possibly in projective geometry), and in the autumn of 1899 he sought to base his logic on the relation of diversity (see, *e.g.*, "Appendix I.5. Logic Founded on Diversity", CP3, p. 559). He did not yet believe, however, that *identity* was a relation (see, *e.g.*, CP3, p. 140, from the draft of *Principles*). His reasoning was that a relation required two terms. It is only in October 1900 that he came to accept identity as a relation (see CP3, pp. 593-594, from the draft of "On the Logic of Relations").

It was in this crucial period that Russell began to discover mathematical logic as an algebra of relations. In 1898 he read Whitehead's newly published *Treatise of Universal Algebra*, and at the same time learned, through reading Whitehead's book, about the work of Boole. In the middle of 1901, while working on the *Principles*, Russell read in both Boole and De Morgan. The *Principles* came to be regarded, at least by Russell, as a study in the system of Peano supplemented and augmented by his "own" invention, the logic of relations (see, *e.g.*, CP3, pp. xxvi, xxviii, 310, and Anellis [10, p. 287]). The complete "Calculus of Logic" for Russell now was precisely this Peanesque system as augmented by the logic of relations. Wilson [69, p. 78] more accurately and astutely notices that Russell merely "has simplified and improved the older work of C. S. Peirce on the theory of relations, adapting it to the system of Peano, and has [thereby] produced a coherent treatment" of the foundations of mathematics. As for Russell himself, he writes in his paper "The Logic of Relations" (CP3 Paper 8, p. 314) and the published French version (CP3, Paper V.2, p. 613) (*q.v.* Russell [49]) that he has simplified the logic of relations of Peirce and Schröder and provided it with a Peanesque translation. (For a discussion of Russell's knowledge of, and borrowings from, Peirce's work in logic, see Anellis [10]).

Under the influence of Whitehead and (especially) Couturat in the period from 1898 to 1900, and thereafter of Peano and his colleagues as well, and having abandoned his Kantian and Hegelian philosophy, Russell came to revise his initially negative, even hostile, attitude towards Cantorian set theory. That is, he no longer regarded it, as he formerly had in 1896 and 1897 when he first came into contact with it through the negative appraisal of Hannequin, as an instance of philosophical confusions and mathematical misunderstandings, and a tissue of contradictions (see, *e.g.*, Anellis [2, 3]). In the manuscript "On the Principles of Arithmetic" which appears to date from 1898 and to have been written after "An Analysis of Mathematical Reasoning", Russell devoted considerable attention to ordinal and cardinal numbers (see Russell [62, pp. 247-260]). This change of heart with respect to Cantor was assisted in the summer and autumn of 1900 by Peano and his school — and more specifically with the aid of Guilio Vivanti (1859-1949), who co-authored with Peano Part VI of the *Formulaire de mathématiques*. It is in Part VI of the *Formulaire* (read by Russell in September 1900) that set theory was translated into the Peanesque notational system of logic. Now Russell studied Cantorian set theory seriously and without the intent of searching for Hegelian contradictions to be displayed, but with a view to understanding its potential as a serious mathematical theory and as a legitimate contender as the foundation of mathematics in which any contradictions which might arise were to be dealt with and extirpated rather than displayed and either celebrated or condemned. It was early in the last month of 1900 that he detected a problem with Cantor's theorem, a problem whose extent he realized to be serious but whose breadth and nature he did not yet fully appreciate or comprehend as a paradox and which was the first hint, if not the first version, of the Russell Paradox (see Anellis [1, pp. 9-11], [3, pp. 23-25], [6]). By the time he was working in earnest on the manuscript draft that became the *Principles*, the Russell Paradox was already near, if not at, the center of his attention and, as we noted that Grattan-Guinness [25] had enunciated, was at the core of his concern in the *Principles*, rather than at its periphery. The antinomy of infinite numbers which one finds in the "Fundamental Ideas" [62, p. 166] places Russell "on the brink of the Paradox of the Largest Cardinal," Moore (CP3, p. xxiii) tells us. The version of Russell's antinomy of numbers in the earliest, 1899-1900, draft of the *Principles* (which is likewise interpretable as the next version of the "The Philosophy of Mathematics" of 1898-1899 and the last effort, on the mathematical side, of the Tiergarten Programme) is, in Moore's words (CP3, p. xx), "very similar in form" to the Paradox of the Largest Cardinal, which he [Russell] formulated in January 1901 and which led him in May 1901 to Russell's Paradox." Alejandro Garciadiego [24, especially chapters III and IV] elucidates the origin and development of the paradox of the largest cardinal as it emerged in the 1899-1900 draft of the *Principles* and follows its course through completion of the final printed version of the *Principles*.

The early drafts of the *Principles* printed in CP3 occupy the first 212 pages of that volume and comprise Part I of the volume. They were written in 1899-1902. The first paper in Part I of CP3 is the draft of 1899-1900 (CP3, pp. 9-180). Paper 2 is the draft of 1901 of the *Principles* (CP3, pp. 185-208); Paper 3 is the 1902 draft outline of Book I of the *Principles*, on the variable (CP3, pp. 211-212).

