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ORTHOGONAL CONTINUOUS FUNCTIONS

Abstract

We consider the question of whether there is an orthonormal basis for L^2 consisting of continuous functions.

1 Introduction

In elementary analysis, the typical orthonormal bases for $L^2[0, 1]$ (trig functions, orthogonal polynomials, etc.) frequently consist of continuous functions. It is natural to ask whether such orthonormal bases must exist if $[0, 1]$ is replaced by a more general space and measure. One commonly studied generalization of $[0, 1]$ is:

Definition 1.1. (X, ν) is a *nice* measure space iff X is a compact Hausdorff space and ν is a regular Borel probability measure on X which is strictly positive (i.e., all non-empty open sets have positive measure).

The assumption that ν is strictly positive is mainly for notational convenience. In general, one can simply delete the union of all open null sets to obtain a strictly positive measure.

Since ν is strictly positive, distinct elements of $C(X)$ do not become equivalent in L^2, so we may regard $C(X)$ as contained in $L^2(X, \nu)$. There are then two well-known situations where there is an $\mathcal{F} \subseteq C(X)$ which forms an orthonormal basis for $L^2(X, \nu)$. The first is whenever $L^2(X, \nu)$ is separable (by Gram-Schmidt). The second is when X is a compact group and ν is Haar measure (by the Peter-Weil Theorem; see, e.g., Folland [1]). However, there

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need not be such an $\mathcal{F}$ in general, since Theorem 3.6 yields an example where $L^2(X,\nu)$ is not separable but every orthogonal family from $C(X)$ is countable.

In the example of Theorem 3.6, X is actually a topological group, since it is a product of two-element spaces, and ν looks a bit like the product measure, which in this case would be Haar measure. Nevertheless, by Theorem 2.5, no such ν can be absolutely continuous with respect to Haar measure.

The proof for the specific example of Theorem 3.6 works equally well whether one considers the scalar field to be $\mathbb{R}$ or $\mathbb{C}$. However, if one starts with an arbitrary nice (X,ν), it is reasonable to ask whether the properties discussed here can depend on the scalar field. They do not, as we show in Corollary 2.2. Of course, any orthogonal family of real-valued functions remains orthogonal when viewed as a subfamily of $L^2(X,\nu,\mathbb{C})$, but Corollary 2.2 explains how to replace orthogonal complex-valued functions by real-valued ones. The familiar method from Fourier series replaces φ and $\overline{\varphi}$ by $(\varphi+\overline{\varphi})/\sqrt{2}$ and $(\varphi-\overline{\varphi})/(i\sqrt{2})$, but this requires assuming that $\varphi\in\mathcal{F} \iff \overline{\varphi}\in\mathcal{F}$.

One might study the following property of X: For every finite regular Borel measure ν on X, there is an $\mathcal{F}\subseteq C(X)$ which forms an orthonormal basis for $L^2(X,\nu)$. We do not know whether this is equivalent to some interesting topological property of X. Note that every compact F-space and every compact metric space has this property.

2 Basics

Throughout, when discussing $C(X)$ and $L^2(X,\nu)$ and general Hilbert spaces, we *always* presume that the scalar field is the complex numbers. We shall show that we can convert a family of orthogonal continuous functions to a family of real-valued orthogonal continuous functions with the same span. To do this, we use the following lemma about Hilbert spaces, which gives us a uniform way to transform an "almost orthogonal" family to an orthogonal one:

Lemma 2.1. *Suppose that $\mathcal{H}$ is a Hilbert space and $\mathcal{E}\subseteq\mathcal{H}$ is such that the closed linear span of $\mathcal{E}$ is $\mathcal{H}$ and $\{g\in\mathcal{E}:(g,f)\neq 0\}$ is countable for all $f\in\mathcal{E}$. Then there is an orthonormal basis $\mathcal{F}$ for $\mathcal{H}$ such that each element of $\mathcal{F}$ is a finite linear combination of elements of $\mathcal{E}$. Furthermore, if for all $g,f\in\mathcal{E}$ (g,f) is real, then the coefficients in these linear combinations are real.*

PROOF. On $\mathcal{E}$, let $\sim$ be the smallest equivalence relation such that $g\sim f$ whenever $(g,f)\neq 0$. Let $\mathcal{E}_j$, for $j\in J$, list all the $\sim$ equivalence classes. Then the $\mathcal{E}_j$ are all countable, and are pairwise orthogonal. For each j, apply Gram-Schmidt to obtain an orthonormal family $\mathcal{F}_j$ with the same linear span, such that the elements of $\mathcal{F}_j$ are linear combinations of elements of $\mathcal{E}_j$. Then, let $\mathcal{F} = \bigcup_j \mathcal{E}_j$. $\square$

Corollary 2.2. *Suppose that (X, ν) is a nice measure space and $\mathcal{G} \subseteq C(X)$ is an orthonormal family. Then there is an orthonormal family $\mathcal{F} \subseteq C(X)$, consisting of real-valued functions, such that the closed linear span of $\mathcal{F}$ contains the closed linear span of $\mathcal{G}$.*

Proof. As usual, write each $G \in \mathcal{G}$ as $G = \Re(G) + i\Im(G)$, where $\Re(G)$ and $\Im(G)$ are real-valued functions. Let $\mathcal{E} = \{\Re(G) : G \in \mathcal{G}\} \cup \{\Im(G) : G \in \mathcal{G}\}$. Then the closed linear span $\mathcal{H}$ of $\mathcal{E}$ contains $\mathcal{G}$, so Lemma 2.1 will apply if we can verify that $\{g \in \mathcal{E} : (g, f) \neq 0\}$, for any $f \in \mathcal{E}$, is countable. To see this, apply Bessel's inequality: $\sum_{G \in \mathcal{G}} |(G, f)|^2 \leq \|f\|^2$. Since f is real-valued, $|(G, f)|^2 = (\Re(G), f)^2 + (\Im(G), f)^2$, so that $(\Re(G), f) = (\Im(G), f) = 0$ for all but countably many $G \in \mathcal{G}$. $\square$

In particular, if $\mathcal{G}$ is an orthonormal basis, we may replace $\mathcal{G}$ by a real-valued orthonormal basis $\mathcal{F}$. Or, if $\mathcal{G}$ is an uncountable orthonormal family, then $\mathcal{F}$ will be a real-valued uncountable orthonormal family. So, the properties of (X, ν) considered in this paper do not depend on the scalar field.

The next definition and lemma give us a way of ensuring that there are no uncountable orthonormal families within $C(X)$.

Definition 2.3. We say $\mathcal{F} \subseteq C(X)$ is *maximal orthogonal* iff $\mathcal{F}$ is orthogonal in $L^2(X, \nu)$ and there is no orthogonal $\mathcal{G}$ with $\