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ULTIMATELY INCREASING FUNCTIONS

Abstract

A function g between directed sets $\langle \Sigma, \succeq' \rangle$ and $\langle \Lambda, \succeq \rangle$ is called *ultimately increasing* if for each $\sigma_1 \in \Sigma$ there exists $\sigma_2 \succeq' \sigma_1$ such that $\sigma \succeq' \sigma_2 \Rightarrow g(\sigma) \succeq g(\sigma_1)$. A subnet of a net a defined on $\langle \Lambda, \succeq \rangle$ [9] is nothing but a composition of the form $a \circ g$ where g is ultimately increasing and $g(\Sigma)$ is a cofinal subset of Λ. While even for linearly ordered sets, an increasing net defined on a cofinal subset of the domain need not have an increasing extension, in complete generality, it must have an ultimately increasing extension, and conversely when the domain is linearly ordered. Applications are given in the context of functions with values in a linearly ordered set equipped with the order topology - in particular, the extended real numbers. For example, we show that a real sequence $\langle a_n \rangle$ converges to the supremum of its set of terms if and only if $\langle a_n \rangle$ is the supremum of the ultimately increasing sequences that it majorizes.

1 Introduction.

While sequences suffice to describe the basic constructs for a metric space topology, they fail to do so in the setting of a general topological space. For example, these two statements which are equivalent in metric spaces

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(1) each open cover of X has a finite subcover;

(2) each sequence in X has a subsequence that converges to a point of X;

are not in the context of a general topological space $\langle X, \mathscr{T} \rangle$ [12]. But such an equivalence holds if we replace sequences by nets, which are functions defined on more general "ordered" sets than the counting numbers $\mathbb{N}$ equipped with the usual order (see, e.g., [5, 8, 9]). Specifically, a *net* is a function defined on a *directed set* $\langle \Lambda, \succeq \rangle$, where $\succeq$ is a reflexive, transitive relation on Λ such that whenever $\lambda_1 \in \Lambda$ and $\lambda_2 \in \Lambda$, there exists $\lambda_3 \in \Lambda$ with both $\lambda_1 \preceq \lambda_3$ and $\lambda_2 \preceq \lambda_3$.

If a is a net defined on $\langle \Lambda, \succeq \rangle$, following the convention used for sequences, we will often write a_λ for $a(\lambda)$ for each index λ. Of course, a net $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ in a topological space $\langle X, \mathscr{T} \rangle$ is declared *convergent* to $x_0 \in X$ provided each neighborhood of x_0 contains the net *residually*, that is, it contains each a_λ for all λ satisfying $\lambda \succeq \lambda_0$ for some index λ_0. With this definition in mind, in a general topological space, condition (1) is equivalent to

($2'$) each net in X has a subnet that converges to a point of X,

where by a *subnet* of a net a with domain $\langle \Lambda, \succeq \rangle$, we mean a composition of the form $a \circ g$ where g is a function defined on a second directed set $\langle \Sigma, \succeq' \rangle$ with values in $\langle \Lambda, \succeq \rangle$ with the following property ($*$):

($*$) for each $\lambda \in \Lambda$, there exists $\sigma_1 \in \Sigma$ such that $\sigma \succeq' \sigma_1 \Rightarrow g(\sigma) \succeq \lambda$.

In the same spirit, a subset of a topological space $\langle X, \mathscr{T} \rangle$ is closed if and only if it is stable under taking limits of convergent nets; a function f defined on $\langle X, \mathscr{T} \rangle$ is continuous at a point $x_0 \in X$ if and only if it takes nets convergent to x_0 to nets convergent to $f(x_0)$; convergent nets have unique limits if and only if $\mathscr{T}$ satisfies the Hausdorff separation property, and so forth. Another adequate theory of convergence can be built around the notion of convergent filters; the two theories are essentially interchangeable (see, e.g., [5, 13]).

Nets seem on the surface to be so much like sequences that they are often called generalized sequences. But there is some annoying pathology lurking in the background. For example, a subnet of a sequence that is also defined on $\mathbb{N}$ need not be a subsequence of the initial sequence: $a_2, a_1, a_4, a_3, \ldots$ is a subnet of the sequence $a_1, a_2, a_3, a_4, \ldots$ because the function $g : \mathbb{N} \to \mathbb{N}$ defined by

$$g(n) = \begin{cases} n - 1, & \text{if } n \text{ is even} \\ n + 1, & \text{if } n \text{ is odd} \end{cases}$$

satisfies condition ($*$).

While a function g satisfying condition $(*)$ need not be increasing in the usual sense - that is, $\sigma_1 \preceq' \sigma_2$ need not ensure $g(\sigma_1) \preceq g(\sigma_2)$ - it is increasing in the following weaker sense.

Definition 1.1. A function $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ is called *ultimately increasing* if for each $\sigma_1 \in \Sigma$ there exists $\sigma_2 \succeq' \sigma_1$ such that $\sigma \succeq' \sigma_2 \Rightarrow g(\sigma) \succeq g(\sigma_1)$.

Intuitively, an ultimately increasing net does not go down in value in the long run (which might be an apt descriptor for typical investment objectives in these troubled economic times). Notice that the same functions are defined by replacing "$\sigma_2 \succeq' \sigma_1$" by "$\sigma_2 \in \Sigma$" in the definition.

It is the purpose of this note to investigate this property of functions between directed sets, but just as importantly, to illustrate how the idea imposes itself on the foundations of analysis. We obtain a number of basic structural results. For example, we show that each ultimately increasing net defined on a cofinal subset of a directed set can be extended to a globally defined ultimately increasing net. In particular, each increasing net defined on a cofinal subset has an ultimately increasing extension, and we show that each ultimately increasing globally defined net is so obtained when the domain is linearly ordered. As applications, we show how the concept is intrinsic to the Monotone Convergence Theorem from calculus, and show how the order topology for a complete chain is naturally induced by ultimately increasing nets, e.g., the extended real numbers equipped with its usual topology.

2 Preliminaries.

We initially review some notation and terminology relative to directed and partially ordered sets. For further details and for unexplained terminology, we invite the reader to consult [6].

