RESEARCH

Real Analysis Exchange
Vol. 40(2), 2014/2015, pp. 403–420

Arlene Ash, University of Massachusetts Medical School, Worcester, MA 01605. email: arlene.ash@umassmed.edu

J. Marshall Ash, Department of Mathematics, DePaul University, Chicago, IL 60614. email: mash@math.depaul.edu

Stefan Catoiu, Department of Mathematics, DePaul University, Chicago, IL 60614. email: scatoiu@math.depaul.edu

NEW DEFINITIONS OF CONTINUITY

Abstract

We classify all generalized $\mathcal{A}$-differences of any order $n \geq 0$ for which $\mathcal{A}$-continuity at x implies ordinary continuity at x. We show that the only $\mathcal{A}$-continuities that are equivalent to ordinary continuity at x correspond to the limits of the form

$$\lim_{h \to 0} A\left[f(x+rh) + f(x-rh) - 2f(x)\right] + B\left[f(x+sh) - f(x-sh)\right],$$

with $ABrs \neq 0$. All other $\mathcal{A}$-continuities truly generalize ordinary continuity.

1 Introduction

A real-valued function f is *continuous* at the real number x if

$$f(x+h) - f(x) = o(1) \text{ as } h \to 0. \tag{1.1}$$

More generally, given a set of $2m$ parameters $\mathcal{A} = \{A_1, \ldots, A_m; a_1, \ldots, a_m\}$, where $m \geq 2$, the first m satisfy $\sum_{i=1}^{m} A_i = 0$ and the second m are distinct, we say that the function f is $\mathcal{A}$-*continuous* at x if

$$\Delta_{\mathcal{A}} f(x,h) := \sum_{i=1}^{m} A_i f(x + a_i h) = o\left(1\right) \text{ as } h \to 0. \tag{1.2}$$

Mathematical Reviews subject classification: Primary: 26A24; Secondary: 26A27
Key words: continuity, generalized continuity, $\mathcal{A}$-continuity, derivatives, generalized derivatives, $\mathcal{A}$-derivatives
Received by the editors March 16, 2015
Communicated by: Brian S. Thomson

Call such kind of continuity an *m-point $\mathcal{A}$-continuity.* Setting $\mathcal{A} = \{1, -1; 1, 0\}$ shows that ordinary continuity is a special case of two point $\mathcal{A}$-continuity. The transformation $A_i \to \lambda A_i$ for all i for a non-zero λ does not disturb relation (1.2); neither does $a_i \to \mu a_i$ for all i for a non-zero μ. In particular, any $\mathcal{A} = \{\lambda, -\lambda; \mu, 0\}$ with $\lambda\mu \neq 0$ also defines ordinary continuity.

If f is continuous at x, then for any $\mathcal{A}$,

$$\begin{aligned}\sum_{i=1}^{m} A_i f(x + a_i h) &= \sum_{i=1}^{m} A_i \left(f(x + a_i h) - f(x)\right) \\ &= \sum_{i=1}^{m} A_i o(1) = o(1)\end{aligned} \tag{1.3}$$

so that f is also $\mathcal{A}$-continuous at x.

The converse of this is false in general. For example, if f is *any* even function, then

$$f(0 + h) - f(0 - h) = 0 = o(1) \text{ as } h \to 0.$$

Thus all even functions are $\mathcal{A} = \{1, -1; 1, -1\}$-continuous at $x = 0$. But of course many even functions are discontinuous at $x = 0$, so $\{1, -1; 1, -1\}$-continuity is a two point ($m = 2$) kind of A-continuity that does not characterize ordinary continuity.

For simplicity of the notation and without loss of generality, assume $x = 0$. On the other hand, suppose $Ar \neq 0$ and f is $\{A, A, 1, -1, -2A; r, -r, 1, -1, 0\}$-continuous at 0, so that as $h \to 0$,

$$Af(rh) + Af(-rh) + f(h) - f(-h) - 2Af(0) = o(1). \tag{1.4}$$

Successively replace h by $-h$, $\dfrac{h}{r}$ and $-\dfrac{h}{r}$ to see that additionally

$$\begin{aligned}Af(-rh) + Af(rh) + f(-h) - f(h) - 2Af(0) &= o(1) \\ Af(h) + Af(-h) + f(r^{-1}h) - f(-r^{-1}h) - 2Af(0) &= o(1) \\ Af(-h) + Af(h) + f(-r^{-1}h) - f(r^{-1}h) - 2Af(0) &= o(1).\end{aligned}$$

Multiply these four relations respectively by $1, -1, \dfrac{1}{A}$ and $\dfrac{1}{A}$, and add. We get

$$
\begin{aligned}
&(A+(-A))\, f\,(rh) + (A+(-A))\, f\,(-rh) + \\
&\left(1-(-1)+\frac{1}{A}A+\frac{1}{A}A\right) f\,(h) + \left(-1+(-1)+\frac{1}{A}A+\frac{1}{A}A\right) f\,(-h) + \\
&\left(\frac{1}{A}+\frac{1}{A}\,(-1)\right) f\left(r^{-1}h\right) + \left(\frac{1}{A}\,(-1)+\frac{1}{A}\right) f\left(-r^{-1}h\right) + \\
&\left(-2A-(-2A)+\frac{1}{A}\,(-2A)+\frac{1}{A}\,(-2A)\right) f\,(0) \\
&= 4\,(f\,(h)-f\,(0)) = o\,(1)\,.
\end{aligned}
$$

So when $|r| \neq 1$ we have examples of $m = 5$ point $\mathcal{A}$-continuity that are not generalizations of ordinary continuity. They are instead characterizations of ordinary continuity; i.e., they are actually nothing more than alternative definitions of ordinary continuity!