Grattan-Guinness's conclusions about the order in which the parts of the *Principles* were written, when he declares that Parts I and II of the *Principles* were written no earlier than the summer of 1901, that they did not exist in 1900, except perhaps possibly as preliminary sketches which however are no longer extant, must be taken *cum grano salis* and indeed with an initial stiff dose of skepticism as we examine the material in Part I of CP3. In the headnote to the draft of 1899-1900 of the *Principles* (Paper 1 of CP3), Moore writes (CP3, p. 9) that "Parts I and II differ the most" between the 1899-1900 draft and the final published version, and he points out the relationships between the draft and the published text of the *Principles*, providing both comparisons and contrasts. He tells us that: Part I of the draft, on "Number", corresponds to Part II of the published *Principles*; Part II of the draft, on "Whole and Part", has been incorporated into Part II of the published text, albeit in reduced form; and that the treatment of arithmetic in

Part I of the draft differs from the treatment found in the published *Principles* and is closer to the treatment found in "Analysis of Mathematical Reasoning." We cannot in any wise conclude from the evidence adduced by Moore that Parts I and II of the *Principles* did not exist prior to the summer of 1901; we can conclude neither more nor less than that the some of the material in Parts I and II of the published text were written in 1899-1900 but were then rewritten and rearranged later, probably in the summer of 1901 or some time thereafter, Grattan-Guinness [25, p. 105] then does slightly mitigate his original claim by admitting Russell had written a book manuscript titled "Principles of Mathematics" that "played an important rôle in the preparation" of the *Principles* "even to the extent of providing some of the folios" of the *Principles*, and he explicitly equates the book manuscript with the material in Part I of CP3. It is perhaps, therefore a matter of interpretation or degree of distinction whether the material presented in Part I of CP3 is an early draft *of* the *Principles* or a draft of the immediate *predecessor* of the *Principles* which was subsequently used in some measure *for preparing* the *Principles*.

Not only Russell himself (in CP3 alone, see, *e.g.*, the quote at pp. xiii, to cite just one of many; also see, *e.g.*, in the *Principles* itself [51, p. viii] and [58, pp. 217-219]), but in his wake mathematicians contemporary with Russell at the turn of the century, such as E. B. Wilson (see, *e.g.*, Wilson [69]), and historians of logic from Hubert Kennedy (see, *e.g.*, [34]) to Gregory Moore (see, *e.g.*, CP3, pp. xxv-xxviii), have heavily stressed the great importance of Peano's influence on Russell. For Russell, in January 1901, writing "Recent Work on the Principles of Mathematics," (Paper 10, pp. 366-379): "The great master of the art of formal reasoning, among the men of our own day, is an Italian, Professor Peano, of the University of Turin. He has reduced the greater part of mathematics (and he or his followers will, in time, have reduced the whole) to strict symbolic form ..." (CP3, p. 368).

The meeting with Peano at the First International Congress of Philosophy in Paris in July 1900 was viewed by Russell as a significant turning point in his intellectual development (see, *e.g.*, his *Autobiography* [58, pp. 217-219]). The meeting, made possible by Louis Couturat, who, as organizer of the congress, invited Russell to participate, brought Russell into contact with the leader of a school whose aim was to provide an axiomatic development of all mathematics (but especially arithmetic, number theory, analysis, and geometry) which would be based upon a small number of axioms. Moreover, Peano provided a convenient notational system, based upon the primitive terms of *element*, *class*, and *number*, and the primitive connective of *class* *membership* or *elementhood*. Russell [56, p. 66] lists this distinction, along with the distinction between a one-element set and the individual comprising that set ([56, p. 67]) as among the two most crucial lessons which he learned from Peano. (Interestingly, even somewhat surprisingly in retrospect, however, Russell [56, p. 67] doubted that Peano was aware that these distinctions had also been made, earlier, by Frege. Surprising because Peano was among the best informed and most appreciative of Frege's work among logicians of the day and because they discussed these issues in their correspondence; in an undated letter to Peano that seems to have been written some time during the period 1891-1894, Frege takes particular note of Peano's differentiation between '$\in$' and '$\supset$', *q.v.* ([23, p. 109]); and Peano, for example, mentions the distinction which he makes in the *Formulaire* between the signs '$\in$' and '$\supset$' in his letter to Frege of 24 October 1895 (see Frege [23, pp. 111-112].) (By contrast, Peirce's student Christine Ladd-Franklin is reported (by then American Mathematical Society secretary Frank Nelson Cole (1861-1926) ([18, p. 59]), as late as the summer of 1918, to regard the $\in$ as a defect in Peano's and Russell's systems; 'there is nothing peculiar in the *relation* concerned — the specificity lies simply in the subject term which is "individual" or "singular.")

Peano's notational system gave Russell the tools that he employed in the 1901-1902 reworking of the final draft of the *Principles*, and Peano's conception of mathematics being drawn from a simple set of axioms deriving from the logical notions of class and class membership complemented and completed Russell's "conversion" to logicism. We may well imagine that the talk delivered at the Paris Congress by Mario Pieri (1860-1913) on "Geometry as a Purely Logical System" (which was published as [39]) clinched logicism's attractiveness in Russell's eyes, given the preoccupation which Russell had with geometry from his youth forward and the centrality of geometry to his thought up to that very moment. For Russell, the calculus of logic could only have arisen through geometry; the axiomatics of geometry and the history of geometry made logic possible; and, as we already said, for Russell, mathematical logic could only have arisen in virtue of the work of Peano and his colleagues in axiomatizing geometry.

