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THERE ARE MEASURABLE HAMEL FUNCTIONS

Abstract

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is a *Hamel function* if f, considered as a subset of $\mathbb{R}^2$, is a Hamel basis of $\mathbb{R}^2$. We show that there is a Marczewski measurable Hamel function. Additionally, we show that there is a Hamel function which is both Lebesgue measurable and with the Baire property.

1 Introduction.

The symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for the sets of all real and all rational numbers, respectively. A basis of $\mathbb{R}^n$ as a linear space over $\mathbb{Q}$ is called a *Hamel basis*. The cardinality of a set X we denote by $|X|$.

A *σ-ideal* $\mathcal{I}$ of subsets of $\mathbb{R}$ is family closed under subsets and countably unions. $\mathcal{N}$ denotes the σ-ideal of Lebesgue null sets. $\mathcal{M}$ denotes the σ-ideal of all sets of first category. Recall that a σ-ideal $\mathcal{I}$ is *Borel generated* if there

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exists a family $\mathcal{J} \subset \mathcal{I}$ of Borel sets such that $\mathcal{I} = \{A : A \subset B, B \in \mathcal{J}\}$. $\mathcal{I}$ is (*ccc*) if every family of disjoint Borel sets which do not belong to the ideal is countable.

A set $A \subset \mathbb{R}$ is *Marczewski measurable* ($A \in (s)$ for short) if for every perfect set $P \subset \mathbb{R}$ either $P \cap A$ or $P \setminus A$ contains a perfect set. (Recall that a perfect set is a non-empty closed set without isolated points.) If every perfect set $P \subset \mathbb{R}$ contains a perfect subset which misses A, then A is called *Marczewski null* ($A \in (s_0)$ for short). It is known that (s) is a σ-field and (s_0) is a σ-ideal of (s). A function $f\colon \mathbb{R} \to \mathbb{R}$ is *Marczewski measurable* if it is measurable with respect to the σ-field (s) (i.e. if the preimage of any open set is Marczewski measurable). In [8], Marczewski introduced the σ-field (s) to show that a function $f\colon \mathbb{R} \to \mathbb{R}$ is Marczewski measurable if and only if every perfect $P \subset \mathbb{R}$ has an uncountable Borel subset Q such that $f \restriction Q$ is continuous. (It is known that we can replace the word "perfect" with "uncountable Borel" in the definition of (s) and (s_0)—we will use this fact in the sequel.)

$\mathcal{B}or$ stands for the σ-field of Borel subsets of $\mathbb{R}$. For every σ-ideal $\mathcal{I}$, $\mathcal{B}or\triangle\mathcal{I}$ stands the σ-field of all sets of the form $A\triangle B$, where $A \in \mathcal{I}$, $B \in \mathcal{B}or$, and $A\triangle B$ denotes the symmetric difference between A and B. It is known that the σ-field of all Lebesgue measurable sets is equal to $\mathcal{B}or\triangle\mathcal{N}$, and the σ-field of all sets with the Baire property equals $\mathcal{B}or\triangle\mathcal{M}$.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is a *Hamel function* if f, considered as a subset of $\mathbb{R}^2$, is a Hamel basis of $\mathbb{R}^2$. The class of Hamel functions was introduced by Płotka and researched in [4], [5], [6], [7] and [2]. In [4], the author proved that every function $f : \mathbb{R} \to \mathbb{R}$ is a sum of two Hamel functions. This implies that there is a Hamel function which is not Lebesgue measurable (without the Baire property, which is not Marczewski measurable, respectively).

2 Main Results.

The aim of this paper is to show that there are Hamel functions which are measurable with respect to some σ-fields. Namely, we show the following theorems—they answer problems posed by T. Natkaniec (oral communication).

Theorem 1. *There exists a Marczewski measurable Hamel function.*

Theorem 2. *Suppose that $\mathcal{I}$ is a σ-ideal of subsets of $\mathbb{R}$ which contains singletons. Suppose that there exists a Borel set $B \in \mathcal{I}$ and a Hamel basis $H \subset B$*

with $|B \setminus H| = 2^\omega$. Then there exists a Hamel function which is measurable with respect to the σ-field $\mathcal{B}or \triangle \mathcal{I}$.

Corollary 3. *Suppose that $\mathcal{I}$ is a Borel generated (ccc) σ-ideal of subsets of $\mathbb{R}$ which contains singletons. Suppose that there exists a Hamel basis $H \in \mathcal{I}$. Then there exists a Hamel function which is measurable with respect to the σ-field $\mathcal{B}or \triangle \mathcal{I}$.*

If we use Corollary 3 in the case $\mathcal{I} = \mathcal{N}$ (or $\mathcal{I} = \mathcal{M}$) we get the following corollary. (Recall that the Cantor ternary set contains a Hamel basis, see e.g. [1].)