Similarly, when $r = 1$ or $r = -1$, relations (1.4) collapse to the $m = 3$ point $\mathcal{A}$-continuity conditions

$$(A+1)\, f(h) + (A-1)\, f(-h) - 2Af(0) = o\,(1)\,. \tag{1.5}$$

We may summarize our findings to this point in the following proposition.

Proposition 1. *The following three cases of $\mathcal{A}$-continuity conditions provide equivalent definitions of ordinary continuity.*

(i) The two parameter family of $m = 5$ point $\mathcal{A}$-continuity conditions $\mathcal{A} = \{A, A, 1, -1, -2A; r, -r, 1, -1, 0\}$ indexed by the real parameters A and r with $Ar \neq 0$ and $|r| \neq 1$;

(ii) the one parameter family of $m = 3$ point $\mathcal{A}$-continuity conditions $\mathcal{A} = \{A+1, A-1, -2A; 1, -1, 0\}$ indexed by the real parameter $A \notin \{0, 1\}$;

(iii) the ordinary $m = 2$ point continuity condition $\mathcal{A} = \{1, -1; 1, 0\}$.

The purpose of this paper is to classify all potential definitions of generalized $\mathcal{A}$-continuity into those that give equivalent definitions of continuity and those that are true generalizations of continuity. This classification is given by the following theorem.

Theorem 1. A *The m point $\mathcal{A}$-continuity conditions*

$$\mathcal{A} = \{A_1, \ldots, A_m; a_1, \ldots, a_m\}$$

which are dilates ($A_i \to \lambda A_i$ for all i for a non-zero λ and/or $a_i \to \mu a_i$ for all i for a non-zero μ) of one of the $\mathcal{A}$-continuity conditions listed in the proposition are equivalent to ordinary continuity.

B *Given any other $\mathcal{A}$-continuity condition, there is a measurable function $f(x)$ such that f is $\mathcal{A}$-continuous at $x = 0$, but not continuous at $x = 0$. Hence $\mathcal{A}$-continuity is a generalization of continuity.*

This is the working result that we will be proving in this article. A more concise version of it is the following:

Theorem 2. *The following are all the generalized continuity definitions that are equivalent to the ordinary definition of continuity of a function f at x.*

$$\lim_{h \to 0} A[f(x + rh) + f(x - rh) - 2f(x)] + B[f(x + sh) - f(x - sh)] = 0,$$

where $ABrs \neq 0$.

In the last section, we give a wide generalization of Theorem 2 which finds for any generalized continuity the set of all generalized continuities that are equivalent to it.

2 Proof of the theorem reduced to two lemmas

Since the dilations mentioned in part A of Theorem 1 applied to any form of $\mathcal{A}$-continuity have no effect on the defining condition, part A is only a restatement of the already derived proposition.

Turning to part B, first notice that χ, the characteristic function of the set $\{0\}$, defined by

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases},$$

is not continuous at $x = 0$, but is $\mathcal{A}$-continuous at $x = 0$ whenever all $a_i \neq 0$, since then $\sum A_i f(a_i h) = \sum A_i \cdot 0 = 0 = o(1)$. Thus all such $\mathcal{A}$-continuities are generalizations of continuity, so we may restrict our considerations to $\mathcal{A}$-continuity where one $a_i = 0$; from now on we assume that $a_m = 0$ and $A_m \neq 0$. Since $\sum_{i=1}^{m} A_i = 0$ and $A_m \neq 0$, we have

$$\sum_{i=1}^{m-1} A_i \neq 0. \tag{2.1}$$

Second, if $g(x)$ is any function discontinuous at zero, then $f(x) = g(x) - g(0)$ is also discontinuous at $x = 0$ and also satisfies $f(0) = 0$. So we only need to study functions which are 0 at $x = 0$.

If there exists a measurable function $f(x)$ such that

$$f(0) = 0, f(s_n) = 1 \text{ for some sequence } \{s_n\} \to 0 \tag{2.2}$$

and such that

$$\sum_{i=1}^{m-1} A_i f(a_i h) = 0 \text{ for all } h \neq 0, \tag{2.3}$$

then f is, of course, $\mathcal{A}$-continuous at $x = 0$, but not continuous at $x = 0$.

For the remainder of this paper, we will always represent $f(x+h) - f(x)$ as $f(h)$ and also write $\Delta_{\mathcal{A}} f(x,h)$ as $\Delta_{\mathcal{A}} f(h)$. With this convention, the definition of ordinary continuity becomes $f(h) \to 0$. Furthermore, given an $\mathcal{A}$-continuity $\{\{A_i\};\{a_i\}\}$, the statement that there exist constants $\{R_j\}$ and $\{r_j\}$ such that for all $h \neq 0$

$$\sum_{j=1}^{n} R_j \left(\sum_{i=1}^{m-1} A_i f(r_j a_i h) \right) = f(h)$$

means that there exist constants $\{R_j\}$ and $\{r_j\}$ such that for all $h \neq 0$ and for every function f and every point x,

$$\sum_{j=1}^{n} R_j \left(\sum_{i=1}^{m-1} A_i \left(f(x + r_j a_i h) - f(x) \right) \right) = f(x+h) - f(x).$$

Consider the following dichotomy. Either there is a finite linear combination of dilates $h \to r_j h$ of $\sum_{i=1}^{m-1} A_i f(a_i h)$ equal to $f(h)$ for every $h \neq 0$ or there is not. This reduces the proof of the Theorem to the following two lemmas.