The 1901 draft, and especially the 1902 draft, of the *Principles* in Part I of CP3, the papers in Part III of CP3, and many of the papers included as appendices of CP3, eminently display the Peanesque influence. The papers on the logic of relations ("Sur la logique des relations avec des applications à la théorie des séries" [49]; CP3, Appendix V.2, pp. 613-627; translated into English as Paper 8 of CP3, pp. 314-349) and order ("Théorie générale des séries bien ordonées" [50]; CP3,

Appendix VII.2, pp. 661-673; translated into English as Paper 12 of CP3, pp. 389-421) were not only published in Peano's journal *Rivista di Matematica / Revue de mathématiques*, but are excellent exemplars of the Peanesque school of mathematical logic. Moreover, Grattan-Guinness [25] contends, the logic presented in these two papers were integral aspects of Russell's conception of the *Principles*, as much as were his thoughts on the Russell Paradox.

Paper 9 of CP3, "Recent Italian Work on the Foundations of Mathematics"; CP3, pp. 352-361), written in 1901, is Russell's summary exposition of, and paean to, the work of Peano and his school. Initially intended for the philosophy journal *Mind* but never before published, it describes the Peanesque notational system and basic concepts in non-technical language before unfavorably comparing Peano's work with Schröder's and from thence proceeding to a discussion of the Peanesque axiomatization of arithmetic (especially Peano's postulates) and geometry. In the context of his comparison between Peano and Schröder, Russell dwells on the clarity of Peano's distinction between set membership and class inclusion and criticizes Schröder's handling of classes and individuals. Cantor and his endeavors here win the sympathy of Russell, via the Peano-Vivanti treatment of set theory. By the time he started work on the draft of the *Principles*, he began taking seriously Cantor's work on ordinals and cardinals (see CP3, p. xxiii). Paper 13 of CP3 (pp. 425-430), "On Finite and Infinite Cardinal Numbers", shows Russell to be an active participant in the enterprise of developing Cantorian set theory, applying the tools of Peano's mathematical logic to present what we have come to know as the Frege-Russell definition of cardinal number. Here, a cardinal is defined as *finite* if it was obtained from 0 by mathematical induction; otherwise, it is *infinite*. This paper was written between January and late June 1901 and appeared as a section III of Whitehead's paper "On Cardinal Numbers" which was published in the *American Journal of Mathematics* in October 1902 ([67]); it is Russell's first joint publication with Whitehead, although their collaboration began in early 1901 (see CP3, p. 422).

What Peano and his students and colleagues devised was a modernization, in symbolic form and in set-theoretic notation which they likewise devised, of an axiomatic system similar in kind to that which Euclid had presented. What was still lacking in Peano was an inference rule; therefore the project undertaken by Peano and his school remained an axiom system and never became a formal deductive system; see, *e.g.* van Heijenoort [64, p. 84], [65, p. 12]. (Borga and Palladino [13, pp. 27-28], however, take issue with this interpretation; they argue that the logical laws in Peano axiomatic system indeed do "play the rôle" of inference rules and they point to an explicit statement to that effect by Peano himself in 1894 (see [13, p. 27]), while admitting that in earlier papers, Peano indeed simply *listed* formulæ. They however ignore the fact that ultimately van Heijenoort agrees with them, asserting in the next passage [64, p. 84] that "[some] of Peano's explanations tend to suggest that his logical laws should perhaps be taken as rules of inference, not as formulas in a logical language," though he thinks that doing so would yield an incoherent interpretation of Peano's system.) That putative lacuna in Peano was not yet noticed by Russell in "Recent Italian Work on the Foundations of Mathematics". He closes his exposition by declaring (CP3, p. 362) that those "who care to know what deductive reasoning is, must henceforth master Peano's system, and read the works of himself and his disciples." When the breach in Peano's system was recognized, it was filled for Russell by Frege, whose *Begriffsschrift* of 1879 provided an inference rule — or, as he called it in van Heijenoort's [1967, 28] English translation, "transformation rule" — and constituted his system as a formal deductive system. Ironically, Russell seems to have first learned about Frege from Peano (see Kennedy [34, p. 368]; see also Nidditch [38, esp. p. 109]). Peano was, indeed, one of the comparatively small number among logicians of the day who, despite their scientific differences, seem to have fully appreciated Frege, as their cordial correspondence shows.

3. 1903-1905.