All directed sets $\langle \Lambda, \succeq \rangle$ will be assumed to contain at least two elements. The notations $\lambda_2 \preceq \lambda_1$ and $\lambda_1 \succeq \lambda_2$ will be used interchangeably, and so forth. A subset Λ_0 of Λ is called *residual* if $\exists \lambda \in \Lambda$ such that $\forall \mu \succeq \lambda,\ \mu \in \Lambda_0$ and *cofinal* if $\forall \lambda \in \Lambda\ \exists \mu \succeq \lambda$ with $\mu \in \Lambda_0$. Clearly, Λ_0 is residual if and only if $\Lambda \backslash \Lambda_0$ is not cofinal. For $\Lambda_0 \subseteq \Lambda$, put

$$\uparrow \Lambda_0 := \{\mu \in \Lambda : \exists \lambda \in \Lambda_0 \text{ with } \mu \succeq \lambda\},$$

and

$$\downarrow \Lambda_0 := \{\mu \in \Lambda : \exists \lambda \in \Lambda_0 \text{ with } \mu \preceq \lambda\},$$

of course writing $\uparrow \lambda$ for $\uparrow \{\lambda\}$ and $\downarrow \lambda$ for $\downarrow \{\lambda\}$. Evidently, Λ_0 is a cofinal subset of Λ if and only if $\Lambda = \downarrow \Lambda_0$. Both arrow operators are evidently idempotent and monotone.

A directed set $\langle \Lambda, \succeq \rangle$ is in addition a *partially ordered set* if the relation $\succeq$ is antisymmetric as well as reflexive and transitive. The set of all nonempty subsets $\mathscr{P}_0(\mathbb{N})$ of $\mathbb{N}$ is directed by $B \succeq A$ provided there exists an injection f with domain A and codomain B, but $\langle \mathscr{P}_0(\mathbb{N}), \succeq \rangle$ fails to be a partially ordered set. Of course, a directed set becomes a partially ordered directed set modulo a natural equivalence relation [7, p. 11]. In a partially ordered directed set $\langle \Lambda, \succeq \rangle$, we will write $\lambda_1 \succ \lambda_2$ provided $\lambda_1 \succeq \lambda_2$ but $\lambda_1 \neq \lambda_2$. A partially ordered directed set $\langle \Lambda, \succeq \rangle$ is called a *linearly ordered set* or a *chain* provided whenever λ_1 and λ_2 are distinct members of Λ, either $\lambda_1 \succ \lambda_2$ or $\lambda_2 \succ \lambda_1$ holds. The *order topology* τ_{ord} on a linearly ordered set $\langle \Lambda, \succeq \rangle$ is generated by all sets of the form $\{\mu \in \Lambda : \mu \succ \lambda\}$ and $\{\mu \in \Lambda : \mu \prec \lambda\}$ where λ runs over Λ. For the extended reals $[-\infty, \infty]$ with the usual order, the order topology coincides with the usual topology.

We call a partially ordered directed set $\langle \Lambda, \succeq \rangle$ a *lattice* if each finite subset has both a greatest lower bound and a least upper bound in Λ. The natural numbers $\mathbb{N}$ with the division order forms a lattice, as does any linearly ordered set. We call a lattice $\langle \Lambda, \succeq \rangle$ a *complete lattice* if each nonempty subset E has both a greatest lower bound and a least upper bound, which we denote by $\inf E$ and $\sup E$ in the sequel. For example, the set of all subsets $\mathscr{P}(X)$ of a nonempty set X directed by inclusion is a complete lattice. Although $[-\infty, \infty]$ is a complete lattice, $\mathbb{R} = (-\infty, \infty)$ while a lattice is not a complete lattice. A linearly ordered set that is a complete lattice will be called a *complete chain* in the sequel.

We close this section with some examples of ultimately increasing functions.

Example 2.1. In the context of real functions, $g(x) = \frac{1}{2}x + \sin x$ while not increasing ($g'(x) < 0$ when $x \in (2\pi/3, \pi)$) nevertheless satisfies a uniform ultimately increasing condition: $\forall x \in \mathbb{R},\ g(w) \geq g(x)$ whenever $w \geq x + 4$.

Example 2.2. For functions into the Euclidean plane $\mathbb{R}^2$ directed by $(a_1, a_2) \preceq (b_1, b_2)$ provided $\sqrt{a_1^2 + a_2^2} \leq \sqrt{b_1^2 + b_2^2}$, the ultimately increasing condition is verified if and only if each function value is at least matched in distance from the origin by a residual set of values.

Example 2.3. Let $\Lambda = \{2^n 7^k : (n, k) \in \mathbb{N} \times \mathbb{N}\}$ equipped with the usual order inherited from $\mathbb{N}$, and define $g : \Lambda \to \mathbb{N}$ by $g(2^n 7^k) = n + k$. While g is ultimately increasing, g fails to be increasing as $g(98) < g(56)$.

Example 2.4. The product $\Lambda := \{a, b\} \times \mathbb{N}$ is linearly ordered by the lexicographic order, a.k.a the dictionary order (see, e.g., [10, 13]). Define $g : \Lambda \to \Lambda$ by

$$g(a, n) = g(b, n) = (a, n) \quad (n \in \mathbb{N}).$$

While g is not increasing, it is ultimately increasing because if $(x, n) \in \Lambda$, we have $g(b, k) \succeq g(x, n)$ whenever $k \geq n$.

Example 2.5. Let $\langle x_n \rangle$ be a strictly increasing convergent real sequence, and for each $n \in \mathbb{N}$, put $y_n := x_n - (x_{n+1} - x_n)$. Then the sequence $x_1, y_1, x_2, y_2, \ldots$ viewed as a function from $\mathbb{N}$ into $\mathbb{R}$ is ultimately increasing but does not satisfy condition $(*)$ of the Introduction as it is bounded above by $\lim_{n\to\infty} x_n$.

3 Structural results.

We first record some obvious properties of ultimately increasing functions that the reader can verify:

- $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ is ultimately increasing if and only if for each $\sigma_1 \in \Sigma$ there exists $\sigma_2 \in \Sigma$ such that $g(\uparrow \sigma_2) \subseteq \uparrow g(\sigma_1)$;
- a subnet of a ultimately increasing net is ultimately increasing; in particular, the restriction of a ultimately increasing net to a cofinal subset of the domain remains ultimately increasing;
- the composition of an increasing net following an ultimately increasing net is ultimately increasing;
- if $\langle \Sigma, \succeq' \rangle$ has a largest member σ_1, then $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ is ultimately increasing if and only if $\forall \sigma \in \Sigma,\ g(\sigma) \preceq g(\sigma_1)$.
- a net with values in a power set $\mathscr{P}(X)$ directed by inclusion is ultimately increasing if and only if whenever $A_1, A_2, \ldots A_n$ are values of the net, then eventually each value contains $\cup_{i=1}^{n} A_i$;
- the real-valued ultimately increasing functions defined on a fixed directed set are closed under addition and multiplication by nonnegative scalars; in particular, a convex combination of ultimately increasing functions is ultimately increasing;
- a continuously differentiable real function g defined on $\mathbb{R}$ is ultimately increasing if and only if $\forall a \in \mathbb{R}, \exists r(a) > a$ such that $\forall t \geq r(a), \int_a^t g'(x)dx \geq 0$.