Lemma 1. *If no finite linear combination of dilates of $\sum_{i=1}^{m-1} A_i f(a_i h)$ equals $f(h)$ for all $h \neq 0$, then there is a function f and a sequence $\{s_n\}$ such that conditions (2.2) and (2.3) hold.*

Lemma 2. *If there is a finite linear combination of dilates of $\sum_{i=1}^{m-1} A_i f(a_i h)$ equal to $f(h)$ for all $h \neq 0$, then $\mathcal{A}$ is necessarily one of the exceptional $\mathcal{A}$-continuities listed in the proposition.*

Remark 1. *Our goal was to find all $\mathcal{A}$-continuities that are equivalent to ordinary continuity. Imagine that the only way to prove that a given $\mathcal{A}$-continuity* *implies ordinary continuity was the direct method of finding that a linear combination of its dilates was identically equal to $f(h)$. In this case, the very simple proof that ordinary continuity implies every $\mathcal{A}$-continuity (see (1.3)), the above mentioned dichotomy and Lemma 2, taken together, would already give the entire answer. Lemma 1 means that whenever the direct method fails, a counterexample must exist, so that **all** methods fail.*

3 Proof of Lemma 1

Here we essentially reproduce the proof of Lemma 2 of [3]. This second exposition of a delicate counterexample may help some readers.

We are given an $\mathcal{A}$-continuity given by

$$\mathcal{A} = \{A_1, \ldots, A_{m-1}, A_m; a_1, \ldots, a_{m-1}, 0\},$$

and our goal is to create a function $f : \mathbb{R} \to \mathbb{R}$ whose values $\{f(h)\}$ satisfy an infinite system of equations S; e.g., one of these equations is $\sum_{i=1}^{m} A_i f(a_i 3) = 0$. These equations will be so numerous that for every real number h, $f(h)$ will appear in at least one of the equations. To create f, we will first assign to each real number h a variable x_h, and then think of the set $\{x_h\}_{h\in\mathbb{R}}$ as a subset of an infinite dimensional real vector space. We will next substitute x_h for $f(h)$ throughout the system S, creating an associated system of linear equations S'; e.g., one of these linear equations will be $\sum_{i=1}^{m} A_i x_{a_i 3} = 0$. Next we will show that the hypothesis of Theorem 1, part B guarantees that there is a solution for the linear system S'. Denote one such solution as $\{x_h = c_h\}_{h\in\mathbb{R}}$, where each c_h is a real number; e.g., $\sum_{i=1}^{m} A_i c_{a_i 3} = 0$. Finally, we will create the required function f by setting $f(h) = c_h$ for every real number h; e.g., $f(3) = c_3$.

Our hypothesis means that no finite linear combination of dilates of the expression $\sum_{i=1}^{m-1} A_i x_{a_i h}$ sums to x_h for any h. Let $s_n \to 0$, where $\{s_n\}$ is an algebraically independent set of numbers over the field $K = \mathbb{Q}(a_1, \ldots, a_{m-1})$. For each $n = 1, 2, \ldots$, let $S_n = \{ps_n : p \in K\backslash\{0\}\}$. Decompose the uncountable system of equations

$$\sum_{i=1}^{m-1} A_i x_{a_i h} = 0, \quad h \in \mathbb{R} \tag{3.1}$$

$$x_{s_n} = 1, \quad n = 1, 2, \ldots$$

into the following countable set of subsystems.

$$\sum_{i=1}^{m-1} A_i x_{a_i h} = 0, \quad h \in \mathbb{R} \backslash \bigcup_{n=1}^{\infty} S_n$$
$$(*)_n = \left\{ \begin{array}{l} \sum_{i=1}^{m-1} A_i x_{a_i h} = 0, \quad h \in S_n \\ x_{s_n} = 1 \end{array} \right. , \quad n = 1, 2, \ldots$$

Solve the first homogeneous subsystem with the trivial solution. In other words, set $x_{a_i h} = 0$ for every $h \notin \bigcup_{n=1}^{\infty} S_n$. Fix a positive integer n. The system $(*)_n$ has a solution: Since an infinite system of linear equations over a field has a solution if every finite subsystem has a solution (see [3] for several proofs of this), it is enough to prove that every finite subsystem has a solution. Take a finite subsystem. Without loss of generality, assume that the subsystem includes the equation $x_{s_n} = 1$. The only possible impediment to a solution is if there can be deduced a contradiction of the form $0 = c$ where c is a nonzero constant. But this could only happen if, for some n, x_{s_n} were a finite linear combination of the homogeneous equations, thereby providing $x_{s_n} = 0$, contrary to the hypothesis of this lemma.

Let $f(h)$ be the function that was created by solving the system (3.1). Then $\{k : f(k) \neq 0\}$ has Lebesgue measure 0 because it is a subset of the countable set $\bigcup_{i=1}^{m-1} \bigcup_{n=1}^{\infty} a_i S_n$, so that f is a measurable function. In particular, because of conditions (2.1) and (2.2), $\sum_{i=1}^{m-1} A_i x_{a_i 0} = \left(\sum_{i=1}^{m-1} A_i\right) x_0 = 0$ implies $f(0) = x_0 = 0$. On the one hand, $\lim_{h\to 0} \Delta_A f(h) = \lim_{h\to 0} 0 = 0$, so that f is $\mathcal{A}$-continuous at $x = 0$. On the other hand, if $a \notin \{0\} \cup \bigcup_{n=1}^{\infty} S_n$, then $\lim_{k\to 0} f\left(\frac{a}{k}\right) = \lim_{k\to 0} 0 = 0$ while $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 1 = 1$, so f is not continuous at $x = 0$.

4 Proof of Lemma 2

4.1 Symmetrizing a system of linear equations

Our assumption is that $f(h)$ is a linear combination with coefficients L_d of r_d-dilates of $\Delta_{\mathcal{A}} = \sum_{p=1}^{m-1} A_p f(a_p h)$:

$$\sum_{d=1}^{n} L_d \left(\sum_{p=1}^{m-1} A_p f(a_p r_d h) \right) = f(h). \tag{4.1}$$

Replacing $\Delta_{\mathcal{A}}$ by one of the dilates in equation (4.1) for which some $a_p r_d = 1$ if necessary, we may assume that some $a_p = 1$.