Corollary 4. *There exists a Lebesgue measurable Hamel function (a Hamel function with the Baire property, respectively).*

3 Proofs.

We will use the following lemma in our proofs.

Lemma 5. *[7, Lemma 2] Let $H_1, H_2 \subseteq \mathbb{R}$ be a Hamel bases. Suppose that $h : \mathbb{R} \setminus H_1 \to H_2$ is a bijection. Then a function $H : \mathbb{R} \to \mathbb{R}$ defined by:*

$$H(x) = \begin{cases} h(x) & \text{if } x \notin H_1 \\ 0 & \text{if } x \in H_1, \end{cases}$$

is a Hamel function.

Proof of Theorem 1. Let H_1 be a Hamel basis which is Marczewski null (see [3]). Let H_2 be a Hamel basis which contains a perfect set (see [1]). Fix a Marczewski null set S of size 2^ω such that $S \cap H_1 = \emptyset$. Choose $\{P_\alpha\}_{\alpha<2^\omega}$ and P, pairwise disjoint perfect sets contained in H_2 and all homeomorphic to the Cantor set 2^ω. Let $\{Q_\alpha\}_{\alpha<2^\omega}$ be an enumeration of all perfect subsets of $\mathbb{R}$.

We will construct by induction a family of sets Q^*_α and functions $f_\alpha : Q^*_\alpha \to \mathbb{R}$ such that Q^*_α is either the empty set or a perfect set. In case of $Q^*_\alpha = \emptyset$ we have also $f_\alpha = \emptyset$.

Assume that we are in the stage $\gamma < 2^\omega$. There are two possibilities:

1. $\forall_{\alpha<\gamma} |Q^*_\alpha \cap Q_\gamma| \le \aleph_0$.
2. $\exists_{\alpha<\gamma} |Q^*_\alpha \cap Q_\gamma| = 2^\omega$.

Case 1: Choose any perfect $Q^*_\gamma \subseteq Q_\gamma \setminus \left[\bigcup_{\alpha<\gamma} Q^*_\alpha \cup H_1 \cup S\right]$ and moreover, such that Q^*_γ is homeomorphic to the Cantor set 2^ω. (This choice is possible since H_1 and S are Marczewski null.) Next, let $f_\gamma : Q^*_\gamma \to P_\gamma$ be any homeomorphism.

CASE 2: Put $Q^*_\gamma = \emptyset$ and $f_\gamma = \emptyset$.

Now define: $f^* = \bigcup_{\gamma<2^\omega} f_\gamma$. f^* is a bijection between $\bigcup_{\gamma<2^\omega} Q^*_\gamma$ and some subset of $\bigcup_{\gamma<2^\omega} P_\gamma$. Since $|\mathbb{R}\setminus\big[H_1\cup\bigcup_{\gamma<2^\omega} Q^*_\gamma\big]| = 2^\omega$ and $|H_2\setminus\big[\bigcup_{\gamma<2^\omega} P_\gamma\big]| = 2^\omega$ we can extend f^* to a bijection $f : \mathbb{R}\setminus H_1 \to H_2$ arbitrary.

Next we use Lemma 5 to obtain a Hamel function $H : \mathbb{R} \to \mathbb{R}$.

This function is Marczewski measurable. Indeed, suppose that $Q \subseteq \mathbb{R}$ is any perfect set. Then there exists $\gamma < 2^\omega$ such that $Q_\gamma = Q$.

If $Q^*_\gamma \neq \emptyset$ then $f_\gamma \subseteq H$ is a continuous function from perfect subset $Q^*_\gamma \subseteq Q_\gamma$ into $\mathbb{R}$.

If $Q^*_\gamma = \emptyset$ then there exists $\alpha < \gamma$ such that $|Q^*_\alpha \cap Q_\gamma| = 2^\omega$ but in this case $f^*_\alpha \restriction (Q^*_\alpha \cap Q_\gamma)$ is a continuous function defined on a Borel subset of Q_γ of size 2^ω. □

PROOF OF THEOREM 2. Let $B \in \mathcal{I}$ be a Borel set and let $H_1 \subset B$ be a Hamel basis with $|B \setminus H_1| = 2^\omega$. Let $H_2 \subseteq \mathbb{R}$ be a Hamel basis which contains some perfect set P. We can also assume that $|H_2 \setminus P| = 2^\omega$.

