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ON ORDER TOPOLOGIES AND THE REAL LINE

Abstract

We find order topologies that are universal for certain topological properties. An order topology T enjoys a given property if and only if there is an order preserving homeomorphism of T into the universal space for this property. We give similar results for order preserving mappings in place of homeomorphisms.

Throughout this note, T will be a linearly ordered set. By the order topology on T [K, 1I] we mean the topology generated by all intervals of the form $\{x \in T : x < a\}$ and $\{x \in T : x > a\}$ where $a \in T$. We will call this the order space T, or the open interval space T.

We desire to find an order space S that is separable (has a countable dense subset) such that any order space T is separable if and only if T is homeomorphic to a subspace of S. In fact we do not succeed, but we do the next best thing. We find a topology T^* on the set T, closely associated with the order topology and containing the order topology on T, such that the order space T is separable if and only if T^* is homeomorphic to a subspace of an appropriate separable order space S.

Let $\mathbb{R}$ denote the real line. Give the product $\mathbb{R} \times \{0, 1\}$ the dictionary order: $(a, n) < (b, m)$ if and only if $a < b$ in $\mathbb{R}$ or $a = b$ and $n = 0$, $m = 1$. We call the resulting order space $\mathbb{R}_0$. Note that every subspace of $\mathbb{R}_0$ is disconnected. Thus $\mathbb{R}$ is not homeomorphic to any subspace of $\mathbb{R}_0$, even though $\mathbb{R}$ is obviously order isomorphic to a subset of $\mathbb{R}_0$.

By the enhanced order space T^* we mean the topology on the set T generated by all intervals of the form $\{x \in T : x < a\}$ and $\{x \in T : x > a\}$ for any $a \in T$, and of the form $\{x \in T : x \leq b\}$ where $b \in T$ is an element that has no immediate predecessor. Thus the enhanced order topology contains the order topology on T.

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Now $\mathbb{R}^*$ is commonly called the half open interval space [K, page 59] or the Sorgenfrey line. Moreover, $\mathbb{R}^*$ is separable; consider the set of rational numbers.

The enhanced order space T^* is the same as the order space T if each element of T has an immediate predecessor or an immediate successor. Thus the spaces $\mathbb{R}_0^*$ and $\mathbb{R}_0$ are the same. Moreover, $\mathbb{R}_0$ is separable; consider the set $\{(r,0) : r \text{ rational}\}$. It is easy to see that $\mathbb{R}^*$ is homeomorphic to a subspace of $\mathbb{R}_0$.

We will prove that T is a separable order space if and only if T^* is homeomorphic to a subspace of $\mathbb{R}_0$ (Theorem III).

It will be easier to prove that T is second countable (has a countable basis) if and only if T is homeomorphic to a subspace of $\mathbb{R}$ (Theorem II).

It will be almost trivial to prove that T is countable if and only if T is homeomorphic to a subspace of $\mathbb{Q}$, the space of rational numbers (Theorem I).

Let T be countable. Then the order space T is obviously a second countable regular space and is metrizable by Urysohn's metrization Theorem [K, p. 125]. Then T is homeomorphic to a subspace of $\mathbb{Q}$ by [S, p. 107].

Observe that an analogous argument proves that T^* is homeomorphic to a subspace of $\mathbb{Q}$ if T is countable.

To sum up; $\mathbb{R}_0$ is our prototype of a separable order space, $\mathbb{R}$ is our prototype of a second countable order space, and $\mathbb{Q}$ is our prototype of a countable order space.

Construction

Let E be a countable linearly ordered set. Adjoin to E points $-\infty$ and ∞ where $-\infty$ is less than any element and ∞ is greater than any element of E.

We will construct a mapping f of $E \cup \{-\infty, \infty\}$ into $\mathbb{Q}$ as follows.
Enumerate $E = \{e_1, e_2, e_3, \ldots\}$. Let $f(-\infty) = -1$ and $f(\infty) = 1$, $f(e_1) = 0$. We define $f(e_n)$ by induction on n. Let f be an order preserving mapping of $\{-\infty, \infty\} \cup \{e_1, \ldots, e_{n-1}\}$ into $\mathbb{Q}$. Make $f(e_n) = \big(f(a) + f(b)\big)/2$ where $e_n \in (a,b)$ and a and b are consecutive points of the set $\{-\infty, \infty\} \cup \{e_1, \ldots, e_{n-1}\}$. It follows that f preserves order on $\{-\infty, \infty\} \cup \{e_1, \ldots, e_{n-1}, e_n\}$. By induction we see that f is an order preserving mapping of E into Q.

Lemma 1. *Let (c,d) be a bounded complementary interval of the closure of $f(E)$. Then $c \in f(E)$ and $d \in f(E)$.*

Proof. Assume $c \notin f(E)$. Choose c_1 and d_1 in $f(E)$ such that $c_1 < c < d \le d_1$ and $d_1 - c_1 < 2(d-c)$. The interval (c_1, d_1) contains infinitely many points of $f(E)$. It follows from the construction that some $e \in E$ satisfies

$f(e) = (c_1 + d_1)/2$. Clearly $(c_1 + d_1)/2 \in (c, d)$, contrary to hypothesis. Thus $c \in f(E)$. Similarly $d \in f(E)$. $\square$

In what follows, the symbol $^-$ will denote closure in the Euclidean topology.

Lemma 2. *Let $c \in f(E)^-$ and let u be a real number with* $\sup f(E) \geq u > c$. *Then there is an $e \in E$ with $f(E) > c$ and*

$$\left\{t \in f(E)^- : t < f(e)\right\} \subset \left\{t \in f(E)^- : t < u\right\}.$$

Proof. If $f(E) \cap (c, u)$ is nonvoid, just make $f(e) \in f(E) \cap (c, u)$. So we assume $f(E) \cap (c, u)$ is void. Then there is a $d \in f(E)^-$ such that $u \in (c, d)$ and (c, d) is a finite complementary interval of $f(E)^-$. By Lemma 1, $d \in f(E)$. Then $f(e) = d$ suffices. $\square$

When $c = \sup f(E)$, Lemma 2 will not be needed in what follows.

Lemma 2′. *Let $c \in f(E)^-$ and let u be a real number with* $\inf f(E) \leq u < c$. *Then there is an $e \in E$ with $f(e) < c$ and*

$$\left\{t \in f(E)^- : t > f(e)\right\} \subset \left\{t \in f(E)^- : t > u\right\}.$$

The proof is like that of Lemma 2 with the inequalities reversed, so we omit it.

Theorem I. *Let T be an order space. Then the following are equivalent.*

(1) T is countable,

(2) there is an order preserving homeomorphism from T into Q,

(3) T is homeomorphic to a subspace of Q,

(4) T is order isomorphic to a subset of Q.

Proof. (1) $\Rightarrow$ (2) Let f be the order preserving mapping of T into Q from the construction (with T in place of E). The subspace topology on $f(T)$ in Q contains the order topology on $f(T)$. Let $x \in T$ and $f(x)$ lie in a subbasic open set of one of the forms $\{w \in f(T) : w > a\}$ or $\{w \in f(T) : w < a\}$ in the subspace topology on $f(T)$. We deduce from Lemmas 2 and 2′ that this set contains an open set in the order topology on $f(T)$ containing $f(x)$. It follows that any open set in the subspace topology on $f(T)$ is an open set in the order topology on $f(T)$. Hence f is a homeomorphism.

Finally, the implications (1) $\Longleftrightarrow$ (2), (1) $\Longleftrightarrow$ (3), (1) $\Longleftrightarrow$ (4) are clear. $\square$

Lemma 3. *T is second countable if and only if T is separable and there are at most countably many immediate predecessors (successors) in T.*

Proof. Suppose T is second countable. Then T is separable. Let $\{U_n\}$ be a countable basis of T. For each immediate predecessor a, there is an open set $V_a \in \{U_n\}$ such that a is the greatest element of V_a. Thus if a_1 is another immediate predecessor in T, distinct from a, then $V_a \neq V_{a_1}$. It follows that there are countably many immediate predecessors in T.

Now suppose that the order space T is separable and there are countably many immediate predecessors in T. Let E be a countable subset of T that is dense and includes all the immediate predecessors and successors in T. The proof that the countable family of intervals $\{(e, e') : e \in E,\ e' \in E\}$ is a basis of T is routine, so we leave it. □

Theorem II. *Let T be an order space. Then the following are equivalent.*

(1) T is second countable.

(2) there is an order preserving homeomorphism of T into $\mathbb{R}$.

(3) T is homeomorphic to a subspace of $\mathbb{R}$.

(4) T is order isomorphic to a subset of $\mathbb{R}$.

Proof. (1) $\Rightarrow$ (2) Let T be second countable. By Lemma 3, there is a countable dense subset E of T that contains all the immediate predecessors and successors in T and also the first and last elements of T if any. Let f be the mapping in the construction. For each $x \in T \setminus E$ let

$$A_x = \left\{t \in E : t > x\right\} \text{ and } B_x = \left\{t \in E : t < x\right\}.$$

Then $E = A_x \cup B_x$.

If $x \in T \setminus E$ and $y \in T \setminus E$ and $x < y$, we claim that $f(A_x) \neq f(A_y)$; for if $f(A_x) = f(A_y)$, then $A_x = A_y$ and the interval (x, y) could contain no element of E or of T, and hence by Lemma 1, $x \in E$, $y \in E$. Likewise $f(B_x) \neq f(B_y)$.

If $x \in T \setminus E$ we claim that the distance between the sets $f(A_x)$ and $f(B_x)$ is zero; for otherwise, $f(A_x)$ would have a least element d, $f(B_x)$ would have a greatest element c by Lemma 1, and hence the singleton set $\{x\}$ would be open in T and $x \in E$.

If $x \in T \setminus E$, we claim that $f(B_x)$ has no greatest element; for if c is the greatest element of $f(B_x)$, then the interval $(f^{-1}(c), x)$ could contain no element of E or of T and consequently x would be an immediate successor and $x \in E$. Likewise $f(A_x)$ has no least element.

We extend f to a function g from T into $\mathbb{R}$ as follows; If $e \in E$, put $g(e) = f(e)$, and if $x \in T \setminus E$, let $g(x)$ be the real number so that $g(x) > f(e)$ for $e \in B_x$ and $g(x) < f(e)$ for $e \in A_x$. It follows that g is an order preserving mapping of T into $\mathbb{R}$.

The subspace topology contains the order topology on $g(T)$. From Lemmas 2 and 2′ we deduce that these topologies are in fact the same on $g(T)$. Thus g is an order preserving homeomorphism of T into $\mathbb{R}$.

(4) $\Rightarrow$ (1) Let S be a subset of $\mathbb{R}$. The subspace topology on S is at least as fine as the order topology on S. But $\mathbb{R}$ is second countable, so the subspace S is second countable and separable. It follows that the order space S is separable. Finally, S contains at most countably many immediate successors and predecessors, because every family of mutually disjoint intervals in $\mathbb{R}$ is countable. By Lemma 3, the order space S is second countable.

Now the implications (1) $\Longleftrightarrow$ (2), (1) $\Longleftrightarrow$ (3), (1) $\Longleftrightarrow$ (4) are clear. $\square$

Before we tackle our last theorem we need an observation about the order space $\mathbb{R}_0$.

Lemma 4. *Let S be a subset of $\mathbb{R}_0$. Then the order space S is separable and the subspace S of $\mathbb{R}_0$ is separable.*

Proof. Put $A = \{a \in \mathbb{R} : \text{either } (a, 0) \text{ or } (a, 1) \text{ is an isolated point of } S \text{ in } \mathbb{R}_0\}$. We claim that A is countable; for otherwise there would be an $a_0 \in A$ that is both a left and a right accumulation point of A in $\mathbb{R}$, and hence $(a_0, 0)$ would be a left accumulation point of S and $(a_0, 1)$ would be a right accumulation point of S in $\mathbb{R}_0$. Thus there are at most countably many isolated points of S. Let J denote the set of isolated points in S.

For any rational numbers $r_1 < r_2$, select an element $s(r_1, r_2)$ in S in the open interval from $(r_1, 0)$ to $(r_2, 0)$ if there is one. We obtain a countable family $P \subset S$ of elements of the form $s(r_1, r_2)$. Then $P \cup J$ is a countable subset of S, and routine arguments prove that $P \cup J$ is dense in the subspace S. Thus the subspace S is separable. But the subspace topology contains the order topology on S. It follows that the order space S is also separable. $\square$

Theorem III. *Let T be an order space. Then the following are equivalent.*

(1) T is separable,

(2) there is an order preserving homeomorphism h of T^* into $\mathbb{R}_0$,

(3) T^* is homeomorphic to a subspace of $\mathbb{R}_0$,

(4) T is order isomorphic to a subset of $\mathbb{R}_0$,

(5) T^* is separable.

PROOF. (1) $\Rightarrow$ (2) Let T be separable. Say E_0 is a countable dense subset of T. Let $E = \{x \in T : x \in E_0$ or x is an immediate successor or predecessor of an element of $E_0\}$. Then E is a countable dense subset of T. Note that if x is the immediate predecessor of $y \in E \setminus E_0$, then either y is the immediate successor of a member of E_0 and $x \in E_0$, or y is the immediate predecessor of a member of E_0 and y is an isolated point necessarily in E_0 contrary to assumption. Thus the immediate predecessor of a member of E must lie in E. Likewise the immediate successor of a member of E must lie in E. Put $T_1 = \{t \in T : t$ has an immediate predecessor but no immediate successor in $T\}$, and $T_2 = T \setminus T_1$.

Observe that if $y \in T_1 \cap E$ and x is the immediate predecessor of y, then $x \in T_2 \cap E$. Thus if $u < v < w < z$ and $u, v, w, z \in T_2$, then the set $(u, w) \cap E$ contains an element e and either $e \in T_2 \cap E$, or $e \in T_1 \cap E$ and the immediate predecessor of e lies in $T_2 \cap E$. It follows that the interval $[u, w)$ meets $T_2 \cap E$. Likewise $[v, z)$ and (u, z) meet $T_2 \cap E$. We deduce from this that any open interval (u, w) with $u \in T_2$, $w \in T_2$ that meets T_2 but not $T_2 \cap E$ has $u \in T_2 \cap E$. Moreover, any two such intervals cannot have the same left endpoint. But $T_2 \cap E$ is countable, so there are at most countably many such intervals. It follows that the order space T_2 is separable.

Now let (s, t) be an open interval with $s \in T_2$, $t \in T_2$ such that the open interval (s, t) does not meet T_2. Then either (s, t) meets T and we have $s \in T_2 \cap E$ by the argument in the preceding paragraph, or t is the immediate successor of s and t must be an isolated point necessarily in $T_2 \cap E$. In any case, the closed interval $[s, t]$ meets $T_2 \cap E$. But E is countable so there are at most countably many immediate successors and predecessors in the order space T_2. By Lemma 3 the order space T_2 is second countable.

Let g be the order preserving homeomorphism from T_2 into $\mathbb{R}$ defined precisely as in the proof of (1) $\Rightarrow$ (2) for Theorem II. Put $h(x) = \big(g(x), 0\big) \in \mathbb{R}_0$ for $x \in T_2$. Observe that if $y \in T_1$, then y has an immediate predecessor that is necessarily in T_2. For $y \in T_1$, put $h(y) = \big(g(x), 1\big) \in \mathbb{R}_0$ where x is the immediate predecessor of y. It follows that h is an order preserving mapping of T into $\mathbb{R}_0$. It remains to prove that h is a homeomorphism from T^* into $\mathbb{R}_0$. This will be deduced from considerations of the following four situations.

(i) $x \in T$ and x has an immediate successor and an immediate predecessor in T. Then $h(x)$ has an immediate successor and an immediate predecessor in $h(T)$. It follows that the singleton set $\{h(x)\}$ is open in the enhanced order topology of $h(T)$ and open in the subspace topology of $h(T)$.

(ii) $x \in T$ has an immediate predecessor y, but no immediate successor, and x is not the greatest element in T. It follows that $x \in T_1$ and $y \in T_2$. It also follows that y has no immediate successor relative to the set T_2 (observe that any element of T_1 exceeding x is the immediate successor of another element of T_2 exceeding x). We deduce that the distance (in $\mathbb{R}$) between $g(y)$ and the set $\{g(t) : t \in T_2,\ t > y\}$ is zero; otherwise $\inf\{g(t) : t \in T_2,\ t > y\}$ would be the immediate successor of $g(y)$ in $g(T_2)$ by Lemma 1, and y would have an immediate successor in T_2. From this it follows that the set $\{W \in h(T)\, W \leq h(x)\}$ is not open in the subspace topology on $h(T)$. But from the definition of the enhanced order topology we see that this set is not open in the enhanced order topology on $h(T)$ either. On the other hand, the set $\{W \in h(T) : W \geq h(x)\}$ is open in the enhanced order topology and open in the subspace topology on $h(T)$.

Let $v \in \mathbb{R}$ with $v > g(y)$. By Lemma 2 there is a $u \in g(T_2)$ with $u > g(y)$ such that

$$\bigl\{w \in T_2 : g(w) < u\bigr\} \subset \bigl\{w \in T_2 : g(w) < v\bigr\}\,.$$

Hence in $\mathbb{R}_0$,

$$\bigl\{W \in h(T) : W < (u,0)\bigr\} \subset \bigl\{W \in h(T) : W < (v,0)\bigr\}\,,$$

where the right member is open in the subspace topology on $h(T)$ and the left member is open in both the enhanced order topology and the subspace topology on $h(T)$.

(iii) $x \in T$ has neither an immediate successor nor an immediate predecessor in T, and x is neither the greatest nor the least element of T. It follows that $x \in T_2$ and $h(x) = \bigl(g(x), 0\bigr)$. Now $\bigl(g(x), 1\bigr) \in \mathbb{R}_0$, so the set $\{W \in h(T) : W \leq h(x)\}$ is open in the subspace topology on $h(T)$. By definition, this set is also open in the enhanced order topology on $h(T)$.

Take any $v' \in \mathbb{R}$ with $v' < g(x)$. It follows from Lemma 2$'$ that there exists a $u' \in g(T)$ such that $u' < g(x)$ and

$$\bigl\{w \in T_2 : g(w) > u'\bigr\} \subset \bigl\{w \in T_2 : g(w) > v'\bigr\}\,.$$

It follows that

$$\bigl\{W \in h(T) : W > (u',0)\bigr\} \subset \bigl\{W \in h(T) : W > (v',0)\bigr\}\,,$$

where the right member is open in the subspace topology on $h(T)$, and the left member is open in both the subspace topology and the enhanced order topology on $h(T)$.

(iv) $x \in T$ has an immediate successor $y \in T$ but no immediate predecessor in T, and T has neither a greatest nor a least element. It follows that $x \in T_2$ and $h(x) = \bigl(g(x), 0\bigr)$. Now $\bigl(g(x), 1\bigr) \in \mathbb{R}_0$, so the set $\{W \in h(T) : W \leq h(x)\}$

is open in the subspace topology on $h(T)$, and is also open in the enhanced order topology on $h(T)$.

Take any $v' \in \mathbb{R}$ with $v' < g(x)$. Just as in the argument for case (iii) we find a $u' \in g(T)$ such that $u' < g(x)$ and

$$\left\{W \in h(T) : W > (u', 0)\right\} \subset \left\{W \in h(T) : W > (v', 0)\right\},$$

where the right member is open in the subspace topology on $h(T)$ and the left member is open in both the subspace topology and the enhanced order topology on $h(T)$.

Let $x \in T$ and $h(x)$ lie in a subbasic open set of one of the forms $\{W \in h(T) : W > A\}$, $\{W \in h(T) : W < A\}$, $\{W \in h(T) : W \leq A\}$ in the enhanced order topology on $h(T)$. Then it follows from (i), (ii), (iii), (iv) that this set contains an open set in the subspace topology on $h(T)$ containing $h(x)$. Thus if T has no greatest or least element, every open set in the enhanced order topology on $h(T)$ is also open in the subspace topology on $h(T)$. Analogous arguments using (i), (ii), (iii), (iv) show that if T has no greatest or least element, every open set in the subspace topology on $h(T)$ is also open in the enhanced order topology on $h(T)$. We can relax the assumption that T have no greatest or least element by adjoining to T a copy of the ordered set of rational numbers, $\mathbb{Q}$, all of whose elements exceed all the elements of T and a copy of all $\mathbb{Q}$ all of whose elements are exceeded by all elements of T. Finally, the subspace topology and the enhanced order topology coincide on $h(T)$, and (1) $\Rightarrow$ (2).

From (1) $\Rightarrow$ (2) we deduce that (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4). From Lemma 4 we deduce that (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). From Lemma 4 we also deduce that (3) $\Rightarrow$ (5). Finally, the enhanced order topology contains the order topology, so it follows that (5) $\Rightarrow$ (1). $\square$

If each element of T is an immediate successor or predecessor, then the enhanced order topology coincides with the order topology on $h(T)$. For such order spaces we deduce the following from Theorem III.

Corollary 1. *Let T be an order space in which each element is an immediate successor or predecessor. Then T is separable if and only if T is homeomorphic to a subspace of $\mathbb{R}_0$.*

From Theorems I, II and III we obtain:

Corollary 2. *Let S_0 be a (nonvoid) subset of $\mathbb{R}_0$, let S_1 be a (nonvoid) subset of $\mathbb{R}$ and let S_2 be a (nonvoid) subset of $\mathbb{Q}$. Then the enhanced order space S_0^* is homeomorphic to a subspace of $\mathbb{R}_0$, the order space S_1 is homeomorphic to a subspace of $\mathbb{R}$, and the order space S_2 is homeomorphic to a subspace of Q.*

We leave the proofs of Corollaries 1 and 2.

References

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