We adopt a more symmetrical notation for $\Delta_{\mathcal{A}}$. Write $\sum A_p f(a_p h)$ as

$$\sum_{p=1}^{P} A_p f(t_p) + B_p f(-t_p),$$

where for convenience we take $h > 0$ and set $t_p = |a_p|\, h$, and for each p, at least one of A_p and B_p is nonzero. Then $0 < t_1 < t_2 < \cdots < t_P$ and some $t_p = h$. The condition (2.1) is now

$$\sum_{p=1}^{P} A_p + B_p \neq 0. \tag{4.2}$$

If we dilate by $r_d > 0$, then the inner summand in equation (4.1) becomes

$$\sum_{p=1}^{P} A_p f(t_p r_d) + \sum_{p=1}^{P} B_p f(-t_p r_d),$$

whereas if we dilate by $-r_d$, that inner summand becomes

$$\sum_{p=1}^{P} A_p f(-t_p r_d) + \sum_{p=1}^{P} B_p f(t_p r_d).$$

These two summands involve the same variables, so to look at all possible ways to attain $f(h)$ in equation (4.1) we must add, say, a λ_d-multiple of the former and a μ_d-multiple of the latter. Equation (4.1) will be written as

$$\sum_{d=1}^{D} \sum_{p=1}^{P} \left(\left(\lambda_d A_p + \mu_d B_p \right) f(t_p r_d) + \left(\lambda_d B_p + \mu_d A_p \right) f(-t_p r_d) \right) = f(h), \tag{4.3}$$

where $(\lambda_d, \mu_d) \neq (0,0)$ for all d, and $(A_p, B_p) \neq (0,0)$ for all p. The equation (4.3) can be written as

$$\sum_{\eta} \left\{ \sum_{t_p r_d = \eta} \left(\lambda_d A_p + \mu_d B_p \right) f(t_p r_d) + \left(\lambda_d B_p + \mu_d A_p \right) f(-t_p r_d) \right\} = f(h), \tag{4.4}$$

where the inner sums are all 0 when $\eta \neq h$. When $\eta = h$ we must have the inner sum equal to $f(h)$. By taking the coefficients of $f(\eta)$ and $f(-\eta)$, this translates into the following system of linear equations:

$$\sum_{t_p r_d = \eta} \lambda_d A_p + \mu_d B_p = \delta_{\eta h} \text{ and } \sum_{t_p r_d = \eta} \lambda_d B_p + \mu_d A_p = 0, \tag{4.5}$$

for all η, where $\delta_{\eta h}$ is the usual Kronecker's delta symbol.

We change the notation so that some $t_p = h$ and some $r_d = 1$. Since the entire sum in (4.3) is $f(h)$, we must have $r_{d_0} a_{p_0} h = h$ for some p_0 and d_0. Then $f(r_{d_0} a_{p_0} h) = f(h)$ is a term in the r_{d_0}-dilate of $\Delta_{\mathcal{A}}$. Next replace $\mathcal{A}$ by $\mathcal{A}_0$, its r_{d_0}-dilate, and rewrite (4.3) in terms of dilates of $\mathcal{A}_0$. The new r_{d_0} equals 1, and therefore $t_{p_0} = h$. We also assume that all r_d are positive and ordered as $0 < r_1 < \cdots < r_D$. We will break down all the possible ways in which equation (4.3) might occur into 4 cases. Cases 1 and 4 will lead to exactly the $\mathcal{A}$-continuities appearing in Proposition 1, while Cases 2 and 3 will be shown to lead to no ways at all. To show this we form a P by D rectangle of lattice points

$$\mathcal{S} = \{(p, d) \mid p = 1, \ldots, P \text{ and } d = 1, \ldots, D\}.$$

So $\mathcal{S}$ is a rectangle of lattice points. We label each point (p, d) of $\mathcal{S}$ by the positive number $t_p r_d$.

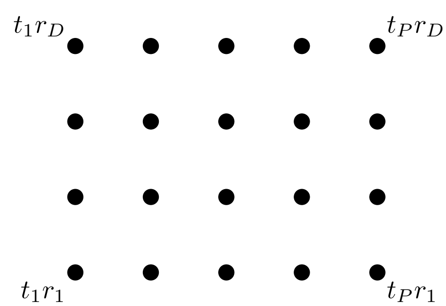

Figure 1: The set $\mathcal{S}$. Each point (p, d) is labeled as $t_p r_d$.

Navigating $\mathcal{S}$ by moving eastward from (p, d) to $(p+1, d)$ or northward from (p, d) to $(p, d+1)$ makes the value of the label strictly increase; i.e., $t_p r_d < t_{p+1} r_d$ and $t_p r_d < t_p r_{d+1}$. This is then the diagram of the partial order of $\mathbb{Z} \times \mathbb{Z}$ restricted to the rectangle $\{1, \ldots, P\} \times \{1, \ldots, D\}$.

4.2 Case 1: either $P = 1$ or $D = 1$

After Case 1 has been treated, we will be able to assume that the rectangle $\mathcal{S}$ is nondegenerate, that is, it has at least two rows and at least two columns.