Since the spaces $\mathbb{R}\setminus B$ and P are Borel isomorphic, let $b : B \to P$ be a Borel bijection.

By virtue of $|B \setminus H_1| = 2^\omega$ and $|H_2 \setminus P| = 2^\omega$ we can extend b to a bijection $b^* : \mathbb{R}\setminus H_1 \to H_2$. Next we use Lemma 5 to obtain a Hamel function $H : \mathbb{R} \to \mathbb{R}$.

We will check that H is $\mathcal{B}or\triangle\mathcal{I}$ measurable. Indeed, suppose that $U \subseteq \mathbb{R}$ is an open set. Then

$$H^{-1}[U] = \begin{cases} (b^*)^{-1}[U] & \text{if } 0 \notin U, \\ (b^*)^{-1}[U] \cup H_1 & \text{if } 0 \in U. \end{cases}$$

But we have $(b^*)^{-1}[U]\triangle b^{-1}[U] \in \mathcal{I}$, therefore $H^{-1}[U] \in \mathcal{B}or\triangle\mathcal{I}$. □

PROOF OF COROLLARY 3. By Theorem 2 it is enough to show that there is a Borel set $B \in \mathcal{I}$ and a Hamel basis $H \subset B$ with $|B \setminus H| = 2^\omega$.

Let $H \in \mathcal{I}$ be a Hamel basis. Let $E \in \mathcal{I}$ be a Borel set with $H \subset E$.

Since $\mathbb{R}\setminus E$ is a Borel set of cardinality 2^ω, so we can find a pairwise disjoint family $\mathcal{B}$ of cardinality 2^ω of Borel subsets of $\mathbb{R}\setminus B$ each of size 2^ω. Since $\mathcal{I}$ is (ccc) there exists a $B_0 \in \mathcal{B}\cap\mathcal{I}$.

Then $B = E\cup B_0 \in \mathcal{I}$ is a Borel set such that $H \subset B$ and $|B\setminus H| = 2^\omega$. □

4 Odds and ends.

Given a set $X \subset \mathbb{R}$, the porosity of X at a real $r \in \mathbb{R}$ is defined by

$$p(X,r) = \limsup_{\varepsilon \to 0^+} \frac{\lambda(X,(r-\varepsilon,r+\varepsilon))}{\varepsilon},$$

where $\lambda(X,I)$ denotes the maximal length of an open subinterval of the interval I which is disjoint from X. A set X is *porous* ($X \in \mathcal{P}$) iff $p(X,a) > 0$ for every $a \in X$. Let $\sigma\mathcal{P}$ denote the sigma-ideal generated by the porous sets. We say that X is a *σ-porous* set iff $X \in \sigma\mathcal{P}$. (For some properties of σ-porous sets see e.g. [9].)

Let $\mathcal{E}$ be a σ-ideal generated by closed Lebesgue null sets, and let $\mathcal{N} \cap \mathcal{M}$ denote the σ-ideal of sets which are both Lebesgue null and of the first category. It is known that $\sigma\mathcal{P}, \mathcal{E} \subset \mathcal{N} \cap \mathcal{M}$, and $\sigma\mathcal{P} \not\subset \mathcal{E}$, $\mathcal{E} \not\subset \sigma\mathcal{P}$.

If we use Theorem 2 in the case $\mathcal{I} = \sigma\mathcal{P}$, $\mathcal{E}$ or $\mathcal{N} \cap \mathcal{M}$ we get the following corollary.

Corollary 6.

1. *There exists a Hamel function which is measurable with respect to the σ-field $\mathcal{B}or \triangle \sigma\mathcal{P}$.*

2. *There exists a Hamel function which is measurable with respect to the σ-field $\mathcal{B}or \triangle \mathcal{E}$.*

3. *There exists a Hamel function which is measurable with respect to the σ-field $\mathcal{B}or \triangle (\mathcal{N} \cap \mathcal{M})$ (i.e. a Hamel function which is both Lebesgue measurable and with the Baire property).*

PROOF. (1). The Cantor ternary set $C \subset [0,1]$ is σ-porous and contains a Hamel basis $H \subset C$. Let $A \subset \mathbb{R} \setminus C$ be a Borel σ-porous set of cardinality 2^ω. Now, we can use Theorem 2 with $B = C \cup A$.

(2). The Cantor ternary set $C \subset [0,1]$ belongs to $\mathcal{E}$ and contains a Hamel basis $H \subset C$. Let $A \subset \mathbb{R} \setminus C$ be a Borel set which belongs to $\mathcal{E}$ and is of cardinality 2^ω. Now, we can use Theorem 2 with $B = C \cup A$.

