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ON THE VECTOR FORM OF THE LAGRANGE FORMULA, THE DARBOUX PROPERTY AND L'HÔPITAL'S RULE

Dedicated to the memory of my colleague Vasile Ene.

Abstract

We prove that the well-known Lagrange formula, the Darboux property and a classical result concerning the connected graph of a differentiable function are specific for $\mathbb{R}$, and surprisingly, the rule of L'Hôpital is also true for the vector case.

Proposition 1. *Let X be a topological vector space. Then the following assertions are equivalent:*

1. *For each $f : [0,1] \to X$, f continuous on $[0,1]$ and differentiable on $(0,1)$, there exists $c \in (0,1)$ such that: $f(1) - f(0) = f'(c)$.*
2. *For each $f : [0,1] \to X$, f continuous on $[0,1]$ and differentiable on $(0,1)$ with $f(1) = f(0)$, there exists $c \in (0,1)$ such that: $f'(c) = O$.*
3. *X is a real topological vector space and $\dim_{\mathbb{R}} X = 1$.*

PROOF. i) or ii) $\Rightarrow$ iii) Let $x \in X$, $x \neq O$. Let us suppose that X is a complex topological vector space. In this case, let

$$f : [0,1] \to X, \quad f(t) = (\cos 2\pi t + i \sin 2\pi t)x\,.$$

The hypotheses from i) or ii) are satisfied, thus there exists $c \in (0,1)$ such that: $f'(c) = O$ (because $f(1) = f(0) = x$). But

$$f'(c) = 2\pi(-\sin 2\pi c + i \cos 2\pi c))x\,,$$

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so $-\sin 2\pi c + i\cos 2\pi c = 0$, which contradicts the fact that $\cos^2 2\pi c + \sin^2 2\pi c = 1$. Hence X is a real topological vector space. In this case, let $y \in X$ and $g : [0,1] \to X$, $g(t) = x\cos 2\pi t + y\sin 2\pi t$. From i) or ii), there exists $c \in (0,1)$ such that $g'(c) = O$, hence $x\sin 2\pi c = y\cos 2\pi c$. If $\cos 2\pi c = 0$, then $\sin 2\pi c = \pm 1$. It follows that $O = y \cdot \cos 2\pi c = \pm x$, so $x = O$, which is impossible. Thus $\cos 2\pi c \neq 0$ and $y = (\tan 2\pi c)x$, hence $\dim_{\mathbb{R}} X = 1$.

iii) $\Rightarrow$ i) or ii) Let X be a topological real vector space, $\dim_{\mathbb{R}} X = 1$, and let $f : [0,1] \to X$ be as in i) or ii). Let $x \in X$, $x \neq O$. Since $f(t) \in X$, there exists an unique $\varphi(t) \in \mathbb{R}$ such that $f(t) = \varphi(t)x$, $\forall t \in [0,1]$. It is easy to prove that φ is continuous on $[0,1]$ and differentiable on $(0,1)$. Then the classical Rolle or Lagrange theorems applied to $\varphi : [0,1] \to \mathbb{R}$ imply i) or ii). □

Proposition 2. *Let X be a normed space. Then the following assertions are equivalent:*

1. *For each function $f : [0,1) \to X$ differentiable on $[0,1)$, it follows that: $f'([0,1)) \subset X$ is a connected subset.*
2. *X is a real normed space and $\dim_{\mathbb{R}} X = 1$.*

PROOF. i) $\Rightarrow$ ii) Let $x \in X$, with $\|x\| = 1$. Suppose that X is a complex normed space. Let $f : [0,1) \to X$,

$$f(t) = \begin{cases} 0 & \text{if } t = 0 \\ (t^2 \sin \frac{1}{t} + it^2 \cos \frac{1}{t})x & \text{if } t \in (0,1) \end{cases}$$

Obviously, f is differentiable on $(0,1)$. For $t > 0$,

$$\left\| \frac{f(t) - f(0)}{t} \right\| = \left\| (t \sin \frac{1}{t} + it \cos \frac{1}{t})x \right\| = t\,.$$

Thus

$$f^{'}(t) = \begin{cases} 0 & \text{if } t = 0 \\ [2t \sin \frac{1}{t} - \cos \frac{1}{t} + i(2t \cos \frac{1}{t} + \sin \frac{1}{t})]x & \text{if } t \in (0,1) \end{cases}.$$

Since $f'([0,1)) \subset X$ is a connected subset then :

$$\inf_{0<t<1} \left\| \left(2t \sin \frac{1}{t} - \cos \frac{1}{t}\right) + i\left(2t \cos \frac{1}{t} + \sin \frac{1}{t}\right)x \right\| = 0\,,$$

or $\|x\| = 1$, $\inf_{0<t<1}(4t^2 + 1) = 0$, a contradiction. Thus X is a real normed space. Let $y \in X$ and let $F : [0,1) \to X$ be defined as follows:

$$F(t) = \begin{cases} 0 & \text{if } t = 0 \\ (t^2 \sin \frac{1}{t})x + (t^2 \cos \frac{1}{t})y & \text{if } t \in (0,1) \end{cases}.$$