Russell was certainly aware of the need for inference rules. We already noted that he criticized Euclid for merely stringing statements together without providing sound logical arguments for passing from one to the next. He also declared in "Recent Work on the Principles of Mathematics" (CP3, pp. 366-367) that: "We start, in pure mathematics, from certain rules of inference, by which we infer that if one proposition is true, then so is some other proposition. These rules of inference constitute the major part of the principles of formal logic. . . . All pure mathematics — Arithmetic, Analysis, and Geometry — is built up by combinations of the primitive ideas of logic, and its propositions are deduced from general axioms of logic, such as the syllogism and the other rules of inference." If van Heijenoort is right about the absence of inference rules in Peano's work, we must ask whether Russell, at this stage — in January 1901 — is still, like Peano, apparently either failing to distinguish logical laws in an axiomatic system and inference rules, and thinks, as Borga and Palladino suggest that Peano did, that the logical laws of Peano's system "play the rôle" of inference rules, or if Russell was conveniently blurring the distinction because "Recent Work on the Principles of Mathematics" is intended as a frivolous piece for popular consumption. The Borga-Palladino interpretation would seem to be supported by the editors of CP4, who write in their "Introduction" that "[i]n 1901, Russell began writing the logical deduction of mathematics from logic." The editors of CP4 (p. xviii) tell us that for his part, Whitehead told Russell that the impression he has of the draft manuscripts that Russell was sending him for their collaboration on *Principia Mathematica* "is of very elaborate definitions which are not used, some proofs very careful, others equally important carried out by common sense in the style of Euclid." We can well imagine that such a comparison must have mortified Russell after he himself had so decisively challenged the logical rigor of Euclid (Paper 17, CP3 and [51, p. 5]).

Russell first read Frege's work in mid-June 1902, reading the *Begriffsschrift* and the *Grundgesetze* between the 17th and 19th, and continued to study Frege's works through the summer. Thereafter, the Fregean influence on Russell continued to increase perceptively, both in regard to technique and selection and theoretical conception of the logical primitives adopted in the final version of *Principia*, while the Peanesque influence decreased monotonically, without however vanishing entirely. By the middle of 1903, for example, Whitehead and Russell finally abandoned the Peanesque method of restricting the domain of antecedents in antecedents of conditionals and adopted Frege's universal domain when setting forth conditional definitions (see CP4, p. xviii). In his preface to the *Principles* six months later, Russell [1903, viii] gives priority of influence on his work to Cantor and Peano, adding that his debt to Frege in the *Principles* would have been greater had he discovered Frege's work sooner than he did. No inferences can be drawn from this assignment of credit on either the credibility or lack thereof of Grattan-Guinness's [25] assertion that the appendices in the *Principles* on Frege's work and on the Russell Paradox to be found therein, and the appendix presenting the theory of types as a means for overcoming the Russell Paradox, were not conceived merely as afterthoughts or ancillaries arisen from Russell's study of Frege, but are central and integral to the conception and focus of Russell's work in the *Principles* and the focal point of the *Principles* itself as a whole. That no such inferences can be drawn one way or another hinges on the indisputable centrality of Cantor's work on set theory in Russell's thought from as early as 1896-1897, and certainly in the period 1899-1902. There is good evidence that the Russell Paradox was conceived at the turn of the century, and that it first arose principally as a generalization of the paradox of the largest cardinal, certainly by mid-1901, if not sooner (see, *e.g.*, Garciadiego [24] and Anellis [7] for contrasting views, as well as Grattan-Guinness [25]; Anellis [7] also includes a summary of all of the main theories of the discovery of the Russell Paradox).

It was in Frege [21, esp. p. 366]; [22] that we first encounter Frege's criticism of Peano's lack of inference rules. We do not know precisely when, or even whether, Russell read either of these two pieces. What is certain is that Russell would have immediately noticed that for Frege the central inference rule was the rule of detachment, and that the rule of detachment plays an important rôle in the *Principles* [51, pp. 11-16], but not yet the central, singular rôle that it does in *Principia Mathematica*. We notice that many of the notes included as appendices in CP3, especially Appendix III, among the *nachgelassene* papers are Russell's efforts to sort out the meaning of implication in Peano and to distinguish between formal implication (*i.e.*, the rule of detachment) and material implication (the difference between these being characterized (see, *e.g.*, van Heijenoort [66, p. 115]) as belonging, in the case of the former to the metasystem, in the case of the latter to the system, of *Principia*). And we see that the rule of detachment, $[\vdash p \;\&\; \vdash (p \supset q)] \supset \vdash q$, is the very heart of the *Principia* system [68, p. 9]. We see very clearly the distinction in "Outlines of Symbolic Logic" (Paper 4, CP4, pp. 80-84), written in late June - early July 1904 for Couturat, where *1.1 in the list of "indefinables" for the logic of implication is: "$p \supset q. = .p$ implies q. This holds whenever q is true or p is not true. Thus in particular it holds if p is not a proposition, whatever q may be" and *2.1 in the list of "indemonstrables" is: "When p is true, and when $p \supset q$, q is true" (CP4, p. 80). In the *Principles*, the distinction between formal and material implication still remains quite Peanesque, where formal implication is characterized as holding between propositional functions when the antecedent implies the consequent for all values of the variable ([51, p. 14]). The difference between material and formal implication is characterized by Russell in the *Principles* ([51, p. 16]) as the difference between being denoted by *implies* in the case of material implication and by *if...then* in the case of formal implication. In the *Principles*, the closest Russell came to providing a rule of detachment was by listing implication as an axiom ([51, p. 16]). Thus, in the *Principles*, whatever Borga-Palladino and van Heijenoort have said about the absence of inference rules in Peano's system and about logical laws or axioms "playing the rôle" of inference rules holds with equal force in discussing the *Principles*, but not *Principia*. Russell's intense investigation of the concept of the conditional and work towards elaboration of the distinction between formal and material implication, between the inference rule of detachment and material implication, is especially significant, since, as Frege [21, p. 373] noted and van Heijenoort [64, p. 84] reiterated, the absence of the rule of detachment in Peano is due to an inadequate analysis or interpretation of the conditional.