It is the not the case that the inverse of an ultimately increasing injective net be ultimately increasing. Define $g : \mathbb{Z} \to \{0, 1, 2, \ldots\}$ by

$$g(n) = \begin{cases} 2|n| - 1 & \text{if } n < 0 \\ 2n & \text{if } n \geq 0. \end{cases}$$

It is clear that g is ultimately increasing and bijective. However, its inverse is not ultimately increasing because whenever $n \in \mathbb{N}$, then whenever k is odd and $k > 2n$, we have $g^{-1}(k) < g^{-1}(2n)$.

As we have noted earlier, a subnet of an increasing net need not be increasing. Further, the composition of an ultimately increasing net following an increasing net need not be ultimately increasing: while $g : \mathbb{N} \to \mathbb{R}$ defined by $g(n) = -\frac{1}{n}$ is increasing and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is ultimately increasing, $f \circ g$ is not ultimately increasing.

In the Introduction, we defined a subnet of a net a defined on a directed set $\langle \Lambda, \succeq \rangle$ as a composition where the outside function was a and the inside function g defined on a separate directed set was to satisfy condition $(*)$. Condition $(*)$ is nothing but the requirement that g be ultimately increasing plus an additional property.

Proposition 3.1. *For a function $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ between directed sets, the following conditions are equivalent:*

(a) g satisfies condition $(*)$;

(b) g is ultimately increasing and its range is cofinal in Λ.

Proof. $(a) \Rightarrow (b)$. If g satisfies $(*)$, clearly $g(\Sigma)$ is cofinal in Λ, and letting λ run over the range of g in $(*)$, by the first bulleted item above, g is also ultimately increasing.

$(b) \Rightarrow (a)$. Fix $\lambda \in \Lambda$. By cofinality of the range, $\exists \sigma_0 \in \Sigma$ with $g(\sigma_0) \succeq \lambda$. Since g is ultimately increasing, $\exists \sigma_1 \succeq' \sigma_0$ such that $\sigma \succeq' \sigma_1 \Rightarrow g(\sigma) \succeq g(\sigma_0) \succeq \lambda$, as required. $\square$

While an increasing function defined on an infinite subset of the positive integers $\mathbb{N}$ can be extended to an increasing function on all of $\mathbb{N}$, it is easy to see that an increasing function defined on a cofinal subset of a general directed set need not extend to an increasing function defined on the entire directed set. Consider $\Sigma = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\Lambda = \{\frac{1}{n} : n \in \mathbb{N}\}$, both equipped with the usual order they inherit from $\mathbb{R}$; then the identity map on Λ does not extend to an increasing function from Σ to Λ. We intend to show that an increasing function does however extend to an ultimately increasing function. This is an immediate consequence of the following comprehensive result.

Theorem 3.2. *Suppose $\langle \Sigma, \succeq' \rangle$ and $\langle \Lambda, \succeq \rangle$ are two directed sets. Suppose $\widehat{\Sigma}$ is a cofinal subset of Σ and $g : \langle \widehat{\Sigma}, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ is ultimately increasing. Then g extends to an ultimately increasing function with domain Σ.*

Proof. We first create a function $h_g : \widehat{\Sigma} \to \widehat{\Sigma}$ that picks for each $\widehat{\sigma} \in \widehat{\Sigma}$ an element $h_g(\widehat{\sigma})$ of $\widehat{\Sigma}$ such that $h_g(\widehat{\sigma}) \succeq' \widehat{\sigma}$ and $g(\{\mu \in \widehat{\Sigma} : \mu \succeq' h_g(\widehat{\sigma})\}) \subseteq \uparrow g(\widehat{\sigma})$.

Let $\mathscr{D}$ be the set of all pairs of functions (f, h_f) where the domains of f and h_f are a common subset $D(f)$ of Σ such that the following properties hold

(1) f extends g and h_f extends h_g on $\widehat{\Sigma}$;

(2) h_f takes values in $\widehat{\Sigma}$ and for each $\sigma \in D(f)$, $h_f(\sigma) \succeq' \sigma$ and $f(\uparrow h_f(\sigma) \cap D(f)) \subseteq \uparrow f(\sigma)$.

Property (2) guarantees that for each $(f, h_f) \in \mathscr{D}$, the function f is ultimately increasing on its domain $D(f)$. Notice that $\mathscr{D}$ is nonempty as it contains (g, h_g). Partially order $\mathscr{D}$ by $(f_1, h_{f_1}) \trianglelefteq (f_2, h_{f_2})$ provided f_2 extends f_1 and h_{f_2} extends h_{f_1}. It is left to the reader to verify that each chain in $\mathscr{D}$ has an upper bound, namely the minimal extension for the function pairs (f, h_f) in the chain. By Zorn's lemma, $\mathscr{D}$ has a maximal element (f_0, h_{f_0}). It remains to show that $D(f_0) = \Sigma$.

Suppose to the contrary that $\mu \in \Sigma \backslash D(f_0)$. By cofinality of $\widehat{\Sigma}$ in Σ, choose $\widehat{\sigma} \in \widehat{\Sigma}$ with $\mu \preceq' \widehat{\sigma}$. Put $f(\mu) := f_0(\widehat{\sigma}) = g(\widehat{\sigma})$, put $h_f(\mu) := h_{f_0}(\widehat{\sigma}) = h_g(\widehat{\sigma})$, and let (f, h_f) restricted to $D(f_0)$ agree with (f_0, h_{f_0}). We claim that this determines an extension of (f_0, h_{f_0}) that lies in $\mathscr{D}$.

