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AN IMPROVEMENT OF A RECENT RESULT OF THOMSON

Abstract

In [5], Brian S. Thomson proved the following result: *Let f be AC^*G on an interval $[a,b]$. Then the total variation measure $\mu = \mu_f$ associated with f has the following properties: a) μ is a σ-finite Borel measure on $[a,b]$; b) μ is absolutely continuous with respect to Lebesgue measure; c) There is a sequence of closed sets F_n whose union is all of $[a,b]$ such that $\mu(F_n) < \infty$ for each n; d) $\mu(B) = \mu_f(B) = \int_B |f'(x)|\,dx$ for every Borel set $B \subset [a,b]$. Conversely, if a measure μ satisfies conditions a)–c) then there exists an AC^*G function f for which the representation d) is valid.* In this paper we improve Thomson's theorem as follows: in the first part we ask f to be only $VB^*G \cap (N)$ on a Lebesgue measurable subset P of $[a,b]$ and continuous at each point of P; the converse is also true even for μ defined on the Lebesgue measurable subsets of P (see Theorem 2 and the two examples in Remark 1).

In [5] Brian S. Thomson proved a theorem that can be written in the following form:

Theorem A.

I. *If $F : [a,b] \to \mathbb{R}$ is AC^*G on $[a,b]$ then $\mu_F^* : \mathcal{P}([a,b]) \to [0,+\infty]$ has the following properties:*

   1) *$\mu_F^* \ll m$;*
   2) *there is a sequence of closed sets $\{P_n\}$ such that $\cup_{n=1}^{\infty} P_n = [a,b]$ and $\mu_F^*(P_n) < +\infty$ for each n.*
   3) *$(\mu_F^*)_{|\mathcal{B}or([a,b])}$ is a measure (see [4, p. 40]);*
   4) *$\mu_F^*(B) = (\mathcal{L}) \int_B |F'(t)|\,dt$ whenever B is a Borel subset of $[a,b]$.*

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II. *Conversely, let* $\mu : \mathcal{B}or([a,b]) \to [0,+\infty]$ *be a measure such that:*

1′) $\mu \ll m$;

2′) *there is a sequence of closed sets* $\{P_n\}$ *such that* $\cup_{n=1}^{\infty} P_n = [a,b]$ *and* $\mu(P_n) < +\infty$ *for each* n;

Then there exists a continuous function $F : [a,b] \to \mathbb{R}$, $F \in AC^*G$ *on* $[a,b]$, *such that* $(\mu_F^*)_{|\mathcal{B}or([a,b])} = \mu$.

In this paper we shall improve Theorem A as it will be shown in Theorem 2 (see also the two examples given in Remark 1).

We denote by m the Lebesgue measure in $\mathbb{R}$. By $\mathcal{O}(f;X)$ we shall mean the oscillation of the function f on the set X, and by $f_{|X}$ the restriction of the function f on the set X. The conditions AC, ACG, AC^* AC^*G, VB^*, VB^*G and Lusin's condition (N) are defined as in [3].

Definition 1. Let $P \subset \mathbb{R}$, $\mathcal{A} \subseteq \mathcal{P}(P) = \{E : E \subset P\}$ and $\alpha : \mathcal{A} \to [0,+\infty]$.

- We say that α is absolutely continuous with respect to m and write $\alpha \ll m$ if $\alpha(Z) = 0$ whenever $Z \in \mathcal{A}$ and $m(Z) = 0$.
- For P a Lebesgue measurable subset of $\mathbb{R}$, we put $\mathcal{L}eb(P) = \{E \subset P : E$ is Lebesgue measurable$\}$.
- For P a Borel measurable subset of $\mathbb{R}$, we put $\mathcal{B}or(P) = \{E \subset P : E$ is Borel measurable$\}$.