Also certain is that the statement of the Russell Paradox in its now familiar form as presented in Russell's famous letter to Frege of 16 June 1902 stems directly from Frege's use in the *Grundgesetze* of functions as arguments for other, higher-order, functions. This shows firstly that Russell now regarded it as more than a paradox of set theory, but as a basic paradox of logic; it also indicates, secondly, that under the influence of Frege, Russell has moved away from the Peanesque notion of a "Logical Calculus" as a propositional calculus based upon class inclusion supplemented by a quantification theory in which variables range over individuals, such as numbers, and can appear within the scope of quantifiers (already a very far cry from his Bradleyan "pre-Peano" stage of 1899-1900 in which logic was regarded as concerning the relation of whole and part), towards Frege's full quantification theory, that is, first- and higher-order functional logic. In the paper "Sur la logique des relations" ([49]; CP3, Paper V.2), written in early October 1900 (see (CP3, p. 310), we find Russell's first explicit expression of dissatisfaction with Peano's treatment of functions. Here, the complaint is still however merely that Peano's notion of function needs to be defined in terms of relations in order to be fully comprehensible and useful (see, *e.g.*, CP3, p. 613). The final paper in CP3 (Appendix XI: "General Theory of Functions", pp. 687-690), an analysis, and search for a way out, of the Russell Paradox, was written between March 1902 and May 1903 and decisively shows the importance for Russell of Frege's influence and the new formulation, in terms of functions, of the Russell Paradox. Here, it is stressed (CP3, p. 690) that "[i]t is the interdependent variation of argument and concept which is dangerous." Here in this manuscript, moreover, we finally clearly see the familiar function-theoretic notation that one would expect in modern first- and higher-order functional logic.

The inclusion of the "General Theory of Functions" manuscript gives one very strong indication that Frege's work was clearly influencing Russell as he was in the final stages of the writing of the *Principles.* It also illustrates that the absence of the published book-length monographs from the collected writings is a serious handicap to the presentation of the evolution and development of Russell's work. This gap is only partially filled in for us by the editor, who notes in the headnote for Appendix XI (CP3, p. 686) that the manuscript presented in Appendix

XI was a direct ancestor of Russell's discussion in the *Principles* [51, p. 104] of "the independent variability of the function and the argument as a characteristic of legitimate propositional functions." The contrast between Appendix XI and the statement as it appears in the *Principles* decidedly shows the decisive and increasing influence of Frege and the corresponding decreasing influence of Peano. Before April 1902, Moore notes (CP3, p. 686), when Russell dealt with functions at all, they were regarded as special cases of relations and relations were regarded as fundamental. From the time of his discovery of the work of Frege, and certainly from mid-June 1902 onward, the propositional function became central for Russell and relations were derived from them. On 19 May 1903, Russell finally and decisively abandons classes in favor of Frege functions and a modification of Frege's course-of-values (*Wertverlauf*), seeking to define numbers without using classes (see CP4, p. xx). Russell's modification comes down to us as the range of a function. He worked thereafter on a function-theoretic solution to the Russell Paradox, and many of the manuscripts in CP4 are the tailings of this effort. The simple theory of types which Russell offered in his appendix to the *Principles* was deemed unsatisfactory.

The papers included in CP4 represent writings which Russell took up after completion of the *Principles* and in part which worked towards and contributed to the development of Russell's work as he, and very soon he and Whitehead together, labored towards the *Principia*. The presence of the editors' "Introduction" is really more crucial in CP4 than in CP3 because of the nature of so much of the material included in CP4, which consists in large measure of Russell's *nachgelassene* notes and drafts rather than complete or completed writings. These manuscripts, and indeed much of the material included in CP4, initially give the feeling — except for the valuable aid of the editors — of being a scattered miscellany, which the introduction helps tie together. The introduction points out, traces, and helps fill in the details of, the evolution of the salient features that dominate Russell's work through this critical period, for example the growing influence of Frege, the increasing sharpening of the conception of the conditional and the concomitant increasingly sharper differentiation between material implication and the rule of detachment Russell's increasing differentiation from Peano regarding the nature and scope of propositional functions, and the movement towards exposition of the theory of definite descriptions which in its best-known form makes its appearance in the famous paper "On Denoting" ([52]) at the end of the period covered in CP4 and which is a cornerstone of CP4 (Paper 16). They also discuss the way in which Peano's notation was modified and simplified for *Principia*.

The materials included in CP4 are illustrative of these developments in Russell's thinking; but the amount of material — despite the thickness of CP4 — is so comparatively thin, that the task of the editors in pointing out the details of the evolution and alterations of these in Russell's work and the correlative task of filling in the details of this history for the readers is quite essential, and generally speaking, is done both well and effectively. (I can think of only one glaringly problematic exception to this: at first, I had some misgivings with the statement by the editors (at CP4, p. xiii) that "[i]n 1901, Russell began writing the logical deduction of mathematics from logic." My discomfiture arises because it is not entirely clear from the context in which the statement is made whether the reference is to the *Principles* or the *Principia* and here speaks to the question, already adumbrated, of *when* Russell makes the shift from constructing an axiomatic system to constructing a formal deductive system.)