To see that f satisfies (2), we consider two cases for $\sigma \in D(f)$: (i) $\sigma = \mu$, and (ii) $\sigma \neq \mu$. In case (i), whether or not $\mu \in \uparrow h_{f_0}(\widehat{\sigma})$ holds, we compute

$$f(\uparrow h_f(\mu) \cap D(f)) \subseteq \{f(\mu)\} \cup f_0(\uparrow h_{f_0}(\widehat{\sigma}) \cap D(f_0)) \subseteq \{f(\mu)\} \cup \uparrow f_0(\widehat{\sigma}) = \uparrow f(\mu).$$

In case (ii) where $\sigma \in D(f_0)$, we have

$$f(\uparrow h_{f_0}(\sigma) \cap D(f_0)) \subseteq \uparrow f_0(\sigma) = \uparrow f(\sigma).$$

But if $\mu \in \uparrow h_{f_0}(\sigma)$, then $h_{f_0}(\sigma) \preceq \mu \preceq \widehat{\sigma}$, and since $\widehat{\sigma} \in D(f_0)$, we get

$$f(\sigma) = f_0(\sigma) \preceq f_0(\widehat{\sigma}) = f(\mu).$$

This shows that

$$f(\uparrow h_f(\sigma) \cap D(f)) \subseteq \uparrow f(\sigma)$$

in the second case. Thus, the maximality of (f_0, h_{f_0}) is violated, and the proof is complete. $\square$

Theorem 3.2 of course yields

Theorem 3.3. *Suppose $\langle \Sigma, \succeq' \rangle$ and $\langle \Lambda, \succeq \rangle$ are two directed sets. Suppose $\widehat{\Sigma}$ is a cofinal subset of Σ and $g : \langle \widehat{\Sigma}, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ is increasing. Then g extends to an ultimately increasing function with domain Σ.*

We now give a partial converse to Theorem 3.3.

Theorem 3.4. *Let $\langle \Sigma, \succeq' \rangle$ be linearly ordered and $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ be ultimately increasing. Then g restricted to some cofinal subset of Σ is increasing.*

Proof. This is obviously true if Σ has a largest element as g restricted to any singleton is trivially increasing, so we assume in the following argument that Σ does not have one.

We define a family of subsets $\mathscr{A}$ of Σ as follows:

$$\mathscr{A} := \{\Gamma \subseteq \Sigma : \text{whenever } \sigma_1 \prec' \sigma_2 \text{ in } \Gamma, \text{ then } \forall \sigma \in \Sigma, \sigma \succeq' \sigma_2 \Rightarrow \ g(\sigma) \succeq g(\sigma_1)\}.$$

Given any $\sigma_1 \in \Sigma$, that g is ultimately increasing ensures $\mathscr{A}$ contains some subset of Σ with exactly two elements containing σ_1: choose $\sigma_2 \succeq' \sigma_1$ with $g(\uparrow \sigma_2) \subseteq \uparrow g(\sigma_1)$ and choosing $\sigma_3 \succ' \sigma_2$, $\{\sigma_1, \sigma_3\}$ does the job. Partially order $\mathscr{A}$ by inclusion. It is a simple exercise to show that if $\mathscr{C}$ is any linearly ordered subset of $\mathscr{A}$, then $\{\sigma \in \Sigma : \exists \Gamma \in \mathscr{C} \text{ with } \sigma \in \Gamma\}$ is an upper bound for $\mathscr{C}$ in $\mathscr{A}$. By Zorn's Lemma, $\mathscr{A}$ has a largest member Γ_0. It remains to show that Γ_0 is cofinal in Σ. If cofinality fails, we consider two cases for Γ_0: (1) Γ_0 has a largest member σ_L; (2) Γ_0 has no largest member.

In case (1), since we are now assuming that Σ has no largest member, σ_L is not the largest member of Σ, and there exists $\alpha \succ' \sigma_L$ such that $g(\uparrow \alpha) \subseteq \uparrow g(\sigma_L)$. But whenever $\sigma \in \Gamma_0$ with $\sigma \prec' \sigma_L$ we have $g(\sigma_L) \in \uparrow g(\sigma)$, and we compute

$$g(\uparrow \alpha) \subseteq \uparrow g(\sigma_L) \subseteq \uparrow (\uparrow g(\sigma)) = \uparrow g(\sigma).$$

This proves that $\Gamma_0 \cup \{\alpha\} \in \mathscr{A}$, in violation of Γ_0 being the largest member of $\mathscr{A}$.

In case (2) take $\mu \notin \downarrow \Gamma_0$ and let $\sigma_1 \in \Gamma_0$ be arbitrary. There exists $\sigma_2 \in \Gamma_0$ with $\sigma_1 \prec' \sigma_2$. Then since $\uparrow \mu \subseteq \uparrow \sigma_2$ we have

$$g(\uparrow \mu) \subseteq g(\uparrow \sigma_2) \subseteq \uparrow g(\sigma_1).$$

This shows that $\Gamma_0 \cup \{\mu\} \in \mathscr{A}$, and we again reach a contradiction. Clearly, g restricted to this cofinal element of $\mathscr{A}$ is increasing, completing the proof. $\square$

Remark 3.5. We note that the proof of Theorem 3.4 breaks down if we simply try to apply Zorn's Lemma to the family of subsets Γ of Σ for which g restricted to Γ is increasing.

Corollary 3.6. *Suppose $\langle \Sigma, \succeq' \rangle$ is linearly ordered and $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ is ultimately increasing. Then g has an increasing subnet defined on a linearly ordered set.*

In view of Proposition 3.1, we may state the following corollary.

Corollary 3.7. *Suppose $\langle \Sigma, \succeq' \rangle$ is linearly ordered and $g : \langle \Sigma, \succeq' \rangle \to \langle \Lambda, \succeq \rangle$ satisfies property $(*)$. Then g has an increasing subnet defined on a linearly ordered set that satisfies $(*)$.*

4 Sequences and nets that are convergent to their suprema.

The elementary Monotone Convergence Theorem of one-variable calculus says that an increasing sequence $\langle a_n \rangle$ of real numbers converges to the extended real number $\sup\{a_n : n \in \mathbb{N}\}$. But clearly a real sequence can converge to the supremum of its set of terms without being increasing. In this section we expose the characteristic features of sequences that converge to the supremum of their set of terms, a theory in which ultimately increasing sequences play a fundamental role. Since most of the analysis can be executed for nets with values in a complete chain, we work at this more general level. We produce multiple characterizations of nets that converge to their suprema.

