INROADS

Real Analysis Exchange
Vol. 36(1), 2010/2011, pp. 169–176

Robert Kantrowitz, Department of Mathematics, Hamilton College, Clinton, NY 13323, U.S.A. email: rkantrow@hamilton.edu

SUBMULTIPLICATIVITY OF NORMS FOR SPACES OF GENERALIZED BV-FUNCTIONS

Abstract

The purpose of this article is to offer a couple of short arguments for results describing the interaction between the norm and pointwise products in certain spaces of functions of generalized bounded variation.

1 Introduction.

Functions of bounded variation have played an important role in various aspects of mathematical analysis for well over the last century. Their main influence is, arguably, in connection with the study of Fourier series. They appear also in the theory of Riemann-Stieltjes integration, and, in particular, in characterizing the dual space of the Banach space of continuous functions on a compact interval.

The notion of bounded variation has undergone generalizations in several directions, and the surrounding literature is vast. The papers of Avdispahić [1] and Pierce and Velleman [7] provide historical summaries and numerous references. It is well-known that, when equipped with certain natural norms involving the variation, linear spaces of functions of generalized bounded variation become Banach spaces. In many cases, the norms are also submultiplicative, and so the function spaces carries the additional structure of unital commutative Banach algebras with respect to pointwise multiplication of the functions. Several papers, among them [2], [5], [8], and [9], deal with submultiplicativity for norms on spaces of functions of generalized bounded variation. This article is another such paper.

Mathematical Reviews subject classification: Primary: 26A45; Secondary: 46J10
Key words: bounded variation, Banach algebra
Received by the editors October 26, 2009
Communicated by: Brian S. Thomson

We first take up the idea of Λ_p-variation, a generalization of Waterman's Λ-variation; see [12]. We establish the submultiplicativity of a canonical norm on the Banach space $\Lambda_p BV$ from first principles. That our choice of norm involves the supremum coupled with the variation of the function benefits us by minimizing the level of technicality. Indeed, the argument hinges on little more than a finite dimensional version of Minkowski's inequality, specifically, the triangle inequality for the p-norm on Euclidean space.

Another fruitful avenue of generalization for functions of bounded variation, originally opened by Young in [14], and subsequently pursued by others – Musielak and Orlicz [6] notable among them – is that of Φ-bounded variation. Here, too, certain linear spaces of functions are known to be Banach spaces under suitable norms involving the Φ-variation, and, in section 3, we present a streamlined argument for submultiplicativity of a norm whose definition again involves both the supremum and the variation.

2 Λ_p-bounded variation.

Throughout the paper, I denotes a compact interval of the real line, and p is a real number that satisfies $1 \leq p$; in this section, $\Lambda = (\lambda_j)_{j=1}^{\infty}$ is a non-decreasing sequence of positive real numbers for which the series $\sum_{j=1}^{\infty} 1/\lambda_j$ diverges. For any natural number n, consider subintervals $I_j = [a_j, b_j]$, $j = 1, 2, ..., n$, of I that are non-overlapping in the sense that their interiors are pairwise disjoint. When f is a real-valued function defined on I, form the sum $\sum_{j=1}^{n} |f(I_j)|^p/\lambda_j$, where the notation $f(I_j)$ represents the difference $f(b_j) - f(a_j)$. The Λ_p-variation of f, denoted $V_{\Lambda_p}(f)$, is defined to be the supremum over all sums of this type, and, if $V_{\Lambda_p}(f) < \infty$, then f is said to be a function of bounded Λ_p-variation. The collection of all functions of bounded Λ_p-variation on I constitutes the linear space $\Lambda_p BV$. When Λ is a bounded sequence, like the sequence whose entries are all 1, the space $\Lambda_p BV$ is Wiener's space of functions of bounded p-variation; see [13]. Krabbe remarks in the next-to-last sentence of [3] that Wiener's space is a Banach algebra under pointwise multiplication and the norm we discuss below. If $p = 1$ and Λ is the constant 1 sequence, then we have the classical space of BV-functions. The case in which $p = 1$ and $\Lambda = (j)_{j=1}^{\infty}$ renders the space HBV of functions of bounded harmonic variation. Whenever $f \in \Lambda_p BV$, it follows that f is bounded (Lemma 1.6 of [11]), and the definition

$$\|f\|_{\Lambda_p} = \|f\|_{\infty} + V_{\Lambda_p}(f)^{1/p}$$

provides a complete norm for $\Lambda_p BV$. As promised, we show that this norm is submultiplicative.

