Uniformly convex operators and martingale type

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Abstract

The concept of uniform convexity of a Banach space was generalized to linear operators between Banach spaces and studied by Beauzamy [1]. Under this generalization, a Banach space X is uniformly convex if and only if its identity map I_X is. Pisier showed that uniformly convex Banach spaces have martingale type p for some $p > 1$. We show that this fact is in general not true for linear operators. To remedy the situation, we introduce the new concept of martingale subtype and show, that it is equivalent, also in the operator case, to the existence of an equivalent uniformly convex norm on X. In the case of identity maps it is also equivalent to having martingale type p for some $p > 1$.

Our main method is to use sequences of ideal norms defined on the class of all linear operators and to study the factorization of the finite summation operators. There is a certain analogy with the theory of Rademacher type.

1. Introduction

Banach spaces admitting an equivalent uniformly convex norm enjoy several equivalent characterizations. Among others, they are the superreflexive Banach spaces, i. e. not only is such a Banach space X reflexive, but every Banach space whose finite dimensional subspaces can be found (uniformly) in X is reflexive.

A connection with martingales was studied by Pisier [14]. For $1 < p \leq 2$, a Banach space X has martingale type p if there exists a constant $c \geq 0$

such that

$$\Big(\int_0^1 \Big\| \sum_{k=1}^n d_k(t)\Big\|^p dt\Big)^{1/p} \leq c\Big(\sum_{k=1}^n \int_0^1 \|d_k(t)\|^p dt\Big)^{1/p}$$

for all X-valued martingale difference sequences $d_1, \ldots, d_n$.

For our purpose the fundamental result of Pisier's paper [14, Thm. 3.2, p. 340] combined with James's and Enflo's investigations [10, Thm. 4, p. 903], [6, p. 281] can be summarized as follows. (See further down for detailed definitions.)

Theorem 1. *For a Banach space X the following properties are equivalent:*

  (i) *X has martingale type p for some $p > 1$,*
 (ii) *X is superreflexive,*
(iii) *X admits an equivalent uniformly convex norm,*
 (iv) *X does not factor the finite summation operators Σ_n uniformly.*

If one is interested in linear operators between Banach spaces instead of just Banach spaces, the above theorem is no longer true, as can be seen by examples (see Example 3 at the end). The aim of this paper is to prove an operator-theoretic replacement of Theorem 1; see Theorem 2 in Section 5.

The main difficulties arising in this setting are the lack of a suitable substitute of J-convexity in the operator case (see Beauzamy [2, p. 265] for a definition of J-convexity) and the fact that the submultiplicativity of the martingale type ideal norms can no longer be exploited.

The following general method has turned out to be useful to generalize Banach space-theoretic results to results about operators.

Given a sequence of parameters $(\boldsymbol{\alpha}_n)$ associating with every operator $T : X \to Y$ a sequence of non-negative numbers $(\boldsymbol{\alpha}_n(T))$, let $\boldsymbol{\alpha}_n(\infty) := \sup \boldsymbol{\alpha}_n(T)$, where the supremum is taken over all operators $T : X \to Y$ of norm 1 and all Banach spaces X and Y. Then the sequence $(\boldsymbol{\alpha}_n(\infty))$ describes the 'worst' behavior that can occur for an operator T. In the Banach space case, one is mostly interested in the boundedness of the sequence $(\boldsymbol{\alpha}_n(I_X))$ of the identity map of a Banach space X, i. e. $\boldsymbol{\alpha}_n(I_X) = O(1)$. In the operator case, the behavior $\boldsymbol{\alpha}_n(T)/\boldsymbol{\alpha}_n(\infty) \to 0$, i. e. $\boldsymbol{\alpha}_n(T) = o(\boldsymbol{\alpha}_n(\infty))$ is much more useful. Another beautiful example for this heuristic in the context of Rademacher and Gauss type is given by Hinrichs in [9].

In particular, taking as $\boldsymbol{\alpha}_n(T)$ the martingale type ideal norm $\boldsymbol{\tau}_n(T)$ formed with n martingale differences, the condition $\boldsymbol{\tau}_n(T) = o(\sqrt{n})$ will be the right replacement for Condition (i) in Theorem 1.

Let us quickly review the contents of this article. In Section 2 we introduce martingale and Haar (co)type ideal norms, which are close relatives. The main result of this section is that all the o-conditions described above for these four types of ideal norms yield equivalent properties. In Section 3 we establish the connection of the martingale type ideal norms with the factorization of the finite summation operators. To do so, we use a variant of the martingale type ideal norms, namely the equal norm martingale type ideal norms. In Section 4 we repeat the definitions of uniform convexity and uniform smoothness of linear operators, introduce the super weakly compact operators and give various characterizations due to Beauzamy [1] connecting the two concepts. Since the main emphasis of this article is on the connection with martingales, we are rather brief here and give mostly references for the proofs. Finally in Section 5 we formulate and prove our main theorem and provide an example of an operator, for which Theorem 1 is false.

To finish this introduction, let us point out some notational conventions used throughout. We write B_X for the unit ball of a Banach space X and I_X for its identity map. Furthermore, the reader has already realized, that we use Landau's big-O and little-o notation.

2. Martingale type and cotype

First of all, we introduce the martingale type and cotype ideal norms. They were first considered by Pisier in [14, Rem. 3.3, p. 346]. We also consider Haar type and cotype ideal norms as restrictions of martingale type and cotype ideal norms to special classes of martingales. An operator is said to have the corresponding subtype or subcotype, if these sequences of ideal norms behave just a little better than the worst case. The main result of this section will be that all four possible subtype and subcotype properties coincide.

Let us start by considering an arbitrary martingale (f_k) defined on $[0, 1)$ with values in a Banach space X and adapted to a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$ of finitely generated σ-algebras. We denote by E_k the operator of conditional expectation with respect to the σ-algebra $\mathcal{F}_k$.

In particular, taking for $\mathcal{F}_k$ the σ-algebra generated by the dyadic intervals

$$\Delta_k^{(j)} := \left[\frac{j-1}{2^k}, \frac{j}{2^k} \right),$$

we obtain the so called *Walsh-Paley* or *dyadic* martingales. Since the corresponding martingales f_n are just linear combinations of the Haar functions

$\chi_k^{(j)}$ for $k = 0, \ldots, n$ and $j = 1, \ldots, 2^{k-1}$, we also use the term *Haar polynomials* to denote the functions f_n in this case. Remember, that the Haar functions $\chi_k^{(j)}$ are defined by

$$\chi_k^{(j)}(t) := \begin{cases} +2^{(k-1)/2} & \text{if } t \in \Delta_k^{(2j-1)}, \\ -2^{(k-1)/2} & \text{if } t \in \Delta_k^{(2j)}, \\ 0 & \text{otherwise.} \end{cases}$$

We let $\chi_0^{(0)} \equiv 1$.