As a first result, we show that an ultimately increasing net with values in a complete chain must converge to the supremum of its values.

Proposition 4.1. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Then each ultimately increasing net $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ in Λ converges in the order topology to sup$\{a_\sigma : \sigma \in \Sigma\}$.*

Proof. If the supremum of the set of terms is the smallest element of Λ, then the net is constant and there is nothing to prove. Otherwise, let $\lambda \prec \sup\{a_\sigma : \sigma \in \Sigma\}$ be arbitrary. By the definition of supremum, there exists σ_1 with $a_{\sigma_1} \succ \lambda$, and then σ_2 such that $\sigma \succeq' \sigma_2 \Rightarrow a_\sigma \succeq a_{\sigma_1}$. Thus $\mu \in \uparrow \sigma_2 \Rightarrow \lambda \prec a_\mu \preceq \sup\{a_\sigma : \sigma \in \Sigma\}$ as required. $\square$

Proposition 4.2. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose that $\{\langle b^i_\sigma \rangle_{\sigma \in \Sigma} : i \in I\}$ is a family of ultimately increasing nets in Λ and for each $\sigma \in \Sigma$, put $a_\sigma := \sup_{i \in I} b^i_\sigma$. Then $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ converges in the order topology to the supremum of its values.*

Proof. Put $\widehat{\lambda} := \sup\,\{a_\sigma : \sigma \in \Sigma\}$, and let $\lambda \prec \widehat{\lambda}$ be arbitrary. Choose an index σ_1 such that $a_{\sigma_1} \succ \lambda$. Next pick $i \in I$ such that $b^i_{\sigma_1} \succ \lambda$. Choosing $\sigma_2 \succ' \sigma_1$ such that $\forall \sigma \succeq' \sigma_2,\ b^i_\sigma \succeq b^i_{\sigma_1}$, we have $\sigma \succeq' \sigma_2 \Rightarrow a_\sigma \succ \lambda$. It now follows that $\liminf_{\sigma \in \Sigma} a_\sigma \succeq \lambda$, and as $\limsup_{\sigma \in \Sigma} a_\sigma \preceq \sup\,\{a_\mu : \mu \in \Sigma\}$, the net converges to $\sup\,\{a_\mu : \mu \in \Sigma\}$. $\square$

We next give our first characterization of those nets that converge to the supremum of their set of values. It is fairly transparent; more interesting ones will follow.

Theorem 4.3. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is a net in Λ. The following conditions are equivalent:*

(1) $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is convergent in the order topology to sup $\{a_\sigma : \sigma \in \Sigma\}$;

(2) $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ majorizes an increasing net $\langle b_\sigma \rangle_{\sigma\in\Sigma}$ for which

$$\textit{sup}\ \{a_\sigma : \sigma \in \Sigma\} = \textit{sup}\ \{b_\sigma : \sigma \in \Sigma\};$$

(3) $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ majorizes an ultimately increasing net $\langle b_\sigma \rangle_{\sigma\in\Sigma}$ for which

$$\textit{sup}\ \{a_\sigma : \sigma \in \Sigma\} = \textit{sup}\ \{b_\sigma : \sigma \in \Sigma\}.$$

Proof. (1) $\Rightarrow$ (2). For each $\sigma \in \Sigma$, put $b_\sigma = \inf\{a_\mu : \mu \succeq' \sigma\}$. Then $\langle b_\sigma \rangle_{\sigma\in\Sigma}$ is an increasing net convergent to sup $\{b_\sigma : \sigma \in \Sigma\}$. But by the definition of the net, it also converges to $\liminf_{\sigma\in\Sigma} a_\sigma = \lim_{\sigma\in\Sigma} a_\sigma = \sup\ \{a_\sigma : \sigma \in \Sigma\}$.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). By Proposition 4.1 and condition (3), $\langle b_\sigma \rangle_{\sigma\in\Sigma}$ converges to sup $\{a_\sigma : \sigma \in \Sigma\}$. But clearly

$$\lim_{\sigma\in\Sigma} b_\sigma \preceq \liminf_{\sigma\in\Sigma} a_\sigma \preceq \limsup_{\sigma\in\Sigma} a_\sigma \preceq \sup\ \{a_\sigma : \sigma \in \Sigma\},$$

from which condition (1) follows. $\square$

Since the infima of the values of each tail of a convergent sequence is finite, we obtain this consequence of Theorem 4.3.

Corollary 4.4. *Let $\langle a_n \rangle$ be a real sequence. The following are equivalent:*

(1) $\lim_{n\to\infty} a_n = \textit{sup}\ \{a_n : n \in \mathbb{N}\}$;

(2) $\langle a_n \rangle$ majorizes an increasing sequence $\langle b_n \rangle$ such that sup $\{a_n : n \in \mathbb{N}\} =$ sup $\{b_n : n \in \mathbb{N}\}$;

(3) $\langle a_n \rangle$ majorizes an ultimately increasing sequence $\langle b_n \rangle$ such that sup $\{a_n : n \in \mathbb{N}\} =$ sup $\{b_n : n \in \mathbb{N}\}$.

That $\langle a_n \rangle$ majorizes an increasing sequence $\langle b_n \rangle$ with $\lim_{n\to\infty} a_n - b_n = 0$ is not enough to guarantee that condition (1) of Corollary 4.4 holds.

Example 4.5. Consider the real sequence $\langle a_n \rangle$ whose list of terms is $\frac{1}{2}, 0, 0, 0, \ldots$. Then the increasing sequence $\langle b_n \rangle$ whose list of terms is $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \ldots$ satisfies $a_n - b_n = \frac{1}{n}$. However, $\langle a_n \rangle$ fails to converge to $\frac{1}{2} = \sup\,\{a_n : n \in \mathbb{N}\}$.

While Theorem 4.3 does not support conferring special status to ultimately increasing nets, the next result and its consequences certainly do.