Theorem 1. *If $f, g \in \Lambda_p BV$, then $fg \in \Lambda_p BV$ and $\|fg\|_{\Lambda_p} \le \|f\|_{\Lambda_p} \|g\|_{\Lambda_p}$.*

PROOF. Let $f, g \in \Lambda_p BV$ be given. For any natural number n and any collection of non-overlapping subintervals $I_1, I_2, \ldots, I_n$ of I, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{|fg(I_j)|^p}{\lambda_j} &= \sum_{j=1}^{n} \frac{|fg(b_j) - fg(a_j)|^p}{\lambda_j} \\
&= \sum_{j=1}^{n} \frac{\big|f(b_j)g(b_j) - f(a_j)g(b_j) + f(a_j)g(b_j) - f(a_j)g(a_j)\big|^p}{\lambda_j} \\
&= \sum_{j=1}^{n} \frac{\big|g(b_j)f(I_j) + f(a_j)g(I_j)\big|^p}{\lambda_j} \\
&\le \sum_{j=1}^{n} \frac{(\|g\|_\infty |f(I_j)| + \|f\|_\infty |g(I_j)|)^p}{\lambda_j} \\
&= \sum_{j=1}^{n} \left(\frac{\|g\|_\infty |f(I_j)|}{\lambda_j^{1/p}} + \frac{\|f\|_\infty |g(I_j)|}{\lambda_j^{1/p}} \right)^p \\
&\le \left[\left(\sum_{j=1}^{n} \frac{\|g\|_\infty^p |f(I_j)|^p}{\lambda_j} \right)^{1/p} + \left(\sum_{j=1}^{n} \frac{\|f\|_\infty^p |g(I_j)|^p}{\lambda_j} \right)^{1/p} \right]^p \\
&= \left[\|g\|_\infty \left(\sum_{j=1}^{n} \frac{|f(I_j)|^p}{\lambda_j} \right)^{1/p} + \|f\|_\infty \left(\sum_{j=1}^{n} \frac{|g(I_j)|^p}{\lambda_j} \right)^{1/p} \right]^p \\
&\le \Big[\|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p} \Big]^p .
\end{aligned}$$

The third-to-last step obtains from Minkowski's inequality applied to the n-tuples

$$\left(\frac{\|g\|_\infty |f(I_1)|}{\lambda_1^{1/p}}, \ldots, \frac{\|g\|_\infty |f(I_n)|}{\lambda_n^{1/p}} \right) \quad \text{and} \quad \left(\frac{\|f\|_\infty |g(I_1)|}{\lambda_1^{1/p}}, \ldots, \frac{\|f\|_\infty |g(I_n)|}{\lambda_n^{1/p}} \right).$$

It follows that $fg \in \Lambda_p BV$ and $V_{\Lambda_p}(fg)^{1/p} \le \|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p}$.

From this inequality, prominently featured in section 4 of [5], it follows that

$$\begin{aligned}
\|fg\|_{\Lambda_p} &= \|fg\|_\infty + V_{\Lambda_p}(fg)^{1/p} \\
&\le \|f\|_\infty \|g\|_\infty + \|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p} \\
&\le \|f\|_\infty \|g\|_\infty + \|g\|_\infty V_{\Lambda_p}(f)^{1/p} + \|f\|_\infty V_{\Lambda_p}(g)^{1/p} + V_{\Lambda_p}(f)^{1/p} V_{\Lambda_p}(g)^{1/p} \\
&= \left(\|f\|_\infty + V_{\Lambda_p}(f)^{1/p}\right)\left(\|g\|_\infty + V_{\Lambda_p}(g)^{1/p}\right) \\
&= \|f\|_{\Lambda_p} \|g\|_{\Lambda_p},
\end{aligned}$$

to establish the result. □

We conclude this section by noting two anomalies. First, for the case that $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and any positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, it is well-known that the definition

$$\|x\| = \left(\sum_{j=1}^{n} \frac{|x_j|^p}{\lambda_j}\right)^{1/p}$$

provides a norm for $\mathbb{R}^n$. However, with pointwise multiplication of n-tuples, simple counterexamples show that these weighted p-norms on Euclidean space fail to be submultiplicative, in general. Thus, while finite sums of this type appear throughout the proof of the theorem, it is not submultiplicativity in the finite dimensional case, but rather Minkowski's inequality that drives the argument.