The manuscripts in Part I, "Early Foundational Work" in particular are notes and drafts for *Principia.* Many of the manuscript papers in the first three parts and Appendix I, on "Frege on the Contradiction" of CP4 exhibit the stages of Russell's excruciating efforts to deal with the Russell Paradox. In 1903, Russell worked on his function-theoretical approach to treating the Russell Paradox (CP4, Papers 1-3). In 1904, work along these lines continued, and were presented with the substitutional theory and, for most of 1904, with the zig-zag theory (CP4, Papers 4-10). The first glimmerings of the theory of definite descriptions first emerge in manuscripts of 1903 (CP4, Papers 11-15). The theory of denoting emerges at last as the theory of definite descriptions and finds its expression in the famous paper "On Denoting" [Russell 1905; CP4, Paper 16, pp. 415-427]. Of the remaining papers included in CP4, many (those in Parts IV and V in particular) were written as temporary respite from the labors on the Russell Paradox and on *Principia.* These include several book reviews, including a double review of two books on Leibniz's logic (Paper 24, CP4), one, *La logique de Leibniz d'après des documents inédits*, by Couturat, the other, *Leibniz' System in seinen wissenschaftlichen Grundlagen*, by the philosopher Ernst Cassierer (1874-1945). Russell finds the latter, operating from a neo-Kantian framework, to have a serious misunderstanding of Leibniz's work. Like Couturat, Cassierer understands that Leibniz's work in logic and the principles of mathematics is the source of his metaphysics; but Cassierer doesn't appreciate the value of symbolic logic, Russell avers (CP4, p. 551). It was the reading of the work of philosopher and psychologist Alexius von Meinong (1853-1920), a leading figure in the Graz school of psychologism, that helped bring Russell to his theory of definite descriptions as a means of avoiding entanglements with such Meinongian "objects of higher order" as round squares. The paper "Meinong's Theory of Complexes and Assumptions" (Paper 17, CP4) written in the first half of 1903 and published in *Mind* in 1904 and the review of the collection by Meinong and his students entitled *Untersuchungen zur Gegenstandstheorie und Psychologie* edited by Meinong (Paper 34, CP4) represent Russell's study of Meinong. In the former paper, Russell points out the necessity of distinguishing between logic and theory of knowledge. Also included are the papers written for the controversy with Hugh MacColl (1837-1909) (Papers 19, 20), who argued that the basic relation of logic is not class inclusion, but implication between propositions and who agreed with Russell that propositions are logically more fundamental than classes, but who, Russell thinks, conflates propositions and propositional functions (see Cavaliere [1996] for an account of MacColl's work), and two reviews of Poincaré's *Science and Hypothesis* (Papers 32, 33). The paper "On the Relation of Mathematics to Symbolic Logic" (Paper 23) published in 1905 in *Revue de métaphysique et de morale* at the instigation of Couturat (see CP4, p. 521), is a defense of logicism against the Kantianism of French philosopher Pierre Léon Boutroux (1880-1922), a nephew of Poincaré.

Van Heijenoort's account ([66, p. 111]) of the history of the writing of *Principia*, set forth without the benefit of the archival information and other documentary available to the editors of CP4, of the *Principia* as being originally conceived in December 1902 as the second volume of Russell's *Principles* turns out to be somewhat inaccurate and oversimplified. The earliest indication that we have of the collaboration between Whitehead and Russell, apart from the article on cardinal numbers published in the *American Journal of Mathematics* [67][Whitehead 1902] for which Russell wrote the third section (and which is included in CP3 as Paper 13) dates from 1901. The conception of producing a joint work arose in late 1900 (see CP4, p. xiv), when it became increasingly clear to both Russell and Whitehead that they were headed for a similar elaboration of very much the same project with the second volume that each planned for the *Principles* and the *Universal Algebra* respectively. Apparently it was Russell who first broached the subject of collaboration, for the editors tell us (CP4, pp. xiv-xv), "Russell enlisted Whitehead as a collaborator in writing his own Volume II; Whitehead put aside work on his book..." The first systematic work on the book, for which there is only fragmentary collateral evidence in the form of correspondence, dates from the latter half of 1902; the relevant manuscripts from this period, however, are apparently no longer extant (see CP4, p. xv). The first surviving systematic work was begun in early 1903, and is included in CP4. The change in title of the work on which they embarked as volume II of the *Principles* to *Principia Mathematica* and the decision to treat it as a separate book must have occurred some time in the summer of 1906 (see CP4, p. xv). Many of the early parts of the *Nachlaß* that comprise CP4 are the extant draft manuscripts of Russell's work in preparation for the *Principia*, in particular his efforts to work his way towards a solution, or at least a way out, of the Russell Paradox.