We will mainly deal with the sequence of martingale differences (d_k) instead of (f_k), where $d_1 = f_1$ and $d_k = f_k - f_{k-1}$ for $k > 1$. The Banach space of square integrable X-valued functions on $[0, 1)$ is denoted by $[L_2, X]$. For $f \in [L_2, X]$ we write

$$\|f|L_2\| := \left(\int_0^1 \|f(t)\|^2 \, dt \right)^{1/2}.$$

The fact that $\|E_k f|L_2\| \leq \|f|L_2\|$ for all $f \in [L_2, X]$ and $k \in \mathbb{N}$ will be frequently used.

With each operator $T : X \to Y$ we associate the operator $[L_2, T] : [L_2, X] \to [L_2, Y]$, defined by

$$[L_2, T]f(t) := T\big(f(t)\big).$$

For any two functions $f \in [L_2, X]$ and $g' \in [L_2, Y']$ we write

$$\big\langle [L_2, T]f, g' \big\rangle := \int_0^1 \big\langle Tf(t), g'(t) \big\rangle \, dt.$$

Definition. For $T : X \to Y$, the n-th *martingale type ideal norm* $\boldsymbol{\tau}(T|\mathcal{M}_n)$ is the smallest constant $c \geq 0$ such that

$$\Big\| \sum_{k=1}^n [L_2, T]d_k \Big| L_2 \Big\| \leq c \Big(\sum_{k=1}^n \|d_k|L_2\|^2 \Big)^{1/2}$$

for all X-valued martingale difference sequences $d_1, \ldots, d_n$ adapted to any filtration on $[0, 1)$.

The n-th *martingale cotype ideal norm* $\boldsymbol{\gamma}(T|\mathcal{M}_n)$ is the smallest constant $c \geq 0$ such that

$$\Big(\sum_{k=1}^n \big\| [L_2, T]d_k \big| L_2 \big\|^2 \Big)^{1/2} \leq c \Big\| \sum_{k=1}^n d_k \Big| L_2 \Big\|$$

for all X-valued martingale difference sequences $d_1, \ldots, d_n$ adapted to any filtration on $[0, 1)$.

Note that, for norm one operators, both sequences, $(\boldsymbol{\tau}(T|\mathcal{M}_n))$ and $(\boldsymbol{\gamma}(T|\mathcal{M}_n))$, are bounded by $2\sqrt{n}$ and therefore the following definition makes sense.

Definition. An operator $T : X \to Y$ has *martingale subtype* or *martingale subcotype* if $\boldsymbol{\tau}(T|\mathcal{M}_n)/\sqrt{n} \to 0$ or $\boldsymbol{\gamma}(T|\mathcal{M}_n)/\sqrt{n} \to 0$, respectively.

For convenience we introduce the following notation for the *dyadic trees:*

$$\mathbb{D}_m^n := \{(k,j) \ : \ k = m, \ldots, n; \ j = 1, \ldots, 2^{k-1}\} \quad \text{for } 1 \le m \le n.$$

We let $\mathbb{D}_0^n := \mathbb{D}_1^n \cup \{(0,0)\}$.

Definition. For $T : X \to Y$, the *Haar type ideal norm* $\boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_m^n))$ associated with the index set $\mathbb{D}_m^n$ is the smallest constant $c \ge 0$ such that

$$\Big\| \sum_{\mathbb{D}_m^n} T x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\| \le c \Big(\sum_{\mathbb{D}_m^n} \|x_k^{(j)}\|^2 \Big)^{1/2}$$

for all $(x_k^{(j)}) \subseteq X$.

The *Haar cotype ideal norm* $\boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_m^n))$ associated with the index set $\mathbb{D}_m^n$ is the smallest constant $c \ge 0$ such that

$$\Big(\sum_{\mathbb{D}_m^n} \|T x_k^{(j)}\|^2 \Big)^{1/2} \le c \Big\| \sum_{\mathbb{D}_m^n} x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\|$$

for all $(x_k^{(j)}) \subseteq X$.

As before, we will be interested in the suboptimal behavior of these sequences.

Definition. An operator $T : X \to Y$ has *Haar subtype* or *Haar subcotype* if $\boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_0^n))/\sqrt{n} \to 0$ or $\boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n))/\sqrt{n} \to 0$, respectively.

It is easily verified that

$$\boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_1^n)) \le \boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_0^n)) \le 2\,\boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_1^n)),$$
$$\boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_1^n)) \le \boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n)) \le 3\,\boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_1^n)).$$

Since for any $(x_k^{(j)}) \subseteq X$ the sequence of functions $d_k := \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)}$ forms a sequence of martingale differences, we obviously have

$$(2.1) \qquad \boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_1^n)) \le \boldsymbol{\tau}(T|\mathcal{M}_n) \quad \text{and} \quad \boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_1^n)) \le \boldsymbol{\gamma}(T|\mathcal{M}_n).$$

No reverse estimate is known. In the limiting case where one of these sequences of ideal norms behaves like $o(\sqrt{n})$ we have, however, equivalence.

Proposition 1. *For any operator $T : X \to Y$ the following properties are equivalent:*

  (i) *T has martingale subtype,*

 (ii) *T has martingale subcotype,*

(iii) *T has Haar subtype,*

(iv) *T has Haar subcotype.*

We postpone the proof of Proposition 1 in order to provide some prerequisites. The main idea of the proof is contained in the following lemma.

Lemma 1. *Let $T : X \to Y$, then we have*

$$\frac{\boldsymbol{\tau}(T|\mathcal{M}_{2^n})}{2^{n/2}} \le 3\,\frac{\boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n))}{n^{1/2}}.$$

Proof. Given any sequence $(d_1, \ldots, d_{2^n})$ of X-valued martingale differences, define a function $F_n : [0, 1) \times [0, 1) \to X$ by $F_n(s, t) := d_i(t)$ if $s \in \Delta_n^{(i)}$.