Definition 2. For $x, y \in \mathbb{R}$, $x \neq y$, let $\langle x,y \rangle$ denote the closed interval with the endpoints x and y. Let $E \subset \mathbb{R}$, $\delta : E \to (0,+\infty)$,

$$\beta^*(E;\delta) = \Big\{ \big(\langle x,y \rangle, x\big) \,:\, x \in E,\ y \in \big(x - \delta(x), x + \delta(x)\big) \Big\}.$$

The finite set $\pi = \big\{ \big(\langle x_i,y_i \rangle, x_i\big) \big\}_{i=1}^{n} \subset \beta^*(E;\delta)$ is said to be a partition if $\{\langle x_i,y_i \rangle\}_{i=1}^{n}$ is a set of nonoverlapping closed intervals. Let $f : \mathbb{R} \to \mathbb{R}$,

$$V_\delta^*(f;E) = \sup \Big\{ \sum_{(\langle x,y \rangle, x) \in \pi} \big| f(y) - f(x) \big| \,:\, \pi \subset \beta^*(E;\delta) \text{ is a partition} \Big\},$$

and

$$\mu_f^*(E) = \inf_\delta V_\delta^*(f;E).$$

Note that this μ_f^* is the same as that of Thomson [5, p. 186], and it is also identical with Thomson's $\mathcal{S}_o$-μ_F of [4].

Lemma 1. *Let $F : [a,b] \to \mathbb{R}$, $E \subset P \subset [a,b]$, $F \in VB^*$ on E, F continuous at each point of P. Then $\mu_F^*(\overline{E} \cap P) \neq +\infty$.*

Proof. By Theorem 7.1 of [3, p. 229], F is VB^* on $\overline{E} \cap P$. Let $X = \overline{E} \cap P$ and $Y = \{x \in X : x \text{ is isolated at least at one side in } X\}$. Since Y is at most countable [3, p. 260], and F is continuous at each point of P, it follows that $\mu_F^*(Y) = 0$. Thomson shows in [4, p. 34] that $\mu_F^*(X \setminus Y) \leq 2V^*(F; X)$. Hence $\mu_F^*(X) \leq 2V^*(F; X) \neq +\infty$. □

Theorem 1. *Let $F : [a,b] \to \mathbb{R}$, and let P be a Lebesgue measurable subset of $[a,b]$. Let $\mu_F^* : \mathcal{P}(P) \to [0, +\infty]$. The following assertions are equivalent:*

(i) $\mu_F^* \ll m$;

(ii) F *is* $VB^*G \cap (N)$ *on* P *and* F *is continuous at each point of* P;

(iii) F *is continuous at each point of* P*, derivable a.e. on* P*, and*

$$\mu_F^*(E) = (\mathcal{L}) \int_E \left|F'(t)\right| dt \,,$$

whenever E *is a Lebesgue measurable subset of* P;

Moreover, each of the three equivalences implies that there exists a sequence of sets P_n such that $\cup_n P_n = P$ and $\mu_F^*(\overline{P}_n \cap P) \neq +\infty$.

Proof. The three equivalences follow from [2, Theorem 13, (ii), (iii), (vii)] (because $\mathcal{S}_o$-$\mu_F = \mu_F^*$). The second part follows by Lemma 1 and (ii). □

Lemma 2. *Let $f : [a,b] \to [0, +\infty)$ a Lebesgue integrable function, P a closed subset of $[a,b]$, $\{(a_i, b_i)\}_i$ the intervals continuous to $P \cup \{a, b\}$, and let $\{\alpha_i\}_i$ be a sequence of positive numbers. Then there is a function $G : [a,b] \to \mathbb{R}$ such that:*

a) $G(t) = 0$ *for* $t \in P \cup \{a, b\}$;

b) $G \in AC$ *on* $[a,b]$;

c) $|G'(t)| = f(t)$ *a.e. on* $\cup_{i=1}^{\infty} (a_i, b_i)$;

d) $G(t) \in [0, \alpha_i)$ *for* $t \in [a_i, b_i]$, $i = 1, 2, \ldots$;

e) $G'(t) = 0$ *a.e. on* P.

