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ON SETS OF CONVERGENCE POINTS OF SEQUENCES OF SOME REAL FUNCTIONS

Abstract

The purpose of this paper is to study a set of convergence points of sequence of real functions from a given class. Here, continuous functions, Borel measurable functions, approximately continuous functions and derivatives are considered.

The investigation of some sets determined by sequences of functions is motivated by the well-known result due to Hahn [1] and also Sierpiński [7] stating that a subset A of a Polish space X is of type $\mathcal{F}_{\sigma\delta}$ iff there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of continuous functions such that $A = \{x \in X : (f_n(x))_n \text{ converges}\}$ (see also [2, Theorem 23.18, p. 185]). It seems interesting to find the analogous characterization of sets of convergence points for sequences of functions from other classes. The same problem and its connections with additional set-theoretic axioms has been considered for transfinite sequences of functions in [4].

In the present paper the sequences of functions of Baire class α, approximately continuous functions and derivatives are examined. All functions considered here are real functions defined on the real line $\mathbb{R}$. Throughout this paper the following abbreviations for some classes of subsets of $\mathbb{R}$ will be used:

Π^0_1 (Σ^0_1) — closed (open) subsets of $\mathbb{R}$;

Π^0_α (Σ^0_α) — the multiplicative (additive) class α of Borel sets, $0 < \alpha < \omega_1$;

$\mathcal{M}$ — the σ-ideal of meager (first-category) subsets of $\mathbb{R}$;

$\mathcal{T}_d$ — the density topology (recall that $\mathcal{T}_d$ consists of all Lebesgue measurable sets having density 1 at each of its points [5]).

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Recall some definitions of various types of functions $f\colon \mathbb{R} \to \mathbb{R}$ which are investigated. Each class of functions is denoted by symbol on the left.

$\mathcal{B}_\alpha$ — f is of Baire class α, where $\alpha < \omega_1$ iff for every open set $U \subset \mathbb{R}$ $f^{-1}(U) \in \Sigma^0_{\alpha+1}$;

l_1 (u_1) — f is lower (resp. upper) semicontinuous iff it is a pointwise limit of non-decreasing (resp. non-increasing) sequence of continuous functions;

l_α (u_α) — f is of Young lower (resp. upper) class α, where $0 < \alpha < \omega_1$ iff it is a pointwise limit of non-decreasing (resp. non-increasing) sequence $\{f_n\colon n \in \mathbb{N}\} \subset \bigcup_{\beta<\alpha} u_\beta$ (resp. $\bigcup_{\beta<\alpha} l_\beta$);

$\mathcal{A}$ — f is approximately continuous iff for every open set $U \subset \mathbb{R}$ $f^{-1}(U) \in \mathcal{T}_d$.

Moreover, we denote by $b\mathcal{A}$ the class of all bounded approximately continuous functions and by Δ the class of all derivatives.

Denote by $L(\{f_n\colon n \in \mathbb{N}\})$ a set of all convergence points of a sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^\mathbb{R}$, i.e.

$$L(\{f_n\colon n \in \mathbb{N}\}) = \{x \in \mathbb{R}\colon \{f_n(x)\colon n \in \mathbb{N}\} \text{ converges}\}.$$

Remark 1.

$$L(\{f_n\colon n \in \mathbb{N}\}) = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \{x \in \mathbb{R}\colon |f_{n+k}(x) - f_n(x)| \le 1/m\}.$$

For a family of functions $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ define

$$\mathcal{L}(\mathcal{F}) = \{L(\{f_n\colon n \in \mathbb{N}\})\colon \{f_n\colon n \in \mathbb{N}\} \subset \mathcal{F}\}.$$

In this language the theorem obtained by Hahn and Sierpiński takes the following form.

Theorem 1. *For the family $\mathcal{C}$ of continuous functions $\mathcal{L}(\mathcal{C}) = \Pi^0_3$.*

To obtain its generalization onto the case of functions of Baire class α we will use some well-known facts.

Theorem 2. [8, Theorem 19, p. 30]
Let $\alpha < \omega_1$ and $f\colon \mathbb{R} \to [-\infty, +\infty]$. Then

- *$f \in l_\alpha$ iff $f^{-1}((a,\infty)) \in \Sigma^0_\alpha$ for each $a \in \mathbb{R}$;*
- *$f \in u_\alpha$ iff $f^{-1}((-\infty,a)) \in \Sigma^0_\alpha$ for each $a \in \mathbb{R}$.*

Remark 2. For $\alpha < \omega_1$, $\mathcal{B}_\alpha = l_{\alpha+1} \cap u_{\alpha+1}$.