Obviously

$$\int_0^1 \int_0^1 \|F_n(s, t)\|^2 \, dt \, ds = \sum_{i=1}^{2^n} \int_{\Delta_n^{(i)}} \int_0^1 \|F_n(s, t)\|^2 \, dt \, ds = \frac{1}{2^n} \sum_{i=1}^{2^n} \|d_i | L_2\|^2.$$

Viewing $F_n(\,\cdot\,, t)$ as a Haar polynomial, we have

$$F_n(s, t) = \sum_{\mathbb{D}_0^n} x_k^{(j)}(t)\,\chi_k^{(j)}(s)$$

where the functions $x_k^{(j)}$ are defined by $x_k^{(j)}(t) := \int_0^1 F_n(s, t)\,\chi_k^{(j)}(s)\,ds$. By the definition of $\boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n))$ we have

$$\sum_{\mathbb{D}_0^n} \|Tx_k^{(j)}(t)\|^2 \le \boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n))^2 \left\| \sum_{\mathbb{D}_0^n} x_k^{(j)}(t)\,\chi_k^{(j)} \Big| L_2 \right\|^2.$$

Integration with respect to $t \in [0, 1)$ yields

$$(2.2) \qquad \sum_{\mathbb{D}_0^n} \|[L_2, T]x_k^{(j)} | L_2\|^2 \le \boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n))^2 \, \|F_n | L_2\|^2$$

$$= \boldsymbol{\gamma}(T|\mathcal{H}(\mathbb{D}_0^n))^2 \, \frac{1}{2^n} \sum_{i=1}^{2^n} \|d_i | L_2\|^2.$$

Letting $\mathbb{N}_k^{(j)} := \{i \; : \; \Delta_n^{(i)} \subseteq \Delta_k^{(j)}\}$ and observing that $\mathbb{N}_{k-1}^{(j)} = \mathbb{N}_k^{(2j-1)} \cup \mathbb{N}_k^{(2j)}$, we obtain for $x_k^{(j)}$

$$x_k^{(j)}(t) = \int_0^1 F_n(s,t)\,\chi_k^{(j)}(s)\,ds = 2^{(k-1)/2}\left(\int_{\Delta_k^{(2j-1)}} F_n(s,t)\,ds - \int_{\Delta_k^{(2j)}} F_n(s,t)\,ds\right)$$

$$= 2^{(k-1)/2-n}\left(\sum_{\mathbb{N}_k^{(2j-1)}} d_i(t) - \sum_{\mathbb{N}_k^{(2j)}} d_i(t)\right).$$

This implies that

$$(2.3) \qquad \|[L_2,T]x_k^{(j)}|L_2\| = 2^{(k-1)/2-n}\left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i - \sum_{\mathbb{N}_k^{(2j)}}[L_2,T]d_i\,\Big|L_2\right\|.$$

Since the conditional expectation operator has norm one in $[L_2,X]$ it now follows that

$$\left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i\,\Big|L_2\right\| \leq \left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i - \sum_{\mathbb{N}_k^{(2j)}}[L_2,T]d_i\,\Big|L_2\right\|$$

and therefore by the triangle inequality

$$\left\|\sum_{\mathbb{N}_{k-1}^{(j)}}[L_2,T]d_i\,\Big|L_2\right\| \leq \left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i - \sum_{\mathbb{N}_k^{(2j)}}[L_2,T]d_i\,\Big|L_2\right\| + 2\left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i\,\Big|L_2\right\|$$

$$(2.4) \qquad \leq 3\left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i - \sum_{\mathbb{N}_k^{(2j)}}[L_2,T]d_i\,\Big|L_2\right\|.$$

Using (2.2), (2.3) and (2.4) the proof completes as follows:

$$\left\|\sum_{i=1}^{2^n}[L_2,T]d_i\,\Big|L_2\right\| = \left(\frac{1}{n}\sum_{k=1}^n\left\|\sum_{j=1}^{2^{k-1}}\sum_{\mathbb{N}_{k-1}^{(j)}}[L_2,T]d_i\,\Big|L_2\right\|^2\right)^{1/2}$$

$$\leq \left(\frac{1}{n}\sum_{k=1}^n\left(\sum_{j=1}^{2^{k-1}}\left\|\sum_{\mathbb{N}_{k-1}^{(j)}}[L_2,T]d_i\,\Big|L_2\right\|\right)^2\right)^{1/2}$$

$$\leq 3\left(\frac{1}{n}\sum_{k=1}^n 2^{k-1}\sum_{j=1}^{2^{k-1}}\left\|\sum_{\mathbb{N}_k^{(2j-1)}}[L_2,T]d_i - \sum_{\mathbb{N}_k^{(2j)}}[L_2,T]d_i\,\Big|L_2\right\|^2\right)^{1/2}$$

$$\leq 3\left(\frac{2^{2n}}{n}\sum_{k=1}^n\sum_{j=1}^{2^{k-1}}\|[L_2,T]x_k^{(j)}|L_2\|^2\right)^{1/2} \leq 3\,\gamma(T|\mathcal{H}(\mathbb{D}_0^n))\left(\frac{2^n}{n}\sum_{i=1}^{2^n}\|d_i|L_2\|^2\right)^{1/2}.$$

$\blacksquare$

We next observe that the martingale and Haar type and cotype ideal norms are dual to each other.

Proposition 2. *For* $0 \le m < n$ *we have*

$$\gamma(T \,|\mathcal{H}(\mathbb{D}_m^n)) \le \tau(T'|\mathcal{H}(\mathbb{D}_m^n)) \le 2\,\gamma(T \,|\mathcal{H}(\mathbb{D}_m^n)),$$
$$\gamma(T'|\mathcal{H}(\mathbb{D}_m^n)) \le \tau(T \,|\mathcal{H}(\mathbb{D}_m^n)) \le 2\,\gamma(T'|\mathcal{H}(\mathbb{D}_m^n)).$$

If $m = 0$, *we can omit the factors* 2 *and have equality. On the other hand*

$$\gamma(T \,|\mathcal{M}_n) \le 2\,\tau(T'|\mathcal{M}_n) \le 4\,\gamma(T \,|\mathcal{M}_n),$$
$$\gamma(T'|\mathcal{M}_n) \le 2\,\tau(T \,|\mathcal{M}_n) \le 4\,\gamma(T'|\mathcal{M}_n).$$

Proof. The proof can be obtained using standard duality techniques and is left to the reader. ∎

We can now prove Proposition 1.