Clearly F is differentiable and

$$F'(t) = \begin{cases} 0 & \text{if } t = 0 \\ x\big(2t\sin\frac{1}{t} - \cos\frac{1}{t}\big) + y\big(2t\cos\frac{1}{t} + \sin\frac{1}{t}\big) & \text{if } t \in (0,1) \end{cases}$$

By i), it follows that $F'([0,1)) \subset X$ is connected, hence

$$\inf_{0<t<1} \big\| xh(t) + yg(t) \big\| = 0\,, \tag{1}$$

where

$$h(t) = 2t\sin\frac{1}{t} - \cos\frac{1}{t}, \quad g(t) = 2t\cos\frac{1}{t} + \sin\frac{1}{t}, \quad t \in (0,1]\,.$$

By (1), there exists a sequence $(t_n)_{n\in N} \subset (0,1)$ such that

$$\big\| xh(t_n) + yg(t_n) \big\| \to 0\,. \tag{2}$$

The sequence $(t_n)_{n\in N}$ being bounded, we can choose a subsequence (which for simplicity will also be denoted by $(t_n)_{n\in N}$) convergent to $t_o \in [0,1]$. If $0 < t_0 \le 1$, using the continuity of the functions h and g on $(0,1)$, by (2), it follows that $\|xh(t_0) + yg(t_0)\| = 0$, hence $xh(t_0) + yg(t_0) = O$. If $g(t_0) = O$, as $h^2(t_0) + g^2(t_0) = 4t_0^2 + 1 > 1$, we obtain: $xh(t_0) = -yg(t_0) = 0$. Then, $h(t_0) \neq 0$, implies $x = 0$, which contradicts the fact that $\|x\| = 1$. Thus $g(t_0) \neq O$ and $y = \lambda x$, with $\lambda = -\frac{h(t_0)}{g(t_0)} \in \mathbb{R}$. If $t_0 = 0$, then $t_n \to 0$, and by (2) we have:

$$\begin{gathered} \Big\| y\sin\frac{1}{t_n} - x\cos\frac{1}{t_n} \Big\| \le \\ \le \big\| xh(t_n) + yg(t_n) \big\| + \Big\| 2t_n \sin\frac{1}{t_n} x \Big\| + \Big\| 2t_n \cos\frac{1}{t_n} y \Big\| \le \\ \le 2|t_n|\big(\|x\| + \|y\|\big) + \big\| xh(t_n) + yg(t_n) \big\| \to 0\,. \end{gathered}$$

Thus:

$$\Big\| y\sin\frac{1}{t_n} - x\cos\frac{1}{t_n} \Big\| \to 0\,.$$

From here we obtain:

$$\inf_{0<t<1} \Big\| y\sin\frac{1}{t} - x\cos\frac{1}{t} \Big\| \le \Big\| y\sin\frac{1}{t_n} - x\cos\frac{1}{t_n} \Big\| \to 0\,,$$

i.e.

$$\inf_{0<t<1} \Big\| y\sin\frac{1}{t} - x\cos\frac{1}{t} \Big\| = 0\,,$$

or using the periodicity of $\sin e$ and $\cos e$,

$$\inf_{0\leq\theta\leq 2\pi} \|y\sin\theta - x\cos\theta\| = 0\,.$$

Let $0 \leq \theta_n \leq 2\pi$ be such that:$\| y\sin\theta_n - x\cos\theta_n \|\to 0$. Extract a convergent subsequence which will be denoted also by $\theta_n \to \theta$, where $\theta \in [0, 2\pi]$. Hence: $\|y\sin\theta - x\cos\theta\| = 0$, i.e. $y\sin\theta = x\cos\theta$. If $\sin\theta = 0$, then $\cos\theta = \pm 1$, from where $0 = y\sin\theta = x\cos\theta = \pm x$, i.e. $x = 0$, contradiction! Thus, $\sin\theta \neq 0$ and hence: $y = \lambda x$, with $\lambda = \cot\theta$, i.e. ii) is proved.