Second, a cousin of the function space $\Lambda_p BV$ is the space consisting of sequences $x = (x_j)_{j=0}^\infty$ for which the sum $\sum_{j=1}^\infty |x_j - x_{j-1}|^p/\lambda_j$ is finite. It is tempting to introduce a norm by defining

$$\|x\| = \|x\|_\infty + \left(\sum_{j=1}^{\infty} \frac{|x_j - x_{j-1}|^p}{\lambda_j}\right)^{1/p}.$$

The simplest case, when $p = 1$ and Λ is the constant sequence of all 1's, is the well-known Banach algebra bv of sequences of bounded variation. In Example 4.1.5 of [4], for instance, Laursen and Neumann characterize the multiplier algebra of one of the subalgebras of bv. On the other hand, when $p = 1$ and $\Lambda = (j)_{j=1}^\infty$, note that the resulting linear space hbv of sequences of bounded harmonic variation contains, for example, the unbounded sequence $x = (0, 1, \sqrt{2}, \sqrt{3}, \ldots)$. Thus, because it involves the supremum, the above candidate for a norm is not available for hbv. Similarly, for $p > 1$ and constant Λ sequence, there are unbounded sequences with bounded p-variation.

Moreover, the square $(0, 1, 2, 3, ...)$ of the sequence x above fails to have finite harmonic variation. Thus, the issue of submultiplicativity of a norm becomes irrelevant for the sequence space *hbv* since it is not even stable under pointwise multiplication.

3 Φ-bounded variation.

The notion of ϕ-variation centers around sums of the type $\sum_j \phi(|f(I_j)|)$ for certain functions ϕ defined on the non-negative real numbers. One readily sees that if $\phi(x) = x^p$, say, then the situation reverts to Wiener's higher variation. Different conditions are imposed on the function ϕ for different purposes in different places in the literature. In this final section, we generalize one step further still and, adopting the notation of Schramm in [10], introduce a sequence $\Phi = \{\phi_j\}_{j=1}^{\infty}$ of increasing, convex functions on the nonnegative real numbers. All of the ϕ_j's are required to satisfy the additional criteria: $\phi_j(0) = 0$ and $\phi_j(x) > 0$ for $x > 0$; $\phi_{j+1}(x) < \phi_j(x)$ for all x; and $\sum_{j=1}^{\infty} \phi_j(x)$ diverges for $x > 0$. In this context, a function f is of Φ-bounded variation if the supremum $V_\Phi(f)$ of sums of the type $\sum_{j=1}^{n} \phi_j(|f(I_j|)$, again taken over the family of all non-overlapping subintervals $\{I_j\}$ of $I = [a, b]$, is finite. In the way of motivation, it is clear that the choices $\phi_j(x) = x^p/\lambda_j$, for example, place us in the setting of $\Lambda_p BV$. The symbol ΦBV denotes the linear space of all functions f such that cf is of Φ-bounded variation for some $c > 0$. Schramm notes that such functions are bounded on I and shows that $\|f\|_0 = \inf\{k > 0 : V_\Phi(f/k) \leq 1\}$ renders the subspace $\Phi BV_0 = \{f \in \Phi BV : f(a) = 0\}$ of ΦBV-functions that are anchored at the left-hand endpoint of I a Banach space. Paralleling the development in the previous section, we endow the space ΦBV_0 with the norm $\|f\|_\Phi = \|f\|_\infty + \|f\|_0$ and establish its submultiplicativity via a short sequence of elementary estimates.

Theorem 2. *If $f, g \in \Phi BV_0$, then $fg \in \Phi BV_0$ and $\|fg\|_\Phi \leq \|f\|_\Phi \|g\|_\Phi$.*

PROOF. Let $f, g \in \Phi BV_0$. If either function is identically zero, the result is trivial. Otherwise, for any natural number n and any collection of non-overlapping subintervals $I_1, I_2, \ldots, I_n$ of I, we have