Proposition 4.6. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is a net in Λ such that $\forall \sigma \in \Sigma,\ a_\sigma \prec \sup\{a_\mu : \mu \in \Sigma\}$. Then $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is τ_{ord}-convergent to $\sup\{a_\sigma : \sigma \in \Sigma\}$ if and only if the net is ultimately increasing.*

PROOF. Sufficiency follows from Proposition 4.1. For necessity, suppose the net fails to be ultimately increasing. Then there exists σ_1 and a cofinal subset Ω of Σ such that $\forall \omega \in \Omega,\ a_\omega \prec a_{\sigma_1}$. As a result, with $\lambda := a_{\sigma_1} \prec \sup\{a_\sigma : \sigma \in \Sigma\}$, it is not the case that a_σ exceeds λ eventually. $\square$

Example 4.7. The sequence defined by $a_1 = 1$ and $a_n = 1 - \frac{1}{n}$ for $n \geq 2$ converges to the supremum of its terms but fails to be ultimately increasing (note that the supremum is achieved).

We now come to our favorite result of this paper.

Theorem 4.8. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is a net in Λ. Then $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ converges to $\sup\{a_\mu : \mu \in \Sigma\}$ if and only if $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is the supremum of the ultimately increasing nets based on Σ that it majorizes.*

PROOF. Sufficiency is a consequence of Proposition 4.2. For necessity, suppose $\lim_{\sigma \in \Sigma}\ a_\sigma = \sup\{a_\mu : \mu \in \Sigma\}$. If the net is already ultimately increasing there is nothing to show. Otherwise, put $\widehat{\lambda} := \sup\{a_\mu : \mu \in \Sigma\}$. Since the net is assumed not to be ultimately increasing, we first notice that $\widehat{\lambda}$ cannot have an immediate predecessor in Λ, else by convergence, the net would assume the value $\widehat{\lambda}$ on a residual subset and thereby be ultimately increasing.

Let $\mathfrak{B}$ denote the set of ultimately increasing nets majorized by $\langle a_\sigma \rangle_{\sigma \in \Sigma}$. Since Λ has a smallest member, $\mathfrak{B}$ is nonempty. It suffices to show that for each $\sigma_0 \in \Sigma$ whenever $\lambda \prec a_{\sigma_0}$, there exists $\langle c_\sigma \rangle_{\sigma \in \Sigma} \in \mathfrak{B}$ with $c_{\sigma_0} \succ \lambda$. For each $\sigma \in \Sigma$, put $b_\sigma = \inf\{a_\mu : \mu \succeq' \sigma\}$, defining an increasing net that is majorized by $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ and that is convergent to $\widehat{\lambda}$. If $a_{\sigma_0} = \widehat{\lambda}$, pick λ_1 with $\lambda \prec \lambda_1 \prec \widehat{\lambda}$ and define $\langle c_\sigma \rangle_{\sigma \in \Sigma}$ by

$$c_\sigma = \begin{cases} \lambda_1 & \text{if } \sigma = \sigma_0 \\ b_\sigma & \text{otherwise,} \end{cases}$$

and in the case that $a_{\sigma_0} \prec \widehat{\lambda}$, let

$$c_\sigma = \begin{cases} a_{\sigma_0} & \text{if } \sigma = \sigma_0 \\ b_\sigma & \text{otherwise.} \end{cases}$$

Either way, since $c_{\sigma_0} \prec \widehat{\lambda}$, $\langle c_\sigma \rangle_{\sigma \in \Sigma}$ is ultimately increasing and has the required property when $\sigma = \sigma_0$. $\square$

Corollary 4.9. *Suppose $\langle a_n \rangle$ is a real sequence. Then $lim_{n\to\infty} a_n = sup\ \{a_n : n \in \mathbb{N}\}$ if and only $\langle a_n \rangle$ is the supremum of the ultimately increasing sequences that it majorizes.*

There is a related characterization of real sequences that are convergent to their suprema that is perhaps more appealing.

Theorem 4.10. *A real sequence $\langle a_n \rangle$ satisfies $lim_{n\to\infty} a_n = sup\ \{a_n : n \in \mathbb{N}\}$ if and only if it is the uniform limit of a sequence of ultimately increasing sequences.*

PROOF. Suppose for each $j \in \mathbb{N}$, $\langle b_n^j \rangle_{n \in \mathbb{N}}$ is an ultimately increasing sequence with $\sup_{n\in\mathbb{N}} |b_n^j - a_n| < \frac{1}{j}$. For each n and j put $c_n^j := b_n^j - \frac{1}{j}$. Then for each n, $a_n = \sup_{j\in\mathbb{N}} c_n^j$, so by Proposition 4.2, $\langle a_n \rangle$ converges to its supremum. Conversely, suppose $\langle a_n \rangle$ converges to its supremum. Let $\varepsilon > 0$ be arbitrary, and for each n put $b_n^\varepsilon = a_n - 2^{-n}\varepsilon$. Then by Proposition 4.6, $\langle b_n^\varepsilon \rangle$ is ultimately increasing and $||\langle b_n^\varepsilon \rangle - \langle a_n \rangle||_\infty < \varepsilon$. $\square$

For a real-valued function f defined on a Hausdorff space that is locally bounded below, there is a largest real-valued lower semicontinuous function g majorized by f (called the *lower envelope* of f) [9, pg. 102]. Under mild conditions on the space, the lower envelope is the supremum of the continuous real-valued functions that f majorizes, provided f majorizes one at all [5, pg. 88]. We now present as a consequence of Theorem 4.8 a parallel result for nets that converge to their supremum.

Proposition 4.11. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is a net in Λ. Then there is a largest net $\langle a_\sigma^* \rangle_{\sigma\in\Sigma}$ majorized by $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ that is convergent to the supremum of its set of values.*

PROOF. Put $\lambda_0 := \inf \{a_\sigma : \sigma \in \Sigma\}$. Then the net with constant value λ_0 is an ultimately increasing net majorized by $\langle a_\sigma \rangle_{\sigma\in\Sigma}$. Letting $\mathfrak{B}$ be as in the proof of Theorem 4.8, we see that $\mathfrak{B}$ is nonempty. Let a^* be the supremum of the set of nets in $\mathfrak{B}$, that is, put $a_\sigma^* := \sup\{b_\sigma : b \in \mathfrak{B}\}$. By Proposition 4.2 or Theorem 4.8, $\langle a_\sigma^* \rangle_{\sigma\in\Sigma}$ converges to the supremum of its values. Suppose $\langle c_\sigma \rangle_{\sigma\in\Sigma}$ were another such net majorized by $\langle a_\sigma \rangle_{\sigma\in\Sigma}$. If for some $\sigma_0 \in \mathbb{N}$, $c_{\sigma_0} \succ a^*_{\sigma_0}$, then again by Theorem 4.8 there exists an ultimately increasing net $\langle d_\sigma \rangle_{\sigma\in\Sigma}$ such that $\forall\sigma$, $d_\sigma \preceq c_\sigma$ and $a^*_{\sigma_0} \prec d_{\sigma_0}$. But clearly $\langle d_\sigma \rangle_{\sigma\in\Sigma} \in \mathfrak{B}$, and a contradiction ensues. $\square$

Some care must be taken to obtain as a special case a result for real sequences, as a real sequence need not majorize an ultimately increasing one. The correct formulation is as follows.