Proof. It follows from Lemma 1 and (2.1) that

$$\frac{\tau(T|\mathcal{M}_{2^n})}{2^{n/2}} \le 3\,\frac{\gamma(T|\mathcal{H}(\mathbb{D}_0^n))}{n^{1/2}} \le 6\,\frac{\gamma(T|\mathcal{M}_n))}{n^{1/2}}.$$

Since the same is true for T', it follows from Proposition 2 that

$$\frac{\gamma(T|\mathcal{M}_{2^n})}{2^{n/2}} \le 6\,\frac{\tau(T|\mathcal{H}(\mathbb{D}_0^n))}{n^{1/2}} \le 12\,\frac{\tau(T|\mathcal{M}_n))}{n^{1/2}}.$$

Hence if one of these quotients tends to zero, all the others tend to zero too, which proves the proposition by virtue of the monotonicity of the involved ideal norms. ∎

3. Equal norm martingale type

Definition. For $T : X \to Y$, the n-th *equal norm martingale type ideal norm* $\tau^\circ(T|\mathcal{M}_n)$ is the smallest constant $c \ge 0$ such that

$$\Big\| \sum_{k=1}^n [L_2, T]d_k \Big| L_2 \Big\| \le c \Big(\sum_{k=1}^n \|d_k|L_2\|^2 \Big)^{1/2}$$

for all X-valued martingale difference sequences $d_1, \ldots, d_n$ adapted to any filtration on $[0, 1)$ under the additional assumption that $\|d_1|L_2\| = \cdots = \|d_n|L_2\|$.

The quantities $\tau^\circ(T|\mathcal{M}_n)$ can also be defined in a different way.

Lemma 2. *For $T : X \to Y$, the ideal norm $\boldsymbol{\tau}^{\circ}(T|\mathcal{M}_n)$ is the smallest constant $c \geq 0$ such that*

$$(3.1) \qquad \Big\| \sum_{k=1}^{n} [L_2, T]d_k \Big| L_2 \Big\| \leq c\, n^{1/2} \sup_{k=1,\dots,n} \|d_k|L_2\|$$

for all X-valued martingale difference sequences $d_1, \dots, d_n$ adapted to any filtration on $[0,1)$.

Proof. For the time being denote by $\boldsymbol{\tau}^{\circ\circ}(T|\mathcal{M}_n)$ the smallest constant such that (3.1) holds. It is obvious that for $\|d_1|L_2\| = \cdots = \|d_n|L_2\|$

$$\Big\| \sum_{k=1}^{n} [L_2, T]d_k \Big| L_2 \Big\| \leq \boldsymbol{\tau}^{\circ\circ}(T|\mathcal{M}_n)\, n^{1/2} \sup_{k=1,\dots,n} \|d_k|L_2\|^2$$

$$= \boldsymbol{\tau}^{\circ\circ}(T|\mathcal{M}_n) \Big(\sum_{k=1}^{n} \|d_k|L_2\|^2 \Big)^{1/2}.$$

Therefore $\boldsymbol{\tau}^{\circ}(T|\mathcal{M}_n) \leq \boldsymbol{\tau}^{\circ\circ}(T|\mathcal{M}_n)$.

On the other hand, let $d_1, \dots, d_n$ be an arbitrary sequence of martingale differences. It follows that for $\tilde{d}_k := d_k/\|d_k|L_2\|$

$$\Big\| \sum_{k=1}^{n} [L_2, T]\tilde{d}_k \Big| L_2 \Big\| \leq \boldsymbol{\tau}^{\circ}(T|\mathcal{M}_n)\, n^{1/2}.$$

But the same is true for $\zeta_k \tilde{d}_k$ instead of $\tilde{d}_k$, where $|\zeta_k| = 1$. An extreme point argument then yields that

$$\Big\| \sum_{k=1}^{n} [L_2, T]\alpha_k \tilde{d}_k \Big| L_2 \Big\| \leq \boldsymbol{\tau}^{\circ}(T|\mathcal{M}_n)\, n^{1/2}$$

whenever $|\alpha_k| \leq 1$. In particular, we may take

$$\alpha_k := \frac{\|d_k|L_2\|}{\displaystyle\sup_{h=1,\dots,n} \|d_h|L_2\|},$$

which shows that $\boldsymbol{\tau}^{\circ\circ}(T|\mathcal{M}_n) \leq \boldsymbol{\tau}^{\circ}(T|\mathcal{M}_n)$. ■

Obviously we have

$$\boldsymbol{\tau}^{\circ}(T|\mathcal{M}_n) \leq \boldsymbol{\tau}(T|\mathcal{M}_n).$$

The main purpose of this section is to prove a reverse estimate. We follow an approach similar to Bourgain/Kalton/Tzafriri in [3, Thm. 3.1., p. 160] where

they show that equal norm Rademacher type 2 is equivalent to ordinary Rademacher type 2. The main idea is contained in the following construction of 'glueing' together m copies of a given martingale of length n, which yields a martingale of length mn with smaller differences. An appropriate blocking of this longer martingale will then give a martingale of length of order n and nearly equal L_2-norms.

Let $\phi_j^m : [\frac{j-1}{m}, \frac{j}{m}) \to [0, 1)$ be defined by $\phi_j^m(t) := mt - j + 1$. Given a function $f : [0, 1) \to X$, we denote by $\Phi_j^m f$ the function

$$\Phi_j^m f(t) := \begin{cases} f(\phi_j^m(t)) & \text{if } t \in [\frac{j-1}{m}, \frac{j}{m}), \\ 0 & \text{otherwise.} \end{cases}$$

Given a sequence of martingale differences (d_k), the sequence of martingale differences

$$\Phi_1^m d_1, \ldots, \Phi_m^m d_1, \Phi_1^m d_2, \ldots, \Phi_m^m d_2, \ldots, \Phi_1^m d_n, \ldots, \Phi_m^m d_n$$

is adapted to the filtration

$$\Phi_1^m \mathcal{F}_1, \ldots, \Phi_m^m \mathcal{F}_1, \Phi_1^m \mathcal{F}_2, \ldots, \Phi_m^m \mathcal{F}_2, \ldots, \Phi_1^m \mathcal{F}_n, \ldots, \Phi_m^m \mathcal{F}_n,$$

where $\Phi_j^m \mathcal{F}_k$ is the σ-algebra generated by all sets $A \subseteq [0, 1)$ such that $\phi_j^m(A) \in \mathcal{F}_k$ and by all its predecessor σ-algebras.

Observe that
(3.2)
$$\Big\| \sum_{k=1}^{n} \sum_{j=1}^{m} \Phi_j^m d_k \Big| L_2 \Big\| = \Big\| \sum_{k=1}^{n} d_k \Big| L_2 \Big\| \quad \text{and} \quad \|\Phi_j^m d_k | L_2\|^2 = \frac{1}{m} \|d_k | L_2\|^2.$$

Moreover, all differences $\Phi_j^m d_k$ in any block of length at most m have disjoint support.

