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A REVERSE BMO-HARDY INEQUALITY

Abstract

This note provides a reverse Hardy inequality associated with BMO-norm.

Throughout this note, denote by $\mathbb{R}$ and $\mathbb{R}_+$ the real axis and the positive half real axis, respectively. For $p \in (1, \infty)$ let $L^p(\mathbb{R}_+)$ be the space of Lebesgue measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with

$$\|f\|_{L^p(\mathbb{R}_+)} = \left[\int_{\mathbb{R}_+} |f(x)|^p \, dx\right]^{\frac{1}{p}} < \infty.$$

As is well known, $L^\infty(\mathbb{R}_+)$ denotes the space of Lebesgue measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\|f\|_{L^\infty(\mathbb{R}_+)} = \text{ess sup}_{x\in\mathbb{R}_+} |f(x)| < \infty.$$

$L^\infty(\mathbb{R}_+)$ may be viewed as a limit space of $L^p(\mathbb{R}_+)$ as $p \to \infty$ in some sense (for instance, duality). However, in most cases, $L^\infty(\mathbb{R}_+)$ is replaced by the space $BMO(\mathbb{R}_+)$ of John and Nirenberg, which consists of those functions $f \in L^1_{loc}(\mathbb{R}_+)$ with bounded mean oscillation

$$\|f\|_{BMO(\mathbb{R}_+)} = \sup_{I\subset\mathbb{R}_+} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty,$$

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where the supremum is taken over all subintervals I of $\mathbb{R}_+$, f_I stands for the mean value of f on I, $f_I = \frac{1}{|I|}\int_I f(x)\,dx$, and $|I|$ denotes the length of I, $|I| = \int_I dx$. It is clear that $L^\infty(\mathbb{R}_+) \subsetneq BMO(\mathbb{R}_+) \not\subseteq L^p(\mathbb{R}_+)$, $p \in (1,\infty)$.

Now, for $f \in L^1_{loc}(\mathbb{R}_+)$ consider the Hardy operator

$$(Pf)(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x \in \mathbb{R}_+.$$

It is well known that this operator plays an important role in the study of weighted norm inequalities for the classical maximal operators of harmonic and complex analysis. It is easy to see that P is bounded on $L^p(\mathbb{R}_+)$, $p \in (1,\infty]$. In fact, the Minkowski inequality for integrals implies Hardy's inequality (cf. [8, p.240])

$$\|Pf\|_{L^p(\mathbb{R}_+)} \le C_p\|f\|_{L^p(\mathbb{R}_+)},$$

where $C_p = p/(p-1)$ or 1 (which is the best possible) if $p \in (1,\infty)$ or $p = \infty$. It is unfortunate that this operator is not invertible on $L^p(\mathbb{R}_+)$, and therefore it is not possible to find a constant c_p depending on p only such that a reverse $L^p(\mathbb{R}_+)$-Hardy inequality

$$\|Pf\|_{L^p(\mathbb{R}_+)} \ge c_p\|f\|_{L^p(\mathbb{R}_+)}$$

holds generally. Nevertheless, suppose f is positive and decreasing in $\mathbb{R}_+$. Then the last inequality is true with $c_p = [p/(p-1)]^{1/p}$ resp. 1 for $p \in (1,\infty)$ resp. $p = \infty$. Moreover, the constant c_p is sharp (cf. [3], [9] and [11]). A short proof for the preceding reverse inequality was given in [10].

The purpose of this note is to extend the above two inequalities from $L^p(\mathbb{R}_+)$ to $BMO(\mathbb{R}_+)$. To the best of our knowledge, the $BMO(\mathbb{R}_+)$ setting is new, nontrivial and of independent interest. The major result is the following

Theorem. *The Hardy operator is bounded on $BMO(\mathbb{R}_+)$; i.e.,*

$$\|Pf\|_{BMO(\mathbb{R}_+)} \le \|f\|_{BMO(\mathbb{R}_+)}.$$

Moreover, the reverse BMO-Hardy inequality

$$\|Pf\|_{BMO(\mathbb{R}_+)} \ge \frac{1}{17}\|f\|_{BMO(\mathbb{R}_+)}$$

is valid for any positive, decreasing function f in $L^1_{loc}(\mathbb{R}_+)$.

PROOF. In what follows, for any function $f : \mathbb{R}_+ \to \mathbb{R}$ let $f_t(x) = f(tx)$, $t > 0$. And, for any subinterval $I = (\alpha,\beta) \subset \mathbb{R}_+$ let tI be the interval $(t\alpha, t\beta)$.