Corollary 4.12. *Let $\langle a_n \rangle$ be a real sequence. The following conditions are equivalent:*

(1) inf $\{a_n : n \in \mathbb{N}\} > -\infty$;

(2) $\langle a_n \rangle$ majorizes an ultimately increasing sequence $\langle b_n \rangle$;

(3) there is a largest sequence $\langle a^*_n \rangle$ majorized by $\langle a_n \rangle$ convergent to sup $\{a^*_n : n \in \mathbb{N}\}$.

We are not enamored with our proof of Proposition 4.11, as it obscures the essential nature of the largest net convergent to its supremum underneath $\langle a_\sigma \rangle_{\sigma\in\Sigma}$. This nature is made clear by our next result.

Theorem 4.13. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is a net in Λ. If we put*

$$a^*_\sigma := \min\, \{a_\sigma, \liminf_{\mu\in\Sigma}\, a_\mu\} \quad (\sigma \in \Sigma),$$

then $\langle a^*_\sigma \rangle_{\sigma\in\Sigma}$ is the largest net majorized by $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ that is convergent to the supremum of its set of values.

PROOF. Suppose $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ majorizes $\langle b_\sigma \rangle_{\sigma\in\Sigma}$ where the latter is convergent to $\sup\{b_\mu : \mu \in \Sigma\}$. Evidently,

$$\sup\{b_\mu : \mu \in \Sigma\} = \lim_{\sigma\in\Sigma} b_\sigma \preceq \liminf_{\sigma\in\Sigma} a_\sigma,$$

and so for each $\sigma \in \Sigma$ we have $b_\sigma \preceq \min\, \{a_\sigma, \liminf_{\mu\in\Sigma} a_\mu\}$. It is left as an easy exercise to verify that for a^*_σ as defined, we have $\sup\, \{a^*_\sigma : \sigma \in \Sigma\} = \lim_{\sigma\in\Sigma} a^*_\sigma = \liminf_{\sigma\in\Sigma} a_\sigma$. $\square$

Theorem 4.13 yields a third characterization of nets convergent to their suprema, and arguably the most important one, as elementary as it may be. The reader is invited to give a self-contained proof.

Corollary 4.14. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Then a net $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ in Λ is convergent to the supremum of its set of values if and only if $\forall\sigma \in \Sigma$, $a_\sigma \preceq \liminf_{\mu\in\Sigma}\, a_\mu$.*

We note that the condition $\forall \sigma \in \Sigma,\ a_\sigma \preceq \liminf_{\mu\in\Sigma} a_\mu$ in the statement of Corollary 4.14 is clearly equivalent to either of the following statements: (i) $\sup\{a_\sigma : \sigma \in \Sigma\} \preceq \liminf_{\sigma\in\Sigma} a_\sigma$, or (ii) $\sup\{a_\sigma : \sigma \in \Sigma\} = \liminf_{\sigma\in\Sigma} a_\sigma$.

In response to the shortcomings of Example 4.7, we present

Proposition 4.15. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is a net in Λ, such that $\sup\{a_\sigma : \sigma \in \Sigma\}$ is not achieved cofinally. Then $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is τ_{ord}-convergent to $\sup\{a_\sigma : \sigma \in \Sigma\}$ if and only if for some $\sigma_0 \in \Sigma$, the net restricted to $\uparrow \sigma_0$ is ultimately increasing and $\sup\{a_\sigma : \sigma \in \uparrow \sigma_0\} = \sup\{a_\sigma : \sigma \in \Sigma\}$.*

Proof. Sufficiency follows immediately from Proposition 4.1 without the initial assumption on the net. For necessity, we can find $\sigma_0 \in \Sigma$ such that the net restricted to $\uparrow \sigma_0$ does not achieve the value $\sup\{a_\sigma : \sigma \in \Sigma\}$. Since the limit of the net so restricted must still be $\sup\{a_\sigma : \sigma \in \Sigma\}$, we must have $\sup\{a_\sigma : \sigma \in \uparrow \sigma_0\} = \sup\{a_\sigma : \sigma \in \Sigma\}$. Apply Proposition 4.6. □

Example 4.16. The sequence $1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, \ldots$ while convergent to 1 fails to be ultimately increasing on any residual subset of $\mathbb{N}$.

Corollary 4.17. *Let $\langle \Sigma, \succeq' \rangle$ be a directed set and let $\langle \Lambda, \succeq \rangle$ be a complete chain. Suppose $\langle a_\sigma \rangle_{\sigma\in\Sigma}$ is a net in Λ that is τ_{ord}-convergent to some $\widetilde{\lambda}$. Then $\widetilde{\lambda} = \sup\{a_\sigma : \sigma \in \Sigma\}$ if and only if one of the conditions below holds:*

(1) $a_\omega = \sup\{a_\sigma : \sigma \in \Sigma\}$ for a cofinal set of indices ω;

(2) for some $\sigma_0 \in \Sigma$, the net restricted to $\uparrow \sigma_0$ is ultimately increasing and $\sup\{a_\sigma : \sigma \succeq' \sigma_0\} = \sup\{a_\sigma : \sigma \in \Sigma\}$.

There is of course a second Monotone Convergence Theorem in real analysis, that of integration theory: if $\langle f_n \rangle$ is an increasing sequence of nonnegative measurable functions pointwise convergent to f almost everywhere on a measurable set E, then $\int_E f = \lim_{n\to\infty} \int_E f_n$. By virtue of Proposition 4.1, the standard proof via Fatou's Lemma (see, e.g. [11, pg. 78]) goes through if we replace the increasing assumption by the requirement that $\forall x \in E, \langle f_n(x) \rangle$ is an ultimately increasing sequence of extended nonnegative reals.