ii) $\Rightarrow$ i) Let X be a normed space with $\dim_{\mathbb{R}} X = 1$, $x \in X$ with $\|x\| = 1$, and let $f : [0, 1) \to X$ be a differentiable function. Then there exists a unique element $\varphi(t) \in \mathbb{R}$ such that $f(t) = \varphi(t)\cdot x$, $\forall t \in [0, 1)$, i.e. there exists $\varphi : [0, 1) \to \mathbb{R}$ such that $f(t) = \varphi(t)x$. Because f is differentiable on $[0, 1)$ and $\|x\| = 1$ we obtain easily that φ is derivable on $[0, 1)$. Using the Darboux theorem for $\varphi : [0, 1) \to \mathbb{R}$, we obtain that $I = \varphi'([0, 1)) \subset \mathbb{R}$ is an interval, i.e. a connected set. Since $f'(t) = \varphi'(t)x$, $\forall t \in [0, 1)$, it follows that $f'([0, 1)) = x\cdot I$ is a connected subset in X. $\square$

Recall that if X is a topological space a subset $A \subset X$ is called path connected if for each x_0, $x_1 \in A$, there exists $\gamma : [0, 1] \to X$, a continuous functions such that: $\gamma(0) = x_0$, $\gamma(1) = x_1$, $Im\gamma \subset A$. In the sequel we need the following, probably well-known result. We give the proof for the completeness.

Proposition 3. *Let X be a metric space, $f : (0,\infty) \to X$, be a continuous function, $a \in X$ and $A = \{(0, a)\} \cup \{(x, f(x)) \mid x > 0\} \subset (0,\infty) \times X$. Then:*

1. *A is connected if and only if there exists a sequence $(x_n)_{n\in N} \subset (0,\infty)$ such that: $x_n \to 0$, and $f(x_n) \to a$.*
2. *A is path connected if and only if: $\lim\limits_{x\to 0,\ x>0} f(x) = a$.*

PROOF. a) If A is connected then: $d((0, a), G) = \inf\limits_{x>0} d((0, a), (x, f(x))) = 0$, $\inf\limits_{x>0}(x+d(f(x), a)) = 0$. (Here G is the graph of f). Conversely, this condition implies: $(0, a) \in \overline{G}$, $\{(0, a)\} \cap \overline{G} = \{(0, a)\} \neq \emptyset$, hence: $A = \{(0, a)\} \cup G$ is connected. But $\inf\limits_{x>0}(x + d(f(x), a)) = 0$, if and only if there exists a sequence $(x_n)_{n\in N} \subset (0,\infty)$ such that: $x_n \to 0$, and $f(x_n) \to a$.

b) Let us suppose that A is path connected. Then for the points $(0, a)$ and $(1, f(1)) \in A$ there exists a continuous path contained in A i.e. $\gamma : [0, 1] \to \mathbb{R}^2$, $\gamma = (g, h)$, continuous such that: $\gamma(0) = (0, a)$, $\gamma(1) = (1, f(1))$, $Im\gamma \subset A$. So: $g(0) = 0$, $g(1) = 1$. Let

$$E = \{t \in [0, 1] \mid g(t) = 0\} = g^{-1}(\{0\})\,,$$

$\alpha = \sup E$. The set E is nonvoid, since $g(0) = 0$. As $g(1) = 1 > 0$, $\alpha \neq 1$, i.e. $0 \leq \alpha < 1$. But g is continuous and E is closed, so $\alpha \in E$, $g(\alpha) = 0$, i.e. $\alpha = \max E$. Hence if $t > \alpha$, $t \in [0, 1]$, $g(t) \neq 0$, so $\gamma(t) = (g(t), h(t)) \in A$, i.e. $g(t) > 0$, $h(t) = f(g(t))$. Thus: $\forall t \in (\alpha, 1]$, $h(t) = f(g(t))$ and using the continuity of h:

$$\lim_{t\to\alpha,\ t>\alpha} h(t) = h(\alpha), \qquad \lim_{t\to\alpha,\ t>\alpha} f(g(t)) = h(\alpha)\,. \tag{3}$$

We prove that:

$$\lim_{x\to 0,\ x>0} f(x) = h(\alpha)\,. \tag{4}$$

If this is not so, then: there exists $\varepsilon_0 > 0$, a sequence $0 < x_n < \frac{1}{n}$, such that:

$$d(f(x_n), h(\alpha)) \geq \varepsilon_0, \quad \forall n \in N\,. \tag{5}$$

From $g(\alpha) = 0 < x_n < 1 = g(1)$, and the Darboux property of g, there is $\alpha < t_n < 1$, such that: $g(t_n) = x_n$. Extract a convergent subsequence: $t_{k_n} \to t \in [\alpha, 1]$. The continuity of g implies: $x_{k_n} = g(t_{k_n}) \to g(t)$, $g(t) = 0$, $t \in E$, $t \leq \alpha$. But $t \geq \alpha$, so $t = \alpha$. Hence: $t_{k_n} \to \alpha$, $t_{k_n} > \alpha$ and (3) gives: $f(g(t_{k_n})) \to h(\alpha)$, $f(x_{k_n})) \to h(\alpha)$, which contradicts (5). Thus (4) is proved. But if A is path connected, then it is connected, so by a) there exists a sequence $(x_n)_{n\in N} \subset (0, \infty)$ such that: $x_n \to 0$, and $f(x_n) \to a$. Using (4):

$$f(x_n) \to h(\alpha), \quad h(\alpha) = a, \text{ i.e. } \lim_{x\to 0,\ x>0} f(x) = a\,.$$

The converse is clear. $\square$

Using the same functions as in the Proposition 2, with the help of Proposition 3, we must show that a classical result, see [3], is also specified to $\mathbb{R}$.

Proposition 4. *Let X be a normed space. Then the following assertions are equivalent:*

1. *For each function $f : [0, 1) \to X$ differentiable on $[0, 1)$, it follows that the graph of f' is a connected subset.*
2. *X is a real normed space and $\dim_{\mathbb{R}} X = 1$.*

Also, minor changes in the above proofs, show that, they are also true if instead of a normed space X, we suppose that X is a vector space endowed with a F-norm, or a p-norm $(0 < p \leq 1)$, see [2] for definitions.

Now a positive result for the classical L'Hôpital rule.

Theorem 1. (The L'Hôpital rule, cases $[\frac{0}{0}]$, resp. $[\frac{\cdot}{\infty}]$)
Let X be a normed space, $(a, b) \subseteq \mathbb{R}$ be an open interval and $f : (a, b) \to X$, $g : (a, b) \to \mathbb{R}$ two functions with the properties:

1. $\lim\limits_{\substack{t \to a \\ t > a}} f(t) = 0$ *and* $\lim\limits_{\substack{t \to a \\ t > a}} g(t) = 0$*; (respectively* $\lim\limits_{\substack{t \to a \\ t > a}} |g(t)| = \infty$*);*
2. *f is differentiable on (a, b) and g is differentiable on (a, b);*
3. $g'(t) \neq 0$, $\forall t \in (a, b)$;
4. *The limit* $\lim\limits_{\substack{t \to a \\ t > a}} \frac{f'(t)}{g'(t)} \in X$ *exists.*

Then there exists

$$\lim_{\substack{t \to a \\ t > a}} \frac{f(t)}{g(t)} \in X \quad \textit{and in addition:} \quad \lim_{\substack{t \to a \\ t > a}} \frac{f(t)}{g(t)} = \lim_{\substack{t \to a \\ t > a}} \frac{f'(t)}{g'(t)} \,.$$

PROOF. Since g' has the Darboux property, by 3) and 2), it follows that g' has a constant sign on (a, b), hence g is strictly monotone on (a, b). Let

$$x = \lim_{\substack{t \to a \\ t > a}} \frac{f'(t)}{g'(t)} \in X \,.$$

Then for $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$\|\frac{f'(t)}{g'(t)} - x\| < \varepsilon \,, \quad \text{hence:} \quad \| f'(t) - xg'(t) \| < \varepsilon |g'(t)| \,,$$

on each compact subinterval $[u, v] \subset (a, a + \delta)$. By the Denjoy-Bourbaki Theorem, it follows that:

$$\|f(v) - xg(v) - f(u) + xg(u)\| \leq \varepsilon \mid g(v) - g(u) \mid \,. \tag{$*$}$$

We prove the case $[\frac{0}{0}]$. By 1) and $(*)$, for $u \searrow a$ it follows that:

$$\|f(v) - xg(v)\| \leq \varepsilon |g(v)| \,, \quad \text{hence:} \quad \|\frac{f(v)}{g(v)} - x\| < \varepsilon \,.$$

Therefore

$$\lim_{\substack{v \to a \\ v > a}} \frac{f(v)}{g(v)} = x \,.$$

We prove the case $[\frac{}{\infty}]$. We may suppose that $g(t) \neq 0$ on $[u, v]$. By 1) and $(*)$, we have:

$$\left\| \frac{f(u)}{g(u)} - x \right\| \leq \frac{\|f(v) - xg(v)\|}{|g(u)|} + \epsilon \left| \frac{g(v)}{g(u)} - 1 \right| ,$$

hence :

$$\limsup_{u\to a, u>0} \left\| \frac{f(u)}{g(u)} - x \right\| \leq \epsilon\,.$$

Since ε is arbitrary, we obtain that:

$$x = \lim_{\substack{u\to a\\ u>a}} \frac{f(u)}{g(u)}\,.$$

$\square$

References

[1] N. Bourbaki, *Fonctions d'une variable reele*, Paris, Herman, 1961.

[2] N. J. Kalton, N. T. Peck, J. W. Roberts, *An F-space sampler*, Cambridge Univ. Press, 1984.

[3] K. Kuratowski, *Topology*, vol. II, Academic Press, 1968.