Lemma 3. *The sequence $(\boldsymbol{\tau}^\circ(T|\mathcal{M}_n))$ is non-decreasing.*

Proof. Let $d_1, \ldots, d_n$ be X-valued martingale differences such that $\|d_k | L_2\| = 1$. For $m := n + 1$ the construction above yields a martingale difference sequence of length $n(n + 1)$. Define a new sequence of martingale differences $\tilde{d}_1, \ldots, \tilde{d}_{n+1}$ by blocking n consecutive terms:

$$\tilde{d}_h := \Phi_{n-h+3}^m d_{h-1} + \cdots + \Phi_{n+1}^m d_{h-1} + \Phi_1^m d_h + \ldots \Phi_{n-h+1}^m d_h.$$

Since $\|\tilde{d}_h | L_2\| = \sqrt{\frac{n}{n+1}}$ it follows that

$$\Big\| \sum_{k=1}^{n} [L_2, T] d_k \Big| L_2 \Big\| = \Big\| \sum_{h=1}^{n+1} [L_2, T] \tilde{d}_h \Big| L_2 \Big\| \leq \boldsymbol{\tau}^\circ(T|\mathcal{M}_{n+1}) \Big(\sum_{k=1}^{n} \|d_k | L_2\|^2 \Big)^{1/2},$$

which proves that $\boldsymbol{\tau}^\circ(T|\mathcal{M}_n) \leq \boldsymbol{\tau}^\circ(T|\mathcal{M}_{n+1})$. ∎

Lemma 4. $\tau(T|\mathcal{M}_n) \leq 16\,\tau^\circ(T|\mathcal{M}_n)$.

Proof. Let $d_1, \ldots, d_n$ be X-valued martingale differences. By scaling we may assume that $\sum_{k=1}^{n} \|d_k|L_2\|^2 = 1$.

Let l be such that $4^l \leq 16\,n < 4^{l+1}$ and $m := 4^l$. For $h = 1, 2 \ldots$ define

$$\mathbb{F}_h := \left\{ k \;:\; \frac{1}{2^h} < \|d_k|L_2\| \leq \frac{2}{2^h} \right\} \quad \text{and} \quad \mathbb{F} := \bigcup_{h=1}^{l} \mathbb{F}_h.$$

First of all, we estimate the sum of all the differences with small norm:

$$\left\| \sum_{k \notin \mathbb{F}} [L_2, T] d_k \Big| L_2 \right\| \leq \tau(T|\mathcal{M}_n) \left(\sum_{k \notin \mathbb{F}} \|d_k|L_2\|^2 \right)^{1/2}$$

(3.3)
$$\leq \tau(T|\mathcal{M}_n)\,(n4^{-l})^{1/2} \leq \frac{1}{2}\tau(T|\mathcal{M}_n).$$

Here we used, that the sequence $(\tau(T|\mathcal{M}_n))$ is obviously non-decreasing and that $16n < 4^{l+1}$.

For the martingale difference sequence (d_k) with $k \in \mathbb{F}$ we apply the glueing technique described above. Then it follows from (3.2) that for $k \in \mathbb{F}_h$

$$\frac{1}{m4^h} < \|\Phi_j^m d_k|L_2\|^2 = \frac{1}{m}\|d_k|L_2\|^2 \leq \frac{4}{m4^h}.$$

Therefore by disjointness, for any subset $\mathbb{L} \subseteq \{1, \ldots, m\}$ of cardinality $|\mathbb{L}| = 4^h$

$$\frac{1}{m} < \left\| \sum_{i \in \mathbb{L}} \Phi_i^m d_k \Big| L_2 \right\|^2 \leq \frac{4}{m}.$$

Writing $\mathbb{L}_j^{(h)} := \{4^h\,(j-1) + 1, \ldots 4^h j\}$ and $\tilde{d}_j^k := \sum_{i \in \mathbb{L}_j^{(h)}} \Phi_i^m d_k$ we obtain a martingale of length

$$N := \sum_{h=1}^{l} \sum_{k \in \mathbb{F}_h} 4^{l-h} \leq 4^l \sum_{h=1}^{l} \sum_{k \in \mathbb{F}_h} \|d_k|L_2\|^2 \leq 4^l \leq 16n.$$

It follows from Lemma 3 that $\tau^\circ(T|\mathcal{M}_N) \leq \tau^\circ(T|\mathcal{M}_{16n})$ and therefore, we obtain from Lemma 2 that

$$\left\| \sum_{h=1}^{l} \sum_{k \in \mathbb{F}_h} \sum_{j=1}^{4^{l-h}} [L_2, T]\tilde{d}_j^k \Big| L_2 \right\| \leq \tau^\circ(T|\mathcal{M}_N) \sup \|\tilde{d}_j^k|L_2\| \sqrt{N}$$

$$\leq \tau^\circ(T|\mathcal{M}_{16n}) \frac{2}{\sqrt{m}} 2^l = 2\,\tau^\circ(T|\mathcal{M}_{16n}).$$

This shows that

(3.4) $$\Big\| \sum_{k \in \mathbb{F}} [L_2, T] d_k \Big| L_2 \Big\| \le 2\, \boldsymbol{\tau}^\circ(T | \mathcal{M}_{16n}).$$

Putting together (3.3) and (3.4) we obtain

$$\boldsymbol{\tau}(T | \mathcal{M}_n) \le \frac{1}{2} \boldsymbol{\tau}(T | \mathcal{M}_n) + 2\, \boldsymbol{\tau}^\circ(T | \mathcal{M}_{16n}),$$

which implies $\boldsymbol{\tau}(T | \mathcal{M}_n) \le 4\, \boldsymbol{\tau}^\circ(T | \mathcal{M}_{16n})$. Finally the assertion follows from the trivial fact that $\boldsymbol{\tau}^\circ(T | \mathcal{M}_{16n}) \le 4\, \boldsymbol{\tau}^\circ(T | \mathcal{M}_n)$. ∎

The *summation operator* $\Sigma : l_1 \to l_\infty$ is defined by

$$\Sigma(\xi_k) := \Big(\sum_{h=1}^{k} \xi_h \Big),$$

while the *finite summation operators* $\Sigma_n : l_1^n \to l_\infty^n$ act between the finite dimensional spaces and are defined in the same way.