5 Ultimately increasing nets and the order topology.

Dualizing the notion of ultimately increasing net, we declare a net g from $\langle \Sigma, \succeq' \rangle$ to $\langle \Lambda, \succeq \rangle$ to be *ultimately decreasing* if for each $\sigma_1 \in \Sigma$ there exists $\sigma_2 \succeq' \sigma_1$ such that $\sigma \succeq' \sigma_2 \Rightarrow g(\sigma) \preceq g(\sigma_1)$. Of course g is called *ultimately monotone* if either g is ultimately increasing or ultimately decreasing. We close this note by describing how the ultimately monotone nets determine the order topology τ_{ord} on a complete chain.

Given a topology $\mathscr{T}$ on a set X, a net $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ in X determines a certain set of points to which it converges. For example, if $\mathscr{T} = \{X, \emptyset\}$, then the set of points to which each net in X converges is X itself. At the other extreme, if $\mathscr{T}$ is the power set of X, then a net $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ converges to $x_0 \in X$ if and only if $a_\lambda = x_0$ eventually, so the set of points to which a net converges is either empty or a singleton. Whatever the topology may be, in general

(1) if $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ is a constant net, then the net converges to the repeated value,

and

(2) if $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ converges to $x_0 \in X$, then the same is true for each subnet.

Abstracting from this situation, by a *convergence* on a set X we mean a rule that assigns to each net in X a possibly empty subset of X - to be viewed as the set of points to which the net converges - in a way such that conditions (1) and (2) above are satisfied. A priori, there is no reason that a particular convergence be topological - that is, be induced by a topology $\mathscr{T}$ on X. There are well-known necessary and sufficient conditions for this to occur, the most subtle of which is an iterated limit condition for the convergence [9, p. 74].

If $\mathbf{Q}$ is a convergence on X we will represent the convergence of $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ to x_0 with respect to $\mathbf{Q}$ by

$$\langle a_\lambda \rangle_{\lambda \in \Lambda} \overset{\mathbf{Q}}{\to} x_0.$$

If $\mathbf{P}$ is a second convergence on X we say that $\mathbf{Q}$ is *finer* than $\mathbf{P}$, or $\mathbf{P}$ is *coarser* than $\mathbf{Q}$, if for each net $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ in X and for each $x \in X$,

$$\langle a_\lambda \rangle_{\lambda \in \Lambda} \overset{\mathbf{Q}}{\to} x \Rightarrow \langle a_\lambda \rangle_{\lambda \in \Lambda} \overset{\mathbf{P}}{\to} x.$$

Identifying a topology $\mathscr{T}$ with the convergence it determines, we can speak of a topology as being either finer or coarser than a particular convergence on X. For two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on X, under this identification, we have $\mathscr{T}_1$ is finer than $\mathscr{T}_2$ as convergences if and only if $\mathscr{T}_2 \subseteq \mathscr{T}_1$ as topologies.

Whether or not a convergence $\mathbf{Q}$ in X is induced by a topology, as perhaps first noticed by Choquet [2, p. 86], there is always a largest topology $\tau\mathbf{Q}$ on X that is coarser than $\mathbf{Q}$. Called either the *modification* or the *topologization* of $\mathbf{Q}$, the closed sets of $\tau\mathbf{Q}$ can easily be shown to consist of those subsets A of X that are stable under taking $\mathbf{Q}$-limits of nets in A (see [3, Lemnma 2.1], and for convergence of filters instead of nets, [1, Thm 1.3.9] or [4, p.

123]). If a convergence is induced by a topology, then that topology must be its modification.

This brings us back to ultimate monotonicity.

Let $\langle \Lambda, \succeq \rangle$ be a complete chain. We intend to show that the convergence $\mathbf{Q}$ on $\langle \Lambda, \succeq \rangle$ described by $\langle a_\sigma \rangle_{\sigma \in \Sigma} \stackrel{\mathbf{Q}}{\to} \lambda$ if and only if $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is ultimately monotone and convergent to λ in the order topology, while not topological, has the order topology τ_{ord} as its modification. Note that if we replace "ultimately monotone" by "monotone" in the prescription, what is obtained fails to be a convergence, as a subnet of a monotone net need not be monotone.

Clearly, τ_{ord} is coarser than $\mathbf{Q}$ because each $\mathbf{Q}$-convergent net converges in the order topology. We next show that if A is a closed set as determined by $\tau\mathbf{Q}$ and $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is a net in A convergent with respect to the order topology to λ, then there is another net in A that is $\mathbf{Q}$-convergent to λ. In view of the structure of the closed sets of the modification, this will imply that $\tau\mathbf{Q} \subseteq \tau_{\text{ord}}$ and hence $\tau\mathbf{Q} = \tau_{\text{ord}}$.

To this end, suppose $\langle a_\sigma \rangle_{\sigma \in \Sigma}$ is a net in A that is τ_{ord}-convergent to λ. For each $\sigma \in \Sigma$, put

$$b_\sigma := \sup\{a_\omega : \omega \in \uparrow \sigma\}.$$

By convergence in the order topology, it is clear that $\forall \sigma \in \Sigma,\ b_\sigma \succeq \lambda$, and that $\langle b_\sigma \rangle_{\sigma \in \Sigma}$ is a decreasing net convergent to λ. But for each $\sigma \in \Sigma$ there is an ultimately increasing net with values in $\{a_\omega : \omega \in \uparrow \sigma\} \subseteq A$ convergent to b_σ: in the case that the supremum is achieved, we can use a constant net, and if not, let the net be defined on the linearly ordered set $\langle \{\mu : \mu \prec b_\sigma\}, \succeq \rangle$, assigning to each $\mu \prec b_\sigma$ some a_ω where $\omega \succeq' \sigma$ and $a_\omega \succ \mu$.

The above argument shows that each b_σ lies in A. As the net $\langle b_\sigma \rangle_{\sigma \in \Sigma}$ is a decreasing net τ_{ord}-convergent to λ, $\langle b_\sigma \rangle_{\sigma \in \Sigma}$ is $\mathbf{Q}$-convergent to λ, as required.

We note in closing that the convergence associated with τ_{ord} is in general properly coarser than $\mathbf{Q}$, because certain τ_{ord}-convergent nets need not be $\mathbf{Q}$-convergent, e.g., in $[-\infty, \infty]$ the sequence $\langle \frac{(-1)^n}{n} \rangle_{n \in \mathbb{N}}$ while convergent in the order topology is not $\mathbf{Q}$-convergent.

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