$$\begin{aligned}
\sum_{j=1}^{n}\phi_j\left(\frac{|fg(I_j)|}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\right) &= \sum_{j=1}^{n}\phi_j\left(\frac{|g(b_j)f(I_j)+f(a_j)g(I_j)|}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\right)\\
&\le \sum_{j=1}^{n}\phi_j\left(\frac{\|g\|_\infty|f(I_j)|+\|f\|_\infty|g(I_j)|}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\right)\\
&= \sum_{j=1}^{n}\phi_j\left(\frac{\|g\|_\infty\|f\|_0\frac{|f(I_j)|}{\|f\|_0}+\|f\|_\infty\|g\|_0\frac{|g(I_j)|}{\|g\|_0}}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\right)\\
&\le \frac{\|g\|_\infty\|f\|_0}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\sum_{j=1}^{n}\phi_j\left(\frac{|f(I_j)|}{\|f\|_0}\right)+\\
&\qquad \frac{\|f\|_\infty\|g\|_0}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\sum_{j=1}^{n}\phi_j\left(\frac{|g(I_j)|}{\|g\|_0}\right)\\
&\le \frac{\|g\|_\infty\|f\|_0}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}V_\Phi\left(f/\|f\|_0\right)+\frac{\|f\|_\infty\|g\|_0}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}V_\Phi\left(g/\|g\|_0\right)\\
&\le \frac{\|g\|_\infty\|f\|_0}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\cdot 1 + \frac{\|f\|_\infty\|g\|_0}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\cdot 1\\
&= 1,
\end{aligned}$$

where the next-to-last step obtains readily from the definition of V_Φ and is the content of part (i) of Lemma 2.1 of [10]. Thus,

$$V_\Phi\left(\frac{fg}{\|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0}\right)\le 1,$$

from which it follows that $fg \in \Phi BV_0$ and $\|fg\|_0 \le \|f\|_\infty\|g\|_0+\|g\|_\infty\|f\|_0$. The proof concludes, *mutatis mutandis,* as in the case of $\Lambda_p BV$ from the previous section:

$$\begin{aligned}
\|fg\|_\Phi = \|fg\|_\infty + \|fg\|_0 &\le \|f\|_\infty\|g\|_\infty + \|f\|_\infty\|g\|_0 + \|g\|_\infty\|f\|_0 + \|f\|_0\|g\|_0\\
&= (\|f\|_\infty + \|f\|_\Phi)(\|g\|_\infty + \|g\|_\Phi) = \|f\|_\Phi\|g\|_\Phi,
\end{aligned}$$

to establish the desired inequality. □

We finish by noting that because $\|f\|_0 \le \|f\|_\infty + \|f\|_0 = \|f\|_\Phi$, it follows as an immediate consequence of the open mapping theorem that $\|\cdot\|_\Phi$ and $\|\cdot\|_0$ are equivalent norms on ΦBV_0.

References

[1] M. Avdispahić, *Concepts of generalized bounded variation and the theory of Fourier series,* Internat. J. Math. & Math. Sci. **9** (1986) 223–244.

[2] P. S. Bullen, *An inequality for variations,* Amer. Math. Monthly **90** (1983) p. 560.

[3] G. L. Krabbe, *Integration with respect to operator-valued functions,* Acta Sci. Math. **22** (1961) 301–319.

[4] K. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, Oxford University Press, Oxford, 2000.

[5] L. Maligranda and W. Orlicz, *On some properties of functions of generalized variation,* Mh. Math. **104** (1987) 53–65.

[6] J. Musielak and W. Orlicz, *On generalized variations I,* Studia Math. **18** (1959) 11–41.

[7] P. B. Pierce and D. J. Velleman, *Some generalizations of the notion of bounded variation,* Amer. Math. Monthly **113** (2006) 897–904.

[8] A. M. Russell, *A commutative Banach algebra of functions of bounded variation,* Amer. Math. Monthly **87** (1980) 39–40.

[9] A. M. Russell, *A commutative Banach algebra of functions of generalized variation,* Pac. J. Math. **84** (1979) 455–463.

[10] M. Schramm, *Functions of Φ-bounded variation and Riemann-Stieltjes integration,* Trans. Amer. Math. Soc. **287** (1985) 49–63.

[11] R. G. Vyas, *On the absolute convergence of small gaps Fourier series of functions of $\Lambda BV^{(p)}$,* J. Inequal. Pure Appl. Math. **6** (1), 23 (2005) 1–6.

[12] D. Waterman, *On Λ-bounded variation,* Studia Math. **57** (1976) 33–45.

[13] N. Wiener, *The quadratic variation of a function and its Fourier coefficients,* J. Math. and Phys. **3** (1924) 72–94.

[14] L. C. Young, *Sur une généralisation de la notion de variation de puissance p-ième bornée au sense de M. Wiener, et sur la convergence des séries de Fourier,* Comptes. Rend. Acad. Sci. Paris **204** (1937) 470–472.