PROOF. We shall use a technique of Thomson [5, p. 190]. For each i, let n_i be a positive integer, and let

$$a_i = a_{i,0} < a_{i,1} < a_{i,2} < \ldots < a_{i,2n_i-1} < a_{i,2n_i} = b_i$$

be such that

$$\int\limits_{a_{i,k}}^{a_{i,k+1}} f(t)\,dt = \frac{1}{2n_i}\int\limits_{a_i}^{b_i} f(t)\,dt < \alpha_i\,.$$

Let $g : [a,b] \to \mathbb{R}$,

$$g(t) = \begin{cases} 0 & \text{if } t \in P \cup \{a,b\} \\ f(t) & \text{if } t \in [a_{i,2k}, a_{i,2k+1}],\ k = \overline{0, n_i - 1},\ i = \overline{1,\infty} \\ -f(t) & \text{if } t \in (a_{i,2k-1}, a_{i,2k}),\ k = \overline{1, n_i},\ i = \overline{1,\infty}\,. \end{cases}$$

Then $G : [a,b] \to \mathbb{R}$, $G(x) = \int_a^x g(t)\,dt$ satisfies our lemma. □

Lemma 3. *Let $f, f_n : [a,b] \to \mathbb{R}$ be such that the series $\sum_{n=1}^{\infty} f_n(x) = f(x)$ for $x \in [a,b]$. Then*

$$\mathcal{O}\big(f;[a,b]\big) \le \sum_{n=1}^{\infty} \mathcal{O}\big(f_n;[a,b]\big)\,. \tag{1}$$

PROOF. Let $x, y \in [a,b]$. Then

$$\big|f(y) - f(x)\big| = \Big|\sum_{n=1}^{\infty}\big(f_n(y) - f_n(x)\big)\Big| \le \sum_{n=1}^{\infty}\big|f_n(y) - f_n(x)\big| \le \sum_{n=1}^{\infty}\mathcal{O}\big(f_n;[a,b]\big)\,.$$

Thus we have (1). □

Theorem 2. *Let P be a Lebesgue measurable subset of $[a,b]$.*

I. *If $F : [a,b] \to \mathbb{R}$ is $VB^*G \cap (N)$ (particularly $F \in AC^*G$) on P and F is continuous at each point of P, then $\mu_F^* : \mathcal{P}(P) \to [0,+\infty]$ has the following properties:*

   1) *$\mu_F^* \ll m$;*
   2) *there is a a sequence of sets P_n such that $\cup_n P_n = P$ and for each n, $\mu_F^*(\overline{P}_n \cap P) \neq +\infty$;*
   3) *$(\mu_F^*)_{|\mathcal{L}eb(P)}$ is a measure;*

4) $\mu_F^*(B) = (\mathcal{L}) \int_B |F'(t)|\, dt$ *whenever* $B \subset \mathcal{L}eb(P)$.

II. *Conversely, let* $\mu : \mathcal{L}eb(P) \to [0, +\infty]$ *be a measure such that:*

1′) $\mu \ll m$;

2′) *there is a sequence of sets* P_n *such that* $\cup_n P_n = P$ *and for each* n, $\mu(\overline{P}_n \cap P) \neq +\infty$.

Then there exists a continuous function $F : [a,b] \to \mathbb{R}$, $F \in AC^*G$ *on* P, *such that* $(\mu_F^*)_{|\mathcal{L}eb(P)} = \mu$.

Proof. I. 1) follows by Theorem 1, (i), (ii).
2) follows by the last part of Theorem 1.
3) Let $\{E_n\}_n \subset \mathcal{L}eb(P)$ be a sequence of pairwise disjoint sets. Then each $E_n = A_n \cup B_n$, where A_n is a Borel set and $m(B_n) = 0$. By 1), $\mu_F^*(B_n) = 0$. Since μ_F^* is a metric outer measure it follows that μ_F^* restricted to the Borel subsets of $[a,b]$ is a measure. Thus we obtain:

$$\begin{aligned}\mu_F^*\big(\cup_n E_n\big) &\le \sum_n^\infty \mu_F^*(E_n) \le \sum_n^\infty \Big(\mu_F^*(A_n) + \mu_F^*(B_n)\Big)\\ &= \sum_n^\infty \mu_F^*(A_n) = \mu_F^*\big(\cup_n A_n\big) \le \mu_F^*\big(\cup_n E_n\big)\,.\end{aligned}$$

Thus $(\mu_F^*)_{|\mathcal{L}eb(P)}$ is a measure.
4) See Theorem 1, (ii), (iii).