When $P = D = 1$, we show that the only possible $\mathcal{A}$-continuities are covered by Proposition 1. Here there is only one t_p, namely $t_1 = h$, and only one r_d, namely $r_1 = 1$, and $\Delta_{\mathcal{A}} = \alpha f(h) + \beta f(-h)$. By (4.2), $\alpha + \beta \neq 0$. Equation (4.3) becomes

$$(\lambda\alpha + \mu\beta) f(h) + (\lambda\beta + \mu\alpha) f(-h) = 1 \cdot f(h) + 0 \cdot f(-h),$$

or

$$\begin{cases} \lambda\alpha + \mu\beta & = & 1 \\ \lambda\beta + \mu\alpha & = & 0 \end{cases},$$

which forces $\alpha \neq \beta$. This means that

$$\begin{aligned} \Delta_{\mathcal{A}} &= \left(\frac{\alpha - \beta}{2}\right) \left\{ \frac{2}{\alpha - \beta} \left(\alpha f(h) + \beta f(-h)\right) \right\} \\ &= \left(\frac{\alpha - \beta}{2}\right) \left\{ (A+1) f(h) + (A-1) f(-h) \right\}, \end{aligned}$$

where $A = (\alpha + \beta)/(\alpha - \beta)$, so that $\Delta_{\mathcal{A}}$ is a non-zero multiple of a difference of the form given in Proposition 1(ii) if $\alpha\beta \neq 0$, and is trivially equivalent to ordinary continuity if one of α or β is zero. We now show that the remaining subcases of Case 1 are impossible. We start with a simple lemma.

Lemma 3. *For every choice of the numbers a, b, c, d with $\begin{pmatrix} c \\ d \end{pmatrix}$ distinct from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, there do not exist two numbers x and y such that both systems*

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are simultaneously true.

Proof. The first system is solvable only if $x^2 - y^2 \neq 0$ and the second system is solvable only if $x^2 - y^2 = 0$. □

When $P = 1$ and $D \geq 2$, with $t_1 = h$, $r_1 = 1$, $r_2 = r \neq 1$, $A_1 = \alpha$ and $B_1 = \beta$, equation (4.3) becomes

$$\begin{gathered} [(\lambda\alpha + \mu\beta) f(h)] + (\lambda\beta + \mu\alpha) f(-h)] + \\ [(\lambda'\alpha + \mu'\beta) f(rh) + (\lambda'\beta + \mu'\alpha) f(-rh)] + \cdots = f(h). \end{gathered}$$

Apply Lemma 3 with $x = \alpha$, $y = \beta$, $a = \lambda$, $b = \mu$, $c = \lambda'$ and $d = \mu'$ to see that this subcase is impossible.

When $D = 1$ and $P \geq 2$, there is only one r_d, namely $r_1 = 1$. One t_p must equal h, call its associated A_p and B_p respectively α and β. Another t_p must be different from h. Call it th and call its associated A_p and B_p respectively α' and β'. Equation (4.3) is now

$$\begin{gathered}[(\lambda\alpha + \mu\beta)f(h) + (\lambda\beta + \mu\alpha)\, f(-h)] + \\ [(\lambda\alpha' + \mu\beta')f(th) + (\lambda\beta' + \mu\alpha')\, f(-th)] + \cdots = f(h)\,.\end{gathered}$$

Apply Lemma 3 with $x = \lambda$, $y = \mu$, $a = \alpha$, $b = \beta$, $c = \alpha'$, and $d = \beta'$ to see that this subcase is also impossible.

For the remaining cases, we assume that both P and D are ≥ 2.

4.3 Case 2: $t_1 r_1 = h$ or $t_P r_D = h$

After this case has been treated, we will be able to assume that $(1, 1)$, the southwest corner of $\mathcal{S}$, has label $< h$ and that (P, D), the northeast corner of $\mathcal{S}$, has label $> h$.

We shall see that Case 2 is impossible. Suppose $t_P r_D = h$; the other case is similar. Because $t_1 r_1 < t_P r_D = h$, from equation (4.3) we have

$$\begin{pmatrix} \lambda_1 & \mu_1 \\ \mu_1 & \lambda_1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that $\lambda_1^2 - \mu_1^2 = 0$ and $\lambda_1 = \pm\mu_1$. Assume $\lambda_1 = -\mu_1$. Then $\lambda_1 (A_1 - B_1) = 0$, so $A_1 = B_1$.

Claim 1. *$\lambda_d = -\mu_d$ for all d and $A_p = B_p$ for all p.*

Proof. Whenever $t_p r_d = \eta$, we say that p (or d) is associated with η. We induct on η to prove that for all $\eta < h$,

$$\text{whenever } p \text{ (resp. } d\text{) is associated with } \eta,\ A_p = B_p \text{ (resp. } \lambda_d = -\mu_d\text{).} \tag{4.6}$$

For the smallest $\eta = r_1 t_1$, (4.6) is true for both $d = 1$ and $p = 1$. Assume (4.6) has been proved for all $\eta' < \eta$. The coefficients of $f(\eta)$ and $f(-\eta)$ are given in (4.5). At most one r_{d_0} that has not appeared in lower indices may appear here multiplied by t_1. Then by the inductive hypothesis, we have

$$\begin{aligned}(\lambda_{d_0} + \mu_{d_0})\, A_1 + \sum_{t_p r_d = \eta, d \neq d_0} \lambda_d A_p + \mu_d B_p = 0 \\ (\lambda_{d_0} + \mu_{d_0 0})\, A_1 - \sum_{t_p r_d = \eta, d \neq d_0} \mu_d B_p + \lambda_d A_p = 0.\end{aligned}$$

Add these two equations, and then note that $(A_1, B_1) \neq (0,0)$ and $A_1 = B_1$ implies $A_1 \neq 0$, so

$$2\left(\lambda_{d_0} + \mu_{d_0}\right) A_1 = 0 \text{ and } \lambda_{d_0} = -\mu_{d_0}.$$

Also, at most one new t_{p_0} may appear, and if it does, it will be multiplied by r_1. In this case the equations (4.5) become

$$\begin{aligned} \lambda_1\left(A_{p_0} - B_{p_0}\right) + \sum_{t_p r_d = \eta, p \neq p_0} \lambda_d A_p + \mu_d B_p = 0 \\ \lambda_1\left(B_{p_0} - A_{p_0}\right) + \sum_{t_p r_d = \eta, p \neq p_0} \lambda_d A_p + \mu_d B_p = 0. \end{aligned}$$

This time subtract to get

$$2\lambda_1\left(A_{p_0} - B_{p_0}\right) = 0 \text{ and } A_{p_0} = B_{p_0}.$$

This proves (*).