Having examined the papers written during the crucial period when Russell came under the influence of Peano, prepared his *Principles* and undertook his collaboration with Whitehead on *Principia*, we are once more brought to the question of Russell's mathematical competence. We have already noted that there has been more than a hint of Whitehead's dissatisfaction with some of Russell's work in his early drafts for *Principia* and an unfavorable comparison of some of Russell's work with the work of Euclid. Moreover, Peirce was extremely critical of Russell (and by misguided implication, of Whitehead), calling him a "blunderer" (see *e.g.*, Anellis [10, p. 286]; an elaboration of the details of Peirce's criticisms of Russell's *The Principles of Mathematics* will be found in Hawkins [29]). How does Russell's work in logic stand when closely examined? How much of the *Principia* is owed to Whitehead? More specifically, to what degree was it necessary for Whitehead to rescue Russell from mathematical mistakes? And were these rescue missions owing to Russell's incompetence, or simply to carelessness? Was Russell merely an astute slogger but unoriginal mathematician who could cleverly mimic, translate, and make minor adjustments and additions or corrections to the work of others, or was he capable of making significant original contributions on his own? Peirce, for example, suggested that Russell not only appropriated the work of others without giving them due credit but assumed credit for that work himself (see, *e.g.*, Anellis [10, p. 287]). It is unlikely that it will be possible to separate carelessness from incompetence, except in those instances where there is clear and incontrovertible evidence that Russell was able to correct careless errors without intervention. And we take must into account that even the best mathematicians make their share of mistakes. Let us recall that the author of the review in the 18 September 1903 edition of the London *Times* of the *Principles* [unsigned; but evidentally G. H. Hardy], thought that Russell in this book was able to "construct a logic of mathematics altogether in advance of any previous system; unless it be that of Professor Frege," while Couturat [20, p. 147] wrote of the *Principles* that it "*contribuant à éclaircir et à préciser* *les principes des Mathématiques*." The best that we can do in assessing Russell's competence is to attempt to determine if there is a *pattern* of mistakes in his work. Do the mistakes that occur when he is under the tutelage of a Whitehead get corrected whereas those which are made when he is on his own continue to stand, undetected and uncorrected? Russell himself in his public pronouncements on the subject, gave the impression, when discussing the work on *Principia*, of carelessness and haste (see Russell [55, p. 138], quoted by the editors of CP4 at p. xi). Is it disingenuity, or frankness or false modesty, when Russell privately admitted to Couturat, in a letter of 14 May 1903 (cited in CP4. p. xl), that Whitehead "has greater mathematical competence than I do, and knows how to develop theories whose technical difficulty is too great for me"? Perhaps the most reliable evidence we have is the entry of 27 January [1903?] in Russell's private diary (as quoted by Monk [37, p. 163]): "My work is second-rate . . . ," he wrote, in reference presumably to the *Principles*.

This question of Russell's mathematical competence in general and in particular of the level and extent of Russell's contribution and rôle in the writing of the *Principia* is raised by the editors of CP4 (pp. xxxviii-xli) in the context of the question of the nature of the collaboration between Whitehead and Russell and on the tendency in much of the literature to give the preponderance of credit for the writing of *Principia* to Russell (see CP4, pp. xxxiii-xli). An example of assignation of greater credit to Russell than to Whitehead is to be found in van Heijenoort's assertion ([66, p. 111]) that the *Principia* was "*d'un effort . . . de la part de Russell*" carried out merely "*avec l'aide de Whitehead*." The editors of CP4 admit (CP4, xxxviii) that "[i]t is clear from the manuscripts and the letters which have survived that Whitehead's expertise . . . made possible" the writing of *Principia*. They further admit (CP4, p. xxxviii) that Whitehead needed to coddle Russell — to give him both emotional support and almost continual reassurance regarding his technical capabilities. That being the case, we must therefore inquire how sincere Whitehead was when he wrote to Russell (in a letter of 28 September 1905; quoted at CP4, p. xl): "I disbelieve in your lack of technical skill . . .," or if he was seeking merely to assuage Russell's ego and pride. Speculations upon motivations and capabilities may seem to be either idle or counterproductive, in a case such as this when much of the correspondence and other relevant material that was in Whitehead's possession was destroyed upon his death in accordance with his instructions and when there seems to have been a deliberate effort on Russell's part to obfuscate the issue by admitting or feigning carelessness in some instances, pleading technical weakness in other instances (especially to Whitehead himself), and, more crucially, the apparent destruction of much of the putatively incriminating correspondence from Whitehead. What we do have is Russell's public disclosure ([55]) of the methodology employed in the preparation of the manuscript of *Principia*. This, we may presume, has greater reliability than any private admissions or "cover-ups" made while the *Principia* was a work in progress, even if Whitehead himself was no longer alive to either contest or verify the claims. The division of labors had Russell responsible for doing the bulk of the actual composition inasmuch as he was not burdened by teaching duties as was Whitehead. That aside, the chores were divided between Russell, whose primary responsibilities were with the more philosophical parts of *Principia* and Whitehead responsible in the main for the mathematical aspects. The primary exception here was with the section on series, for which Russell assumed the major responsibility. The methodology for writing the *Principia*, which Russell outlined in Russell [55] and which is familiar at least to Russell scholars, if not to historians of logic and philosophy at large, was as follows: when a first draft was written, each author would send it to his co-author for comments, corrections, revisions; these were then passed back to the original author, who would take the proposed changes under advisement for the preparation of the next draft. Thus there would be at least three stages to the writing of each part of the *Principia* and both Russell and Whitehead would have a hand in every part of the book at least once, if not twice. However, because many of the early drafts are no longer extant, it is more difficult in the case of the *Principia* than in the case of the *Principles* and the remainder of Russell's earlier pre-*Principia* writings to pinpoint any specifics.