The significance of the ideal norms $\boldsymbol{\tau}^\circ(T | \mathcal{M}_n)$ is due to the following fact, which establishes the connection with the factorization of the finite summation operators.

Proposition 3. *There exists a factorization $\Sigma_n = B_n[L_2, T]A_n$ of the finite summation operator Σ_n, such that $\|B_n\| \, \|A_n\| \le 6\sqrt{n}/\boldsymbol{\tau}^\circ(T | \mathcal{M}_{2n})$.*

Proof. There is nothing to prove for $T = 0$. For $T \ne 0$, by definition, for all $0 < \delta < 1$, there exists a sequence of martingale differences $d_1, \ldots, d_{2n}$ such that $\|d_k | L_2\| = 1$ and

$$\Big\| \sum_{k=1}^{2n} [L_2, T] d_k \Big| L_2 \Big\| > \delta\, \boldsymbol{\tau}^\circ(T | \mathcal{M}_{2n}) \sqrt{2n}.$$

Choose $g' \in [L_2, Y']$ such that $\|g' | L_2\| = 1$ and

$$\Big\langle \sum_{k=1}^{2n} [L_2, T] d_k, g' \Big\rangle > \delta\, \boldsymbol{\tau}^\circ(T | \mathcal{M}_{2n}) \sqrt{2n}.$$

Let

$$\mathbb{F} := \Big\{ k \ : \ \langle [L_2, T] d_k, g' \rangle > \boldsymbol{\tau}^\circ(T | \mathcal{M}_{2n}) \frac{\delta}{4\sqrt{2n}} \Big\}.$$

It follows from Lemma 3 that

$$\Big\langle \sum_{k\in\mathbb{F}}[L_2,T]d_k, g'\Big\rangle \le \Big\|\sum_{k\in\mathbb{F}}[L_2,T]d_k\Big|L_2\Big\|\, \|g'|L_2\| \le \boldsymbol{\tau}^\circ(T|\mathcal{M}_{2n})|\mathbb{F}|^{1/2}$$

and therefore

$$\delta\,\boldsymbol{\tau}^\circ(T|\mathcal{M}_{2n})\sqrt{2n} < \Big\langle \sum_{k\in\mathbb{F}}[L_2,T]d_k, g'\Big\rangle + \sum_{k\notin\mathbb{F}}\Big\langle [L_2,T]d_k, g'\Big\rangle$$

$$\le \boldsymbol{\tau}^\circ(T|\mathcal{M}_{2n})|\mathbb{F}|^{1/2} + 2n\,\boldsymbol{\tau}^\circ(T|\mathcal{M}_{2n})\frac{\delta}{4\sqrt{2n}}.$$

This shows that $m := |\mathbb{F}| \ge \delta^2\, 9n/8$. Choosing δ appropriately, we may arrange that $m \ge n$. In particular, we find elements $i_1 < \cdots < i_n$ in $\mathbb{F}$.

We can now define $A_n : l_1^n \to [L_2, X]$ by

$$A_n e_k := \frac{d_{i_k}}{\big\langle [L_2,T]d_{i_k}, g'\big\rangle}$$

and $B_n : [L_2, Y] \to l_\infty^n$ by

$$B_n f := \big(\, \langle f, E_{i_k}g'\rangle\,\big)_{k=1}^n. \qquad\blacksquare$$

4. Uniform convexity and smoothness and super weakly compact operators

We now show, how the concepts above connect to the theory of weakly compact operators.

First of all, we repeat the classical definitions of uniform convexity and uniform smoothness of Banach spaces in the more general case of linear operators between Banach spaces. See e. g. Beauzamy [1, Def. 7, p. 121].

Definition. An operator $T : X \to Y$ is *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

for $\|x_\pm\| = 1$ with $\dfrac{\|x_+ + x_-\|}{2} \ge 1 - \delta$ it follows that $\dfrac{\|Tx_+ - Tx_-\|}{2} \le \varepsilon$.

The operator T is *uniformly smooth* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

for $\|y\| = 1$ and $\|x\| \le \delta$ it follows that $\dfrac{\|y + Tx\| + \|y - Tx\|}{2} \le 1 + \varepsilon\,\|x\|$.

Using an equivalent norm on the target space Y does not spoil uniform convexity. This is, however, not the case for the source space X. Similarly, uniform smoothness depends on the special choice of the norm on the target space. We are therefore rather interested in uniformly convex renormable and uniformly smooth renormable operators.

Definition. An operator $T : X \to Y$ is *uniformly convex renormable* if there exists an equivalent norm $||| \cdot |||$ on X such that $T : [X, ||| \cdot |||] \to Y$ is uniformly convex.

An operator $T : X \to Y$ is *uniformly smooth renormable* if there exists an equivalent norm $||| \cdot |||$ on Y such that $T : X \to [Y, ||| \cdot |||]$ is uniformly smooth.

The above properties are equivalent to T factoring through a uniformly convex or uniformly smooth Banach space, respectively; see [13, Prop. 7.10.11].

As in the case of Banach spaces, we can easily show that the concepts of uniform convexity and smoothness are dual to each other. See Lindenstrauss [11, Thm. 1] for a proof in the Banach space case, which can easily be carried over to linear operators.

Proposition 4. *An operator $T : X \to Y$ is uniformly convex if and only if its dual $T' : Y' \to X'$ is uniformly smooth.*

The operator T is uniformly smooth if and only if T' is uniformly convex.

Next we introduce super weakly compact operators, whose Banach space counterparts are the superreflexive Banach spaces, and show their connection with the factorization of the summation operators. Again, this basically follows from the, by now, classical proof that superreflexive Banach spaces do not factor the summation operators uniformly, cf. James [10], Beauzamy [2, Prop. 7, p. 236], or Heinrich [7, Thm. 5.1, p. 29].

Definition. An operator $T : X \to Y$ is *weakly compact*, if the image of the closed unit ball of X under T is relatively weakly compact in Y.

For the theory of ultraproducts of Banach spaces and linear operators, we refer to Heinrich's papers [7, 8]. We mainly use the notation of Pietsch/ Wenzel [13].

Definition. An operator $T : X \to Y$ is *super weakly compact*, if all its ultrapowers $T^{\mathcal{U}} : X^{\mathcal{U}} \to Y^{\mathcal{U}}$ are weakly compact.

The summation operator is the typical non-weakly compact operator.

Example 1. The summation operator Σ is not weakly compact.

Definition. An operator $T : X \to Y$ is said to *factor the summation operator* Σ, if there exist operators $A : l_1 \to X$ and $B : Y \to l_\infty$ such that $\Sigma = BTA$.