The claim follows, since every p is associated with some $\eta < h = t_P r_D$, say $\eta = t_p r_1$, and every d is associated with some $\eta < h$, say $\eta = t_1 r_d$. □

Finally, because of the claim, every term in curly brackets in (4.4) is 0, even when $\eta = h$. This is a contradiction, since that term is $f(h)$.

Similarly, when $\lambda_1 = \mu_1$, we have $A_1 = -B_1$. This implies $\lambda_d = \mu_d$ for all d and $A_p = -B_p$ for all p. This, in turn, leads to the same contradiction as in the $\lambda_1 = -\mu_1$ case.

4.4 Case 3: either $t' r_d < h < t'' r_d$ for some d and $t', t'' \in \{t_1, \ldots, t_P\}$, or $t_p r' < h < t_p r''$ for some p and $r', r'' \in \{r_1, \ldots, r_D\}$.

After treating this case, we will be able to assume that the points on any fixed row of $\mathcal{S}$ are either all labeled by values $\leq h$ or all have values $\geq h$. The same is true for the columns of $\mathcal{S}$.

We will show that Case 3 is also impossible. Suppose $r_{d_0} t' < h < r_{d_0} t''$, for some d_0, t' and t''. The other case is similar. As in the proof of Case 2, we must have either

(1_{low}): $\lambda_d = -\mu_d$ and $A_p = B_p$ for all (d,p) such that $r_d t_p < h$ or
(2_{low}): $\lambda_d = \mu_d$ and $A_p = -B_p$ for all (d,p) such that $r_d t_p < h$.

Also we must have

(1_{high}): $\lambda_d = -\mu_d$ and $A_p = B_p$ for all (d,p) such that $r_d t_p > h$ or
(2_{high}): $\lambda_d = \mu_d$ and $A_p = -B_p$ for all (d,p) such that $r_d t_p > h$.

Conditions (1_{low}) and (2_{high}) are incompatible, since $r_{d_0}t' < h$ implies $\lambda_{d_0} = -\mu_{d_0}$, while $h < r_{d_0}t''$ implies $\lambda_{d_0} = \mu_{d_0}$, a contradiction since $(\lambda_d, \mu_d) \neq (0,0)$ for all d. Similarly, conditions (1_{high}) and (2_{low}) are incompatible. So we may assume either

(1): $\lambda_d = -\mu_d$ and $A_p = B_p$ for all (d,p) such that $r_d t_p \neq h$ or
(2): $\lambda_d = \mu_d$ and $A_p = -B_p$ for all (d,p) such that $r_d t_p \neq h$.

Now suppose $r_{d_0}t_{p_0} = h$. Since there are at least two t_p, there is at least one t for which $r_{d_0}t \neq h$; similarly there is at least one r such that $rt_{p_0} \neq h$. This shows that we may assume either

(1′): $\lambda_d = -\mu_d$ and $A_p = B_p$ for all (d,p) or
(2′): $\lambda_d = \mu_d$ and $A_p = -B_p$ for all (d,p).

Just as in the proof of Case 2, condition (1′) leads to a contradiction; similarly, condition (2′) leads to a contradiction. Thus Case 3 is impossible.

4.5 Reduction to the case $P = D = 2$ and $t_1r_2 = t_2r_1 = h$

We show that this is the only possible case left after discarding all previous cases.

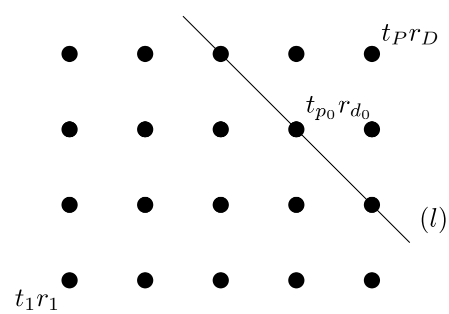

Figure 2: The set $\mathcal{S}$ and line (l). Each point (p,d) is labeled as $t_p r_d$.

The elimination of Case 1 amounts to $P \geq 2$ and $D \geq 2$. Thus the set of integer coordinate points in the plane

$$\mathcal{S} = \{(p,q) \mid p = 1, \ldots, P \text{ and } d = 1, \ldots, D\}$$

forms a non-degenerate rectangle. As before, we label the points (p, d) of $\mathcal{S}$ by the positive numbers $t_p r_d$.

By Section 3.1, we have $h = t_{p_0} r_{d_0}$ for some p_0 and d_0, and the elimination of Case 2 means that $t_1 r_1 < h = t_{p_0} r_{d_0} < t_P r_D$. This clearly implies that $1+1 < p_0+d_0 < P+D$. This has the geometric interpretation that the line (l) given by the equation $p + d = p_0 + d_0$, for variables p, d, leaves the SW-corner $(1, 1)$ below, and the NE-corner (P, D) above. The line passes through the point (p_0, d_0) of $\mathcal{S}$, so it must intersect the rectangle upon a segment. Having slope -1, the line must intersect $\mathcal{S}$ upon a set of at least two $\sqrt{2}$-equidistant points.

Claim 2. *The set $\mathcal{S} \cap (l)$ consists of all points in $\mathcal{S}$ that are labeled by h. In other words, this is the set of all (p, d) for which $t_p r_d = h$.*

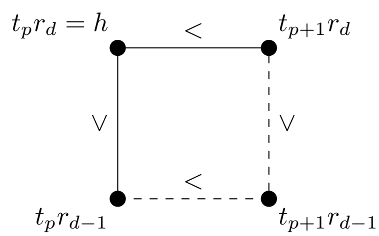

Figure 3: Proof of Claim 2.