Theorem 3. *For $\alpha < \omega_1$, $\mathcal{L}(\mathcal{B}_\alpha) = \Pi^0_{\alpha+3}$.*

Proof. Fix $\alpha < \omega_1$. The inclusion " $\subset$ " follows from Remark 1 and the definition of $\mathcal{B}_\alpha$ functions. So it remains to show that if $A \in \Pi^0_{\alpha+3}$, then $A = L(\{f_n : n \in \mathbb{N}\})$ for some sequence $\{f_n : n \in \mathbb{N}\} \subset \mathcal{B}_\alpha$. This follows by the same method as in [2, Theorem 23.18, p. 185]. Since $A \in \Pi^0_{\alpha+3}$, we have $A = \bigcap_{m\in\mathbb{N}} A_m$, where $A_m \in \Sigma^0_{\alpha+2}$. First, suppose that

(1) for every $m \in \mathbb{N}$ and $A_m \in \Sigma^0_{\alpha+2}$ there is a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathcal{B}_\alpha$ such that

- $|f^m_n(x)| \leq 1/m$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$,
- $A_m = L(\{f^m_n : n \in \mathbb{N}\}) = \{x \in \mathbb{R} : \lim_n f^m_n(x) = 0\}$.

Then rewrite $\bigcup_{m\in\mathbb{N}}\{f^m_n : n \in \mathbb{N}\}$ as a single sequence $\{f_i : i \in \mathbb{N}\}$. Of course, $\lim_i f_i(x) = 0$ for all $x \in A$. To see this fix $x \in A$, $\varepsilon > 0$ and take $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Then there is a positive integer $i_0 = i_0(\varepsilon)$ such that $|f^m_n(x)| < \varepsilon$ for $n > i_0$ and $m \leq k$. But for $m > k$ $|f^m_n(x)| \leq 1/m < 1/k < \varepsilon$, so $|f_i(x)| < \varepsilon$ for $i > i_0$. On the other hand, if $x \notin A$, then $x \notin A_{m_0}$ for some $m_0 \in \mathbb{N}$ and consequently $\{f^{m_0}_n(x) : n \in \mathbb{N}\}$ diverges, so $\{f_i(x) : i \in \mathbb{N}\}$ diverges too.

What is left is to show (1). Fix $m \in \mathbb{N}$ and $A_m \in \Sigma^0_{\alpha+2}$. Then $A_m = \bigcup_{n\in\mathbb{N}} F^m_n$, where $F^m_n \in \Pi^0_{\alpha+1}$ and $F^m_n \subseteq F^m_{n+1}$ for $n \in \mathbb{N}$. Consider the following real function $g : \mathbb{R} \to [1, +\infty]$:

$$g(x) = \begin{cases} 1 & \text{if } x \in F^m_1 \\ n & \text{if } x \in F^m_n \setminus F^m_{n-1} \text{ for } n \geq 2 \\ +\infty & \text{if } x \in \mathbb{R} \setminus A_m. \end{cases}$$

For $a \in \mathbb{R}$ we have $\{x \in \mathbb{R} : g(x) > a\} = \mathbb{R}$ if $a < 1$ or $\{x \in \mathbb{R} : g(x) > a\} = \mathbb{R} \setminus F^m_n \in \Sigma^0_{\alpha+1}$ if $n \leq a < n + 1$ for $n \geq 1$ and consequently $g \in l_{\alpha+1}$, by Theorem 2. It follows that there is a non-decreasing sequence $\{g_n : n \in \mathbb{N}\} \subset \mathcal{B}_\alpha$ pointwise convergent to g. For each $n \in \mathbb{N}$ put $\varphi_n = \min\{n, \max\{g_n, 1\}\}$. Then $1 \leq \varphi_n \leq n$ and $\varphi_{n+1} - \varphi_n \leq n$. Clearly, $\varphi_n \in \mathcal{B}_\alpha$ and $\lim_n \varphi_n = g$. Moreover, we can interpolate between φ_n and φ_{n+1} the functions $p^n_k = \varphi_n + \frac{k}{2n}(\varphi_{n+1} - \varphi_n)$ for $k = 0, 1, ..., 2n$. By renumbering we have a sequence $\{p_n : n \in \mathbb{N}\} \subset \mathcal{B}_\alpha$ such that $1 = p_0 \leq p_1 \leq p_2 \leq ...$, $p_{n+1} - p_n \leq 1/2$ and $f = \lim_n p_n$. Putting $f^m_n = \frac{1}{m}\sin(\pi p_n)$ we obtain the sequence satisfying (1). $\square$

The next result deals with the case of approximately continuous functions.