The operator T is said to *factor the finite summation operators Σ_n uniformly*, if there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$ we can find factorizations $\Sigma_n = B_n T A_n$ such that $\|B_n\| \, \|A_n\| \le c$.

The next proposition connects the above concepts.

Proposition 5. *Let $T : X \to Y$, then the following properties are equivalent:*

(i) *T is super weakly compact,*

(ii) *$T^{\mathcal{U}}$ does not factor the summation operator Σ for any ultrafilter $\mathcal{U}$,*

(iii) *T does not factor the finite summation operators Σ_n uniformly.*

Proof. The equivalence of (i) and (ii) is due to Lindenstrauss/Pełczyński [12].

Assume that $T : X \to Y$ factors the finite summation operators uniformly and let $\mathcal{U}$ be any non-trivial ultrafilter on $\mathbb{N}$.

Let J be the canonical map from l_1 into $l_1^{\mathcal{U}}$ induced by the map $x \mapsto (x, x, \ldots)$. Note moreover, that the map $(x_n) \mapsto \text{w}^*\text{-}\lim_{\mathcal{U}} x_n$ induces a well defined operator Q from $l_\infty^{\mathcal{U}}$ onto l_∞. The last fact is due to the weak-$*$-compactness of the closed unit ball of l_∞. Obviously $\|J\| = \|Q\| = 1$.

By the assumption, there exists a constant c, such that we find for all $n \in \mathbb{N}$ operators $A_n : l_1^n \to X$ and $B_n : Y \to l_\infty^n$ satisfying $\|A_n\| \le 1$, $\|B_n\| \le c$, and $\Sigma_n = B_n T A_n$.

Denote furthermore by J_n the canonical embedding of l_∞^n into l_∞ and by Q_n the projection from l_1 onto l_1^n.

We obtain a factorization of the summation operator Σ via $T^{\mathcal{U}}$ as

$$\Sigma = Q(J_n \Sigma_n Q_n)^{\mathcal{U}} J = Q(J_n B_n)^{\mathcal{U}} T^{\mathcal{U}} (A_n Q_n)^{\mathcal{U}} J.$$

This proves that (ii) $\Rightarrow$ (iii).

If on the other hand (ii) does not hold, then $T^{\mathcal{U}}$ factors Σ for some ultrafilter $\mathcal{U}$. In the sense of Heinrich [7, Def. 1.1, p. 7], $T^{\mathcal{U}}$ is finitely representable in T, i. e. for all $\varepsilon > 0$ and each finite dimensional subspace $\widehat{X}_0 \subseteq X^{\mathcal{U}}$ and finite codimensional subspace $\widehat{Y}_0 \subseteq Y^{\mathcal{U}}$ there are a finite dimensional subspace $X_0 \subseteq X$ and a finite codimensional subspace $Y_0 \subseteq Y$ and maps $R : \widehat{X}_0 \to X_0$ and $S : Y/Y_0 \to Y^{\mathcal{U}}/\widehat{Y}_0$ such that

$$\left\| \widehat{Q} T^{\mathcal{U}} \widehat{J} - SQTJR \right\| \le \varepsilon,$$

where J and $\widehat{J}$ are the canonical embedding maps of X_0 and $\widehat{X}_0$ and Q and $\widehat{Q}$ are the canonical quotient maps of Y_0 and $\widehat{Y}_0$.

It follows, that for all $n \in \mathbb{N}$ there is a subspace $X_0 \subseteq X$ and a subspace $Y_0 \subseteq Y$ and operators $R : l_1^n \to X_0$ and $S : Y/Y_0 \to l_\infty^n$, such that

$$\left\| \Sigma_n - SQTJR \right\| \leq \varepsilon.$$

Moreover, since Σ_n is injective, by the remark following Definition 1.1 in [7, p. 8], we can even arrange that $\varepsilon = 0$. This implies that T factors Σ_n uniformly. ∎

It was shown in Beauzamy [1, Thm. I.1, p. 111] that uniformly convex renormability and super weak compactness are in fact equivalent properties. See also Heinrich's paper [7, Thm. 5,p. 29] and the detailed presentation in Pietsch/Wenzel [13, Sect. 7.6].

One obtains the following equivalences.

Proposition 6. *Let $T : X \to Y$, then the following properties are equivalent:*

(i) *T is super weakly compact,*

(ii) *T is uniformly convex renormable,*

(iii) *T does not factor the finite summation operators Σ_n uniformly.*

5. Main theorem

Before we can formulate and prove the main theorem some more preparations are required.

The following classical result provides the missing link between the operators T and $[L_2, T]$. It was first proved by Day [5, Thm. 2, p. 504] for Banach spaces, but Day's proof can straightforwardly be extended to the operator case.

Proposition 7. *The operator T is uniformly convex if and only if $[L_2, T]$ is uniformly convex.*

Finally, we will need the Haar cotype ideal norms of the finite summation operators.

Example 2. $\frac{1}{2}(n+1)^{1/2} \leq \boldsymbol{\gamma}_n(\Sigma_{2^n}|\mathcal{H}(\mathbb{D}_0^n)) \leq (n+1)^{1/2}.$

Proof. Obviously, for any operator $T : X \to Y$

$$\Big\| \sum_{\mathbb{D}_0^n} T x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\| \leq (n+1)^{1/2} \Big(\sum_{k=0}^{n} \Big\| \sum_{j=1}^{2^{k-1}} T x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\|^2 \Big)^{1/2}$$

$$= (n+1)^{1/2} \Big(\sum_{k=0}^{n} \sum_{j=1}^{2^{k-1}} \| T x_k^{(j)} \|^2 \| \chi_k^{(j)} | L_2 \|^2 \Big)^{1/2}$$

$$\leq (n+1)^{1/2} \| T \| \Big(\sum_{\mathbb{D}_0^n} \| x_k^{(j)} \|^2 \Big)^{1/2},$$

where we have used that for fixed k the Haar functions $\chi_k^{(1)}, \ldots, \chi_k^{(2^{k-1})}$ have disjoint support. This proves the upper estimate by virtue of Proposition 2.