Proof. Since points below the line are labeled by smaller numbers, and points above the line are labeled by larger numbers, it suffices to show that all points of $\mathcal{S}$ that lie on the line are labeled by h. For this, it suffices to show that whenever a point (p, d) of (l) is labeled by h, its (l)-neighbor points $(p-1, d+1)$ and $(p + 1, d - 1)$, when they belong to $\mathcal{S}$, are also labeled by h. Indeed, assume $t_p r_d = h$. Then $t_p r_{d-1} < h$ and $t_{p+1} r_d > h$. We have a strictly increasing sequence $t_p r_{d-1} < t_{p+1} r_{d-1} < t_{p+1} r_d$, with h being strictly between the extreme terms. If the middle term $t_{p+1} r_{d-1} > h$, then $t_p r_{d-1} < h < t_{p+1} r_{d-1}$, which contradicts the elimination of Case 3; while if the middle term is less than h, then $t_{p+1} r_{d-1} < h < t_{p+1} r_d$, which also contradicts the elimination of Case 3. By trichotomy, $t_{p+1} r_{d-1} = h$. A similar proof shows that $t_{p-1} r_{d+1} = h$. $\square$

Claim 3. *The set $\mathcal{S} \cap (l)$ consists of exactly two points.*

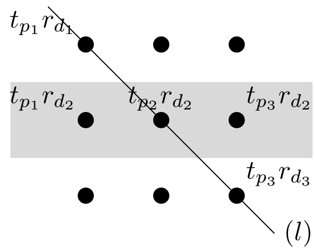

Figure 4: Proof of Claim 3.

PROOF. That the set has at least two points was argued before. Suppose the set has three distinct points (p_1, d_1), (p_2, d_2) and (p_3, d_3). These can be taken so that $p_1 = p_2 - 1$, $p_3 = p_2 + 1$, $d_1 = d_2 + 1$ and $d_3 = d_2 - 1$. The obvious inequality of labels $t_{p_1}r_{d_2} < h = t_{p_2}r_{d_2} < t_{p_3}r_{d_2}$ is ruled out by the elimination of Case 3, so this case does not exist. □

Claim 4. $\mathcal{S} \cap (l) = \{(1, D), (P, 1)\}$. *This means that* $t_1 r_D = h = t_P r_1$.

PROOF. We argue as in the proof of Claim 2. Take the increasing sequence $t_1r_1 < t_1r_D < t_P r_D$. Since h is strictly between the extreme terms, and the elimination of Case 3 rules out its being strictly between consecutive terms, h must equal the middle term $t_1 r_D$. A similar proof is given for $h = t_P r_1$. One inclusion is then proved. The reverse is forced by Claim 3. □

Finally, since (l) has slope -1, we must have $P = D$. This common value is 2, by Claim 3, since together with the two points of Claim 4, $\mathcal{S} \cap (l)$ must contain all the lattice points in between.

4.6 Case 4: $P = D = 2$ and $t_1r_2 = t_2r_1 = h$.

We show that this case leads ineluctably to the situation of the first part of Proposition 1.

PROOF. This case means that the set $\{t_1, t_2\}$ can be written as $\{h, rh\}$, with $r \neq 1$. The only possible pair of distinct positive dilates that can lead to a solvable system of four equations is $\{1, r^{-1}\}$. Our set of differences becomes:

$$Af(rh) + Cf(-rh) + Bf(h) + Df(-h)$$
$$Cf(rh) + Af(-rh) + Df(h) + Bf(-h)$$
$$Af(h) + Cf(-h) + B(r^{-1}h) + Df(-r^{-1}h)$$
$$Cf(h) + Af(-h) + Df(r^{-1}h) + Bf(-r^{-1}h).$$

Multiply the four differences by $\lambda_1, \mu_1, \lambda_2, \mu_2$ and add to produce

$$\begin{aligned}&\{\lambda_1 A + \mu_1 C\} f(rh) + \{\lambda_1 C + \mu_1 A\} f(-rh)\\&+ \{\lambda_1 B + \mu_1 D + \lambda_2 A + \mu_2 C\} f(h) + \{\lambda_1 D + \mu_1 B + \lambda_2 C + \mu_2 A\} f(-h)\\&+ \{\lambda_2 B + \mu_2 D\} f\left(r^{-1}h\right) + \{\lambda_2 D + \mu_2 B\} f\left(-r^{-1}h\right).\end{aligned}$$

Our assumption is that the quantities in curly brackets are, respectively, $0, 0, 1, 0, 0$, and 0. In particular, we get the equations

$$\lambda_1 A + \mu_1 C = \lambda_1 C + \mu_1 A = 0.$$

Since $(A, C) \neq (0, 0)$, one of these cases must hold:

$$\begin{aligned}&(1)\colon A = C \quad \text{and } \lambda_1 = -\mu_1\\&(2): A = -C \text{ and } \lambda_1 = \mu_1.\end{aligned}$$

Similarly, we also have one of these two cases:

$$\begin{aligned}&(3): B = D \quad \text{and } \lambda_2 = -\mu_2\\&(4): B = -D \text{ and } \lambda_2 = \mu_2.\end{aligned}$$

The third curly bracketed term is 1, producing another equation:

$$(5): \lambda_1 B + \mu_1 D + \lambda_2 A + \mu_2 C = 1.$$

Assume $A = C$ and $B = D$. Substitute the values from equations (1) and (3) into equation (5) to obtain the contradiction

$$\lambda_1 B - \lambda_1 B + \lambda_2 A - \lambda_2 A = 1.$$

The assumption $A = -C$ and $B = -D$ leads to the same contradiction. So, one of two cases must hold. Either $A = C$ and $B = -D$, which is the $\mathcal{A}$-continuity $A(f(rh) + f(-rh)) + B\left(f\left(r^{-1}h\right) - f\left(-r^{-1}h\right)\right)$, also written as

$$A(f(x + rh) + f(x - rh) - 2f(x)) + B\left(f\left(x + r^{-1}h\right) - f\left(x - r^{-1}h\right)\right),$$

or the case $A = -C$ and $B = D$, which is the $\mathcal{A}$-continuity

$$B\left(f\left(x + r^{-1}h\right) + f\left(x - r^{-1}h\right) - 2f(x)\right) + A(f(x + rh) - f(x - rh)).$$

Both of these are of the form given in Proposition 1(i). $\square$

Notice that our proof of Lemma 2 explains the existence of the small, initially unexpected and for quite some time mysterious (at least to the authors), family of $\mathcal{A}$-continuities that are equivalent to ordinary continuity in a natural way.

5 The classification of generalized continuities

Let $\Delta_{\mathcal{A}} = \sum A_i f(x + a_i h)$ and $\Delta_{\mathcal{B}} = \sum B_i f(x + b_i h)$, where $\sum A_i = \sum B_i = 0$, be generalized $\mathcal{A}$-continuities. Say $\mathcal{A}$ is *equivalent* to $\mathcal{B}$ if, for every function f and every point x, $\Delta_{\mathcal{A}} f(h, x) \to 0$ as $h \to 0$ if and only if $\Delta_{\mathcal{B}} f(h, x) \to 0$ as $h \to 0$. By Theorem 2, the equivalence class $[f(x+h) - f(x)]$ of ordinary continuity is

$$\{A(f(x+rh)+f(x-rh)-2f(x)) + B(f(x+sh) - f(x-sh)) : ABrs \neq 0\}.$$

Associate to any generalized continuity $\mathcal{A}$ its even part $\mathcal{A}_e$, defined by the expression $\Delta_{\mathcal{A}_e} f(h, x) = (\Delta_{\mathcal{A}} f(h, x) + \Delta_{\mathcal{A}} f(-h, x))/2$, and its odd part $\mathcal{A}_o$, defined by $\Delta_{\mathcal{A}_o} f(h, x) = (\Delta_{\mathcal{A}} f(h, x) - \Delta_{\mathcal{A}} f(-h, x))/2$.

In general,

$$\Delta_{\mathcal{A}} = \Delta_{\mathcal{A}_e} + \Delta_{\mathcal{A}_e},$$

and, in particular,

$$f(x+h) - f(x) = \frac{1}{2}(f(x+h) + f(x-h) - 2f(x)) + \frac{1}{2}(f(x+h) - f(x-h)).$$

A generalized continuity $\mathcal{A}$ is even (resp. odd) if $\mathcal{A} = \mathcal{A}_e$ (resp. $\mathcal{A} = \mathcal{A}_o$). Two examples of even continuity are given by the differences $f(x+h) + f(x-h) - 2f(x)$ and $3f(x+2h) + 3f(x-2h) + f(x+h) + f(x-h) - 8f(x)$. Ordinary continuity is an example of a continuity that is neither odd nor even.

Theorem 3. *Let $\mathcal{B}$ be a generalized continuity. Then the equivalence class of $\mathcal{B}$ is given by the following set of differences.*

$$[\Delta_{\mathcal{B}}] = \{A\Delta_{\mathcal{B}_e} f(rh, x) + B\Delta_{\mathcal{B}_o} f(sh, x) : ABrs \neq 0\}.$$

The proof of this theorem follows very closely the methods developed in reference [2]. Two special cases are especially interesting. If $\mathcal{B}$ is even, then $\mathcal{B} = \mathcal{B}_e$ and $\mathcal{B}_o = 0$, so that the equivalence class of $\mathcal{B}$ is given by exactly the differences that are trivially equivalent to $\mathcal{B}$, that is

$$[\Delta_{\mathcal{B}}] = \{A\Delta_{\mathcal{B}} f(rh, x) : Ar \neq 0\}.$$

Similarly if $\mathcal{B}$ is odd, this happens again. Only purely even and purely odd equivalence classes fail to include non-trivially equivalent differences.

Recall that ordinary continuity implies every generalized continuity. A generalization of this is the following theorem.

Theorem 4. *Let $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{B}}$ be generalized $\mathcal{A}$-continuities. Then $\mathcal{B}$-continuity implies $\mathcal{A}$-continuity if and only if for every function f and point x, $\Delta_{\mathcal{A}_e} f(h,x)$ and $\Delta_{\mathcal{A}_o} f(h,x)$ are finite linear combinations*

$$\Delta_{\mathcal{A}_e} f(h,x) = \sum_i U_i \Delta_{\mathcal{B}_e} f(u_i h, x)$$
$$\Delta_{\mathcal{A}_o} f(h,x) = \sum_i V_i \Delta_{\mathcal{B}_o} f(v_i h, x)$$

of non-zero u_i-dilates of $\Delta_{\mathcal{B}_e} f(h,x)$ and v_i-dilates of $\Delta_{\mathcal{B}_o} f(h,x)$.

The proof of this theorem also follows very closely the methods developed in reference [2].

References

[1] J. M. Ash, *Generalizations of the Riemann derivative*, Trans. Amer. Math Soc., **126** (1967), 181–199.

[2] J. M. Ash, S. Catoiu and W. Chin, *The classification of generalized Riemann derivatives*, preprint.

[3] J. M. Ash, S. Catoiu and M. Csörnyei, *Generalized vs. ordinary differentiation*, preprint.

[4] J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometric series*, Fund. Math. **26** (1936), 1–43.