(3). Since $\mathcal{E} \subset \mathcal{N} \cap \mathcal{M}$, so every function which is $\mathcal{B}or \triangle \mathcal{E}$ measurable is also $\mathcal{B}or \triangle (\mathcal{N} \cap \mathcal{M})$ measurable. $\square$

Remark. In case of $\mathcal{N} \cap \mathcal{M}$ we can also use Corollary 3 (since this ideal is Borel generated and (*ccc*)). However, in case of $\sigma\mathcal{P}$ and $\mathcal{E}$ we cannot use Corollary 3 since it is known that these ideals are not (*ccc*).

We can also construct a Hamel function which is measurable in one sense and non-measurable in another.

Proposition 7. *1. There exists a Lebesgue measurable Hamel function without the Baire property.*

2. There exists a Lebesgue nonmeasurable Hamel function with the Baire property.

Proof. We will show the first case and the second one can be shown similarly.

Let H_1 be a Hamel basis which is Lebesgue null and does not have the Baire property. Let $B \in \mathcal{N}$ be a Borel set with $H \subset B$ and $|B \setminus H| = 2^\omega$. Now, we proceed as in the proof of Theorem 2 and construct a Hamel function $H : \mathbb{R} \to \mathbb{R}$. Then H is Lebesgue measurable. On the other hand, $H^{-1}(\{0\}) = H_1$, so H does not have the Baire property. $\square$

Finally, we show that Theorem 1 does not follow from Theorem 2.

Proposition 8 (folklore)**.** $(s) \neq \mathcal{B}or \triangle \mathcal{I}$ *for every* σ*-ideal* $\mathcal{I}$*.*

Proof. We provide a proof for the completeness. Suppose, for the sake of contradiction, that there is a σ-ideal $\mathcal{I}$ with $(s) = \mathcal{B}or \triangle \mathcal{I}$. We have two cases.

1. $\mathcal{I} \setminus (s_0) \neq \emptyset$.
2. $\mathcal{I} \subset (s_0)$.

In the first case, take an $A \in \mathcal{I} \setminus (s_0)$. Since $A \in (s)$ so there is a perfect set $P \subset A$. Now, take a set $B \subset P$ with $B \notin (s)$. On the other hand, $B \in \mathcal{I} \subset \mathcal{B}or \triangle \mathcal{I}$, a contradiction.

Now, we consider the second case. Let $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be a Borel isomorphism. It is not difficult to show, that $A \in (s)$ iff $f(A) \in (s^2)$ and $A \in (s_0)$ iff $f(A) \in (s_0^2)$. Here (s^2) and (s_0^2) stand for the σ-field of Marczewski measurable subsets of the plane and σ-ideal of Marczewski null subsets of the plane, which are defined similarly to (s) and (s_0).

Let $\mathcal{K} = \{f(A) : A \in \mathcal{I}\}$. Then $\mathcal{K}$ is a σ-ideal and $(s^2) = \mathcal{B}or^2 \triangle \mathcal{K}$. Here $\mathcal{B}or^2$ stands for the σ-field of Borel subsets of the plane.

Let $Z \in (s_0)$ be a set of cardinality 2^ω. Since $\{X \times \mathbb{R} : X \subset Z\} \subset (s^2)$ is of cardinality 2^{2^ω}, $|\mathcal{B}or| = 2^\omega$, and for each $X \subset Z$ there is a Borel set $B_X \subset \mathbb{R}$ and an $A_X \subset \mathbb{R}$, $A_X \in \mathcal{I}$ with $X \times \mathbb{R} = f(B_X) \triangle f(A_X)$, so there are two distinct sets $X_1, X_2 \subset Z$, a Borel set $B \subset \mathbb{R}$ and two sets $A_1, A_2 \in \mathcal{I}$ such that $X_1 \times \mathbb{R} = f(B) \triangle f(A_1)$ and $X_2 \times \mathbb{R} = f(B) \triangle f(A_2)$. Let $W = (X_1 \times \mathbb{R}) \triangle (X_2 \times \mathbb{R}) = f(A_1) \triangle f(A_2) = f(A_1 \triangle A_2)$. Then $f^{-1}(W) = A_1 \triangle A_2 \in \mathcal{I} \subset (s_0)$. On the other hand, since $X_1 \neq X_2$, so $W \notin (s_0^2)$. Thus $f^{-1}(W) \notin (s_0)$, a contradiction. $\square$

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