II. Let $Q_0 = \emptyset$ and $Q_n = \cup_{i=1}^n \overline{P}_i \cup \{a,b\}$, $n = 1, 2, \ldots$. For $n \ge 1$, let $\{(a_{nj}, b_{nj})\}$ be the intervals contiguous to Q_n. Clearly $\mu(Q_n \cap P) \neq +\infty$. We shall use Thomson's technique of [5, p. 189-190]. Since μ is absolutely continuous on $P \cap (Q_n \setminus Q_{n-1})$ and $\mu\big(P \cap (Q_n \setminus Q_{n-1})\big) \neq +\infty$, by the Radon-Nicodym Theorem, there exists a Lebesgue integrable function $g_n : P \cap (Q_n \setminus Q_{n-1}) \to [0, +\infty)$ such that

$$\mu(B) = (\mathcal{L}) \int_B g_n(t)\, dt\,,$$

whenever B is a Lebesgue measurable subset of $P \cap (Q_n \setminus Q_{n-1})$. We may consider $g_n : [a,b] \to \mathbb{R}$, if we put $g_n(x) = 0$ for $x \in [a,b] \setminus \big(P \cap (Q_n \setminus Q_{n-1})\big)$. Let

$$F_1(x) = (\mathcal{L}) \int_a^x g_1(t)\, dt\,.$$

Then $F_1 \in AC$ on $[a,b]$ and $F_1' = g_1$ *a.e.* on $[a,b]$. Clearly F_1 is constant on each (a_{1j}, b_{1j}). Let $\{\alpha_{nj}\}_j$ be a sequence of positive numbers such that

$$\sum_{j=1}^{\infty} \alpha_{nj} < \frac{1}{2^n}\,. \tag{2}$$

By Lemma 2, there exists $F_{n+1} : [a,b] \to \left[0, \frac{1}{2^n}\right)$ such that

a) $F_{n+1}(t) = 0$ for $t \in Q_n$;

b) $F_{n+1} \in AC$ on $[a,b]$;

c) $\left|F'_{n+1}(t)\right| = g_{n+1}(t)$ *a.e.* on each (a_{nj}, b_{nj});

d) $F_{n+1}(t) \in [0, \alpha_{nj})$ on $[a_{nj}, b_{nj}]$;

e) $F'_{n+1}(t) = 0$ *a.e.* on Q_n.

Let $F : [a,b] \to \mathbb{R}$, $F(x) = \sum_{n=1}^{\infty} F_n(x)$. Then F is continuous on $[a,b]$ (see d), b) and (2)). Let $R_n(x) = \sum_{k=1}^{\infty} F_{n+k}(x)$. Since each

$$\left(a_{nj}, b_{nj}\right) \subset [a,b] \setminus Q_n \subset [a,b] \setminus \left(P \cap (Q_n \setminus Q_{n-1})\right),$$

it follows that $g_n(t) = 0$ on (a_{nj}, b_{nj}). By c) we have that $F_n'(t) = 0$, so F_n is constant on each (a_{nj}, b_{nj}). Thus

$$F_1(x) + \ldots + F_n(x) = \text{constant} \ \text{ on each } (a_{nj}, b_{nj}) \tag{3}$$

(because $Q_1 \subset Q_2 \subset \ldots$). Since $F_{n+k}(t) = 0$ on Q_{n+k-1} for $k = \overline{1,\infty}$ (see a)), and $Q_n \subset Q_{n+1} \subset Q_{n+2} \subset \ldots$ it follows that $R_n(t) = 0$ on Q_n. Thus $F(x) = F_1(x) + \ldots + F_n(x)$ for $x \in Q_n$. Hence F and R_n are AC on Q_n. By Lemma 3 and (3), we have

$$\begin{aligned}
\sum_j \mathcal{O}\big(F; [a_{nj}, b_{nj}]\big) &= \sum_j \mathcal{O}\big(R_n; [a_{nj}, b_{nj}]\big) \\
&\le \sum_j \Big(\mathcal{O}\big(F_{n+1}; [a_{nj}, b_{nj}]\big) + \mathcal{O}\big(F_{n+2}; [a_{nj}, b_{nj}]\big) + \ldots\Big) \\
&= \sum_j \mathcal{O}\big(F_{n+1}; [a_{nj}, b_{nj}]\big) + \sum_j \mathcal{O}\big(F_{n+2}; [a_{nj}, b_{nj}]\big) + \ldots \\
&< \frac{1}{2^n} + \sum_j \mathcal{O}\big(F_{n+2}; [a_{n+1,j}, b_{n+1,j}]\big) + \ldots \\
&< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots = \frac{1}{2^{n-1}}\,.
\end{aligned}$$

(see a), d) and (2)). By [3, p. 232], F and R_n are AC^* on Q_n. Clearly $R_n'(x) = 0$ *a.e.* on Q_n and F is AC^*G $(\subset VB^*G \cap (N))$ on P. It follows that

$$F'(x) = F_1'(x) + \ldots + F_n'(x) + R_n'(x) = F_1'(x) + \ldots + F_n'(x) \ a.e. \ \text{ on } Q_n .$$

Thus $F'(x) = F_1'(x) = g_1(x)$ on Q_1 and

$$\bigl|F'(x)\bigr| = \bigl|F_1'(x) + F_2'(x)\bigr| = \bigl|F_2'(x)\bigr| = g_2(x) \ \text{ on } Q_2 \setminus Q_1$$

(because F_1 is constant on each (a_{1j}, b_{1j})). Continuing, it follows that

$$\bigl|F'(x)\bigr| = \bigl|F_1'(x) + \ldots + F_{n-1}'(x) + F_n'(x)\bigr| = \bigl|F_n'(x)\bigr| = g_n(x) \ \text{ on } Q_n \setminus Q_{n-1}$$

(because $F_1, \ldots, F_{n-1}$ are constant on each $(a_{n-1,j}, b_{n-1,j})$). By 3), for any Lebesgue measurable subset B of P, we have

$$\mu_F^*(B) = \sum_{n=1}^{\infty} \mu_F^*\bigl(B \cap (Q_n \setminus Q_{n-1})\bigr) = \sum_{n=1}^{\infty} (\mathcal{L}) \int\limits_{B \cap (Q_n \setminus Q_{n-1})} \bigl|F'(t)\bigr|\, dt =$$

$$= \sum_{n=1}^{\infty} (\mathcal{L}) \int\limits_{B \cap (Q_n \setminus Q_{n-1})} g_n(t)\, dt = \sum_{n=0}^{\infty} \mu\bigl(B \cap (Q_n \setminus Q_{n-1})\bigr) = \mu(B)\,.$$

Thus $\mu_F^*(B) = \mu(B)$. □

Remark 1.

- Theorem 2 contains Theorem A of Thomson.
- We recall the following example of [2]:

  Let C be the Cantor ternary set and $\varphi : [0,1] \to [0,1]$ the Cantor ternary function (see for example [1], pp. 213-214). Then C contains a G_δ-set B such that $m^*(\varphi(B)) = 0$, hence $\varphi \in VB^*G \cap (N)$ on B. But $\varphi \notin ACG$ on B, so $\varphi \notin AC^*G$ on B.

  From this example it follows that $AC^*G \subsetneq VB^*G \cap (N)$, so in Theorem 2, I., the particular case with AC^*G is genuine. Moreover $\mu_\varphi^* = \mu_f^* = 0$ on $\mathcal{L}eb(B)$ whenever $f : [0,1] \to \mathbb{R}$ is $VB^*G \cap (N)$ on B and continuous at each point of B (see Theorem 1).
- We consider the following example:

Let C be the Cantor ternary set. Let $\{(a_{ni}, b_{ni})\}$, $n = 1, 2, \ldots$, $i = 1, 2, \ldots, 2^{n-1}$, be the intervals contiguous to C of length $\frac{1}{3^n}$, and let $c_{ni} = \frac{a_{ni}+b_{ni}}{2}$. Let $F : [0, 1] \to [0, 1]$,

$$F(x) = \begin{cases} 0 & \text{if } x \in C \\ \frac{1}{n} & \text{if } x = c_{ni} \\ \text{linear} & \text{on each } [a_{ni}, c_{ni}] \text{ and } [c_{ni}, b_{ni}] \end{cases}$$

Let $P = \cup_{n=1}^{\infty} \cup_{i=1}^{2^{n-1}} \big(a_{ni}, b_{ni}\big)$. Clearly F is continuous on $[a, b]$, F is AC^*G on P, but F is not AC^*G on $[0, 1]$.

This example shows that the particular case of Theorem 2, I., also strictly contains Thomson's Theorem A, I. because our theorem holds for the function F, but Thomson's theorem doesn't.

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