Our task in determining the degree of Russell's technical competence must rely, in view of the paucity of materials extant and the scantiness of the materials so far published, largely upon two types of documentation, neither wholly reliable in these cases: (1) the surviving materials; and (2) a comparison between Russell's published output before (and insofar as possible, after) his collaboration with Whitehead. Of the surviving material, we must resort especially to comparisons between the private admissions and public pronunciations made by Russell himself, that is, to the private self-depreciations made by Russell to Whitehead and Couturat which survive, for example, compared with the more public confessions of mere "carelessness" and "haste", as in Russell [55]. And in the case of examining and comparing the work of Russell with Whitehead and the work of Russell without Whitehead, we must keep in mind that some, if not all, of the infelicities and missteps in the work are part of Russell's evolution and the development of his thought, and owe as much to the fact that he is at work in developing a new science, or at least at work in a developing science within which other researchers are also still groping or "feeling their way." The work of *Principia* can, after all, be understood as the unification and systematization of all of the work in logic from Boole to Peano and Frege, and the reconciliation of the work of Boole, Peirce and Schröder, Grassmann, [pre-1900] Whitehead, and their colleagues, with the work of Peano, Frege and [pre-1902] Russell. And this is an immense and difficult undertaking. But this must also be balanced against the other side of the same coin, that whether in the *Principles*, on his own or in the *Principia* with the assistance — and under the direction — of Whitehead, Russell is copying and synthesizing the work of others, in a sense (and clearly radically oversimplifying the historical details of the developments) merely putting a new coat of paint and some planks and plaster on an edifice that Peirce, Schröder, Peano, and Frege had already built, or at least otherwise nearly completed, in a sense (again clearly radically oversimplifying the historical details of the developments), building a connecting hallway between two edifices previously constructed, the one by Boole, Peirce, Schröder, Grassmann, Whitehead and their colleagues, the other by Peano and Frege. Therefore, after examining the work which Russell carried out which one finds in the *Principles*, in the items included in CP3 and CP4 and in the *Principia* and comparing and contrasting these materials with Russell's work from 1896 to the end of 1900, we are still left with the nagging — and ultimately, probably unanswerable — questions: How did Russell become such a brilliant logician after falling under the tutelage of Whitehead and the influences of Peano and Frege respectively? Why *really* did Russell to all practical purposes give up work in logic after completing the work on *Principia* and without the collaboration with Whitehead? How well, *really*, could Russell have managed on his own to produce significant work in logic without the collaboration and intervention of a Whitehead? Could the brilliant logical technician who co-authored the *Principia* really have thought that Gödel's incompleteness results means that in school-boy arithmetic, $2 + 2 = 4.001$, that is, that "school-boy arithmetic" is *inconsistent* rather than incomplete — even after all he had ostensibly learned while working on the *Principia*?[3] There is just the remaining powerful insinuation of the

[3]On 26 March 1963, Leon Henkin [31] wrote to Russell, accompanying the letter with a copy of his paper [30], which traced the history of logic from the time of *Principia* forward, from the context of the work of *Principia* and his thesis that logic is a branch of mathematics, not, as the logicists, in particular Russell, would assert, that logic is all of mathematics. The exposition presented a discussion of Gödel's

existence of something grossly incongruent between the Russell of the mid-1890s and the mid-1910s to the 1960s on the one had and the Russell of *Principles* to *Principia* period on the other. The mathematical ineptitude or confusion of the one appears, despite — or even because of — everything we have seen in CP3 and CP4 to belie the mathematical brilliance of the other. Depending upon the extent of one's generosity towards Russell, they who do not wish to ascribe the sudden brilliance of the *Principles*-to-*Principia* Russell to the assistance of Whitehead may describe the *Principles*-to-*Principia* Russell either as a "flash in the pan" in contrast with the pre-*Principles*-and-post-*Principia* Russell, or as one whose brilliant meteoric rise was inspired and stimulated by Couturat, Peano, Frege and (especially) Whitehead but was soon burned out by the immense drain of mathematical energy of working on the *Principles* and especially on the *Principia*. Perhaps future volumes of the *Collected Papers* will supply the materials that allow us to select a cogent answer. It may yet prove possible to revise and rescue our opinion of Russell.

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incompleteness results and work on the decision problem that resulted. Russell [1963] replied to Henkin in a letter of 1 April 1963 and from his comments about his reading of Gödel appears to have concluded that that Gödel's results showed that the axioms [of *Principia*] must be inconsistent, if they lead to a contradiction, and therefore that at least one of the axioms must be false. He then asks Henkin whether these conclusions also apply to "school-boy arithmetic" and, if so, whether this should lead us to think that 2 + 2 equals, not 4, but 4.001. Reading these remarks, Henkin [1963a] replied to Russell at length with an explanation of Gödel's incompleteness results, in a letter of 17 July 1963, specifically explaining that Gödel showed, not the inconsistency, but the incompleteness, of the [*Principia*] system.

In 1982, Henkin [1982], clearly mindful of the date on which Russell had replied to his initial letter, asked me whether Russell might have been joking when he seemed to have concluded that Gödel's results proved inconsistency rather than completeness. I responded that I did not think that Russell was joking, given the entire tenor of Russell's reply and in part inasmuch as Russell was employing an example he had learned from Bradley. (See Anellis [10, p. 11].)

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