Lemma 1. (cf. [3, Lemma 5]) *Every set $A \in \Pi^0_2$ is a countable union of Π^0_2 sets closed in the density topology.*

PROOF. There are closed (in the usual sense) sets F_n, $n \in \mathbb{N}$ such that $F = \bigcup_{n\in\mathbb{N}} F_n \subset A$ and $A \setminus F$ has Lebesgue measure zero. Put $A_n = F_n \cup (A \setminus F)$. Then every A_n is closed in $\mathcal{T}_d$ and $A = \bigcup_{n\in\mathbb{N}} A_n$. Moreover, $A_n = F_n \cup (A \cap (\mathbb{R} \setminus F)) \in \Pi^0_2$. □

Corollary 1. *Every set $A \in \Pi^0_4$ can be represented in the form $A = \bigcap_{m\in\mathbb{N}} A_m$, where for $m \in \mathbb{N}$ A_m is a countable union of Π^0_2 sets closed in the density topology.*

Lemma 2. [9, Lemma 11, p. 26] *For every set $A \in \mathcal{T}_d \cap \Sigma^0_2$ there is an approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in (0, 1]$ if $x \in A$ and $f(x) = 0$ if $x \notin A$.*

Lemma 3. [6] *If $g : \mathbb{R} \to \mathbb{R}$ is a positive function such that for every $a \in \mathbb{R}$ $\{x \in \mathbb{R} : g(x) > a\} \in \mathcal{T}_d \cap \Sigma^0_2$, then g is a pointwise limit of a non-decreasing sequence of approximately continuous functions.*[1]

PROOF. Enumerate the set of all positive rational numbers as $\{q_k : k \in \mathbb{N}\}$. Fix $k \in \mathbb{N}$ and define $E_k = \{x \in \mathbb{R} : g(x) > q_k\}$. Then $E_k \in \mathcal{T}_d \cap \Sigma^0_2$ and by Lemma 2 there is an approximately continuous function $p_k : \mathbb{R} \to \mathbb{R}$ such that $0 < p_k \le 1$ on E_k and $p_k = 0$ on $\mathbb{R} \setminus E_k$. For $n \in \mathbb{N}$ put $p^k_n = \min\{q_k, np_k\}$. Then $q_k \cdot \chi_{E_k}$ is a pointwise limit of non-decreasing sequence $\{p^k_n : n \in \mathbb{N}\} \subset \mathcal{A}$ and $p^k_n < g$. Let $g_n = \max_{1\le k\le n} p^k_n$. It is easy to check that:

(i) $g_n \in \mathcal{A}$,

(ii) $g_n \le g_{n+1} < g$,

(iii) if $g(x) > q$ for some positive rational q, then $\lim_n g_n \ge q$.

It follows that $\lim_n g_n = g$ and $\{g_n : n \in \mathbb{N}\}$ is a required sequence. □

Remark 3. Lemma 3 holds also for $g : \mathbb{R} \to [0, +\infty]$

Now, we are able to prove the following.

Theorem 4. $\mathcal{L}(\mathcal{A}) = \Pi^0_4$

[1] This fact was announced during the conference Summer School on Real Functions Theory in Liptovský Ján (Slovakia) in 1998.

Proof. It is well-known that $\mathcal{A} \subset \mathcal{B}_1$ (see, e. g. [3]), so the inclusion " $\subset$ " follows from Theorem 3. Now, fix $A \in \Pi_4^0$. We claim that $A = L(\{f_n : n \in \mathbb{N}\})$ for some sequence $\{f_n : n \in \mathbb{N}\}$ of approximately continuous functions. To show this we apply the method similar to that in the proof of Theorem 3. By Corollary 1 $A = \bigcap_{m\in\mathbb{N}} A_m$, where for $m \in \mathbb{N}$ $A_m = \bigcup_{n\in\mathbb{N}} F_n^m$, $F_n^m \in \Pi_2^0$, $F_1^m \subseteq F_2^m \subseteq ...$ and $\mathbb{R} \setminus F_n^m \in \mathcal{T}_d$. Fix $m \in \mathbb{N}$. Then $g\colon \mathbb{R} \to [1, +\infty]$ given by the formula

$$g(x) = \begin{cases} 1 & \text{if } x \in F_1^m \\ n & \text{if } x \in F_n^m \setminus F_{n-1}^m \text{ for } n \geq 2 \\ +\infty & \text{if } x \in \mathbb{R} \setminus A_m \end{cases}$$

satisfies the assumptions of Lemma 3. Consequently, there is a non-decreasing sequence $\{g_n : n \in \mathbb{N}\} \subset \mathcal{A}$ pointwise convergent to g. The same construction as before gives us a sequence $\{f_n^m : n \in \mathbb{N}\} \subset \mathcal{A}$ such that $|f_n^m(x)| \leq 1/m$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $A_m = L(\{f_n^m : n \in \mathbb{N}\}) = \{x \in \mathbb{R} : \lim_n f_n^m(x) = 0\}$. Finally, it is enough to rewrite $\bigcup_{m\in\mathbb{N}}\{f_n^m : n \in \mathbb{N}\}$ as a single sequence $\{f_i : i \in \mathbb{N}\}$ to obtain the one we claimed. □

The last result is a consequence of Theorems 3 and 4.

Theorem 5. $\mathcal{L}(\Delta) = \Pi_4^0$

Proof. Note that the construction in Theorem 4 uses bounded approximately continuous functions, so we have actually proved that $\mathcal{L}(b\mathcal{A}) = \Pi_4^0$. Since $b\mathcal{A} \subset \Delta \subset \mathcal{B}_1$ (see e. g. [3]), the proof is complete. □

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