To see the lower estimate, let $f : [0,1) \to l_1^{2^n}$ be defined by

$$f(t) := e_i \quad \text{if } t \in \Delta_n^{(i)},$$

where e_i is the i-th unit vector in $l_1^{2^n}$. Write f as a Haar polynomial

$$f = \sum_{\mathbb{D}_0^n} x_k^{(j)} \chi_k^{(j)}, \qquad \text{where} \quad x_k^{(j)} = \int_0^1 f(t) \chi_k^{(j)}(t)\, dt.$$

Obviously

$$\| f | L_2 \| = \Big\| \sum_{\mathbb{D}_0^n} x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\| = \Big(\frac{1}{2^n} \sum_{i=1}^{2^n} \| e_i \|^2 \Big)^{1/2} = 1.$$

Writing $\mathbb{N}_k^{(j)} := \{ i \ : \ \Delta_n^{(i)} \subseteq \Delta_k^{(j)} \}$, it follows that

$$x_k^{(j)} = \frac{1}{2^n} 2^{(k-1)/2} \big(0, \ldots, 0 \underbrace{+1, \ldots, +1}_{\mathbb{N}_k^{(2j-1)}}, \underbrace{-1, \ldots, -1}_{\mathbb{N}_k^{(2j)}}, 0, \ldots, 0 \big)$$

and

$$\Sigma_{2^n} x_k^{(j)} = \frac{1}{2^n} 2^{(k-1)/2} \big(0, \ldots, 0 \underbrace{1, 2, \ldots, 2^{n-k}}_{\mathbb{N}_k^{(2j-1)}}, \underbrace{2^{n-k}-1, \ldots, 1, 0}_{\mathbb{N}_k^{(2j)}}, 0, \ldots, 0 \big),$$

which in turn yields $\| \Sigma_{2^n} x_k^{(j)} \| = 2^{-(k+1)/2}$. Hence

$$\Big(\sum_{\mathbb{D}_0^n} \| \Sigma_{2^n} x_k^{(j)} \|^2 \Big)^{1/2} = \frac{1}{2} (n+1)^{1/2},$$

which proves the assertion. $\blacksquare$

We are now ready to state our most important result.

Theorem 2. *For an operator $T : X \to Y$ the following properties are equivalent:*

(i) *T has martingale subtype,*

(ii) *T has martingale subcotype,*

(iii) *T has Haar subtype,*

(iv) *T has Haar subcotype,*

(v) *T is super weakly compact,*

(vi) *T does not factor the finite summation operators Σ_n uniformly,*

(vii) *T is uniformly convex renormable,*

(viii) *$[L_2, T]$ is uniformly convex renormable,*

(ix) *T is uniformly smooth renormable.*

Proof The equivalence of (i)–(iv) was shown in Proposition 1. The equivalence of (v)–(vii) was shown in Proposition 6. Proposition 7 shows that (vii) and (viii) are equivalent, hence all the properties (v)–(vii) for T and (v)–(vii) for $[L_2, T]$ are equivalent.

Assume that (i) does not hold. Then for some $c > 0$ there are infinitely many numbers n, such that

$$\frac{\boldsymbol{\tau}(T | \mathcal{M}_n)}{\sqrt{n}} \geq c.$$

It follows from Lemma 4 that for these numbers

$$\frac{6\sqrt{n}}{\boldsymbol{\tau}^{\circ}(T | \mathcal{M}_{2n})} \leq \frac{6\sqrt{n}}{\boldsymbol{\tau}(T | \mathcal{M}_{2n})} \leq \frac{6}{c}$$

and therefore, by Proposition 3 there exist factorizations $\Sigma_n = B_n [L_2, T] A_n$ such that $\|B_n\| \, \|A_n\| \leq 6/c$. Of course, this yields such factorizations for *all* $n \in \mathbb{N}$ and therefore (vi) cannot hold for $[L_2, T]$.

Hence we have shown that (vi) implies (i). On the other hand Example 2 shows that (iv) implies (vi).

So far, we have shown that (i)–(viii) are equivalent. Finally, since by Proposition 2 properties (i)–(iv) for T are equivalent to the same properties for T', it follows from Proposition 4 that also (ix) is equivalent to (vii) and hence to all other properties. ∎

Remark. Note that Pisier's Theorem 1 can easily be obtained from our Theorem 2 using the submultiplicativity of the ideal norms $\boldsymbol{\tau}(I_X | \mathcal{M}_n)$. Namely,

it follows from $\boldsymbol{\tau}(I_X|\mathcal{M}_n) = o(n^{1/2})$ and the submultiplicativity, that there is a constant c and a number $p_0 > 1$ such that

$$\boldsymbol{\tau}(I_X|\mathcal{M}_n) \leq cn^{1/p_0-1/2} \quad \text{for all } n \in \mathbb{N}.$$

By Pisier [14], this implies that X has martingale type p for all $p < p_0$.

To formulate the last example, we will need one more definition.

Definition. For $T : X \to Y$, the *Haar type p ideal norm* $\boldsymbol{\tau}_p(T|\mathcal{H}(\mathbb{D}_1^n))$, is the smallest constant $c \geq 0$ such that

$$\Big\| \sum_{\mathbb{D}_1^n} [L_2, T] x_k^{(j)} \chi_k^{(j)} \Big| L_p \Big\| \leq c \Big(\sum_{k=1}^n \Big\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \Big| L_p \Big\|^p \Big)^{1/p}$$

for all $(x_k^{(j)}) \subseteq X$.

In particular, $\boldsymbol{\tau}_2(T|\mathcal{H}(\mathbb{D}_1^n)) = \boldsymbol{\tau}(T|\mathcal{H}(\mathbb{D}_1^n))$. An operator T is said to have martingale type p if the sequence $(\boldsymbol{\tau}_p(T|\mathcal{H}(\mathbb{D}_1^n)))$ is bounded.

The following example shows that an analogue of the Davis–Figiel–Johnson–Pełczyński Theorem on weakly compact operators (they factor through a reflexive Banach space; see [4, Cor. 1, p. 314]) cannot hold for super weakly compact operators, since if T factors through a superreflexive Banach space then it follows already that it has martingale type p for some $p > 1$; see [14, Thm. 3.2, p. 340].

Let $t = (\tau_n)$ be a non-increasing sequence of positive numbers. We consider the diagonal operator $D_t : l_1 \to l_1$ defined by $D_t(\xi_k) := (\tau_k \xi_k)$.

The following fact is proved in [15].

Example 3. $\quad \boldsymbol{\tau}_p(D_t|\mathcal{H}(\mathbb{D}_1^n)) = \Big(\sum_{k=1}^n |\tau_k|^{p'} \Big)^{1/p'}.$

Corollary. *If $\tau_k = 1/(1+\log k)$ then the operator D_t is super weakly compact but does not have martingale type p for any $p > 1$ and hence does not factor through a superreflexive Banach space.*

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