First of all, one verifies the boundedness of P acting on $BMO(\mathbb{R}_+)$. If $f \in BMO(\mathbb{R}_+)$, then for any $x \in \mathbb{R}_+$,

$$(Pf)(x) = \int_0^1 f_x(u)du. \tag{1}$$

With (1), one has for any subinterval $I \subset \mathbb{R}_+$ and $C = (f_u)_I$,

$$\begin{aligned} \frac{1}{|I|}\int_I |(Pf)(x) - C|\,dx &= \frac{1}{|I|}\int_I \left|\int_0^1 [f_x(u) - C]\,du\right|\,dx \\ &\leq \int_0^1 \left[\frac{1}{|I|}\int_I |f_u(x) - C\,|dx\right]\,du \\ &\leq \int_0^1 \|f_u\|_{BMO(\mathbb{R}_+)}\,du. \end{aligned} \tag{2}$$

Note that $\|f_u\|_{BMO(\mathbb{R}_+)} \leq \|f\|_{BMO(\mathbb{R}_+)}$ for $u > 0$. As a matter of fact,

$$\begin{aligned} \|f_u\|_{BMO(\mathbb{R}_+)} &= \sup_{I \subset \mathbb{R}_+} \frac{1}{|I|}\int_I \left|f(ut) - \frac{1}{|I|}\int_I f(ux)\,dx\right|\,dt \\ &= \sup_{I \subset \mathbb{R}_+} \frac{1}{|uI|}\int_{uI} \left|f(y) - \frac{1}{|uI|}\int_{uI} f(x)\,dx\right|\,dy \\ &\leq \|f\|_{BMO(\mathbb{R}_+)}. \end{aligned} \tag{3}$$

Thus, by (2) and (3) one gets $\|Pf\|_{BMO(\mathbb{R}_+)} \leq \|f\|_{BMO(\mathbb{R}_+)}$.

Next, one shows the opposite direction. Assume that $f \in L^1_{loc}(\mathbb{R}_+)$ is positive and decreasing. By definition,

$$\begin{aligned} (Pf)(2x) &= \frac{1}{2x}\int_0^x f(t)\,dt + \frac{1}{2x}\int_x^{2x} f(t)dt \\ &= \frac{1}{2}(Pf)(x) + \frac{1}{2x}\int_x^{2x} f(t)\,dt \\ &\leq \frac{1}{2}(Pf)(x) + \frac{1}{2}f(x). \end{aligned}$$

Obviously, $(Pf)(x) \geq f(x)$. Thus,

$$(Pf)(2x) \leq \frac{1}{2}[(Pf)(x) + f(x)] \leq (Pf)(x).$$

Put $g(x) = \frac{1}{2}[(Pf)(x) + f(x)]$. Then for any subinterval $I \subset \mathbb{R}_+$,

$$(Pf)(2x) - (Pf)_I \leq g(x) - (Pf)_I \leq (Pf)(x) - (Pf)_I. \tag{4}$$

Once observing an elementary fact that if $a \le c \le b$, then $|c| \le |a| + |b|$, with the help of (4), one obtains

$$\begin{aligned} &|g(x) - (Pf)_I| \\ &\le |(Pf)(x) - (Pf)_I| + |(Pf)(2x) - (Pf)_I| \\ &\le |(Pf)(x) - (Pf)_I| + |(Pf)(2x) - (Pf)_{2I}| + |(Pf)_{2I} - (Pf)_I| \\ &\le |(Pf)(x) - (Pf)_I| + |(Pf)(2x) - (Pf)_{2I}| + 2\|Pf\|_{BMO(\mathbb{R}_+)}. \end{aligned} \tag{5}$$

From (5) it follows that $\|g\|_{BMO(\mathbb{R}_+)} \le 8\|Pf\|_{BMO(\mathbb{R}_+)}$. Since $f = 2g - Pf$, $\|f\|_{BMO(\mathbb{R}_+)} \le 17\|Pf\|_{BMO(\mathbb{R}_+)}$. □

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $|E|$ the Lebesgue measure of the set $E \subset \mathbb{R}^n$. In their paper [4] Bennett, DeVore and Sharpley introduced the weak-L^∞ space W of functions $f : \mathbb{R}^n \to \mathbb{R}^1$ obeying the condition that $f^{**} - f^*$ is bounded in $\mathbb{R}_+$, where f^* means the decreasing (or non-increasing) rearrangement of f

$$f^*(t) = \inf\{s > 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \le t\}$$

and $f^{**} = Pf^*$ (cf. [1, p.60]). An application of our theorem to W produces immediately and interesting consequence.

Corollary. *Let $f \in W$. Then $f^* \in BMO(\mathbb{R}_+)$ if and only if $f^{**} \in BMO(\mathbb{R}_+)$.*

PROOF. It is easy. □

Remark. In fact, as long as $f^* \in L^1_{loc}(\mathbb{R}_+)$, $f^* \in BMO(\mathbb{R}_+)$ is equivalent to $f^{**} \in BMO(\mathbb{R}_+)$. It would be very interesting to compare our theorem with Theorem 4.2 in [4]. In addition, it is not hard to finger out that the Hardy operator acts boundedly on the BMO-type spaces of Milman and Sagher (cf. [2]). Finally, one would mention that the constant $1/17$ in the reverse BMO-Hardy inequality is not sharp and it can be improved in a similar fashion. One conjectures that the best constant is 1. But the previous approach cannot prove this point. And so, it seems necessary that some other ideas will be involved in order to reach the conjecture.

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