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AN INTEGRAL MEAN VALUE THEOREM FOR REGULATED FUNCTIONS

An elementary but useful mean value theorem for integrals asserts that if f is a non-negative integrable function on $[a, b]$ and g is continuous there, then there is a $\theta \in (a, b)$ such that

$$g(\theta) \int_a^b f(t)dt = \int_a^b g(t)f(t)dt.$$

It does not seem to have been observed that this result has an equally useful extension to regulated g, i.e, g which have right and left limits at each point. Let

$$\overline{g}(t) = \max[g(t+), g(t-)] \text{ and } \underline{g}(t) = \min[g(t+), g(t-)].$$

We assume for simplicity in the statement of our result that g has no external status, i.e., for every t, $g(t)$ is in the interval $[\underline{g}(t), \overline{g}(t)]$. Our result is the following:

Theorem 1 *If g is regulated and f is non-negative and integrable on $[a, b]$, then there is a $\theta \in (a, b)$ such that*

$$g^*(\theta) \int_a^b f(t)dt = \int_a^b g(t)f(t)dt,$$

where $g^*(\theta) \in [\underline{g}(\theta), \overline{g}(\theta)]$.

If $g(t) = \sum_1^n g_i(t)$, $g_i(t)$ regulated, then there is a $\theta \in (a, b)$ such that

$$\sum g_i^*(\theta) \int_a^b f(t)dt = \int_a^b g(t)f(t)dt.$$

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Interesting examples of applications of this result occur in a forthcoming paper with P. Pierce which is concerned with the class of regulated functions whose Fourier series converge after any change of variable. We indicate one such application below.

Let g be a regulated 2π-periodic function. We introduce the symbol $\sum(\breve{g}, k, n, x+\theta)$ to denote

$$\sum_{i=1}^{k} \frac{1}{i}\left[\breve{g}\left(\frac{(2i)\pi}{n}+x+\theta_n\right)-\breve{g}\left(\frac{(2i-1)\pi}{n}+x+\theta_n\right)\right],$$

where $\theta_n \in \left(0, \frac{\pi}{n}\right)$, $\breve{g}\left(\frac{j\pi}{n}+x+\theta_n\right)$ is in the interval

$$\left[\underline{g}\left(\frac{j\pi}{n}+x+\theta_n\right), \overline{g}\left(\frac{j\pi}{n}+x+\theta_n\right)\right]$$

and

$$2\sum(\breve{g}, k, n, x+\theta) =$$
$$\int_0^{\pi} \sum \frac{1}{i}\left[g\left(x+\frac{t+2i\pi}{n}\right)-g\left(x+\frac{t+(2i-1)\pi}{n}\right)\right]\sin t dt.$$

We then obtain the following useful lemma.

Lemma 1 *For every regulated function g and every $x \in [0, \pi]$ there exists a sequence $\{\theta_n\}$ with $0 < \theta_n < \frac{\pi}{n}$ and a sequence of integers $\{k_n\}$ with $\lim_{n\to\infty} k_n = \infty$, $\lim_{n\to\infty} \frac{k_n}{n} = 0$ and such that*

$$\int_0^{\pi} (g(x+t)-g(x+))\frac{\sin nt}{t}dt \text{ and } \frac{1}{\pi}\sum(\breve{g}, k_n, n, x+\theta)$$

are equiconvergent.

PROOF OF THE THEOREM We observe that if $\underline{m} = \inf g(t)$ and $\overline{m} = \sup g(t)$, we have

$$\underline{m}\int_a^b f(t)dt \le \int_a^b g(t)f(t)dt \le \overline{m}\int_a^b f(t)dt$$

and there is an $m \in [\underline{m}, \overline{m}]$ such that

$$m\int_a^b f(t)dt = \int_a^b g(t)f(t)dt.$$

Suppose we "complete" the graph of $y = g(t)$ by inserting vertical segments at each jump, i.e., if g is discontinuous at t_0, we adjoin to the graph of g the segment

$$\{(t_0, y) \mid \underline{g}(t_0) \leq y \leq \overline{g}(t_0)\}.$$

We claim that the projection of the completed graph on the Y-axis is a closed segment. This is easily verified from the following considerations.

Let h be a strictly increasing function on $[a, b]$ with $h(a) = 0,\ h(b) = 1$ and let h and g have common points of discontinuity of the same kind, i.e., $h(t+) = h(t)$ if and only if $g(t+) = g(t)$, and $h(t-) = h(t)$ if and only if $g(t-) = g(t)$.

Let us denote the unique continuous extension of $g \circ h^{-1}$ to the closure A of $h([a, b])$ by $g \circ h^{-1}$. We extend $g \circ h^{-1}$ to $\overline{g \circ h^{-1}}$ defined on $[0, 1]$ by setting

$$\overline{g \circ h^{-1}} = g \circ h^{-1}$$

for $t \in A$ and defining $\overline{g \circ h^{-1}}$ to be linear on the closure of each component interval of $[0, 1] \setminus A$.

It is clear that $\overline{g \circ h^{-1}}$ is continuous and the projection of its graph on the Y-axis, a closed segment, is identical to the projection of the completed graph of g. Thus there is a $\theta \in [a, b]$ such that

$$\underline{g}(\theta) \leq m \leq \overline{g}(\theta).$$

We now show that θ may be chosen to be an interior point of $[a, b]$.

The sets

$$\{t|g(t) > m, f(t) > 0\}, \{t|g(t) < m, f(t) > 0\}$$

are either both of positive measure or both of measure zero. In the latter case, if we exclude the trivial case $f = 0$ a.e., we have $g(t) = m$ on a set of positive measure. In the former case there is an interval $[t_0, t_1] \subseteq (a, b)$ such that $g(t_0) - m$ and $g(t_1) - m$ are of opposite sign. The argument used above shows that there is a $\theta \in [t_0, t_1]$ such that

$$\underline{g}(\theta) - m \ \leq\ 0 \ \leq\ \overline{g}(\theta) - m,$$

and, setting $g^*(\theta) = m$, we see that this is the θ sought in the theorem.

If $g(t) = \sum_1^n g_i(t)$, each $g_i(t)$ regulated, we may determine θ and $g^*(\theta)$ as above. If g is continuous at θ, we may set $g_i^*(\theta) = g_i(\theta-)$. Otherwise, there is a unique $\tau_0 \in [0, 1]$ such that

$$\tau_0 g(\theta-) + (1 - \tau_0) g(\theta+) = g^*(\theta).$$

Then we may choose

$$g_i^*(\theta) = \tau_0 g_i(\theta-) + (1-\tau_0) g_i(\theta+)$$

and have

$$g^*(\theta) = \sum g_i^*(\theta)$$

and

$$\underline{g_i}(\theta) \;\leq\; g_i^*(\theta) \;\leq\; \overline{g_i}(\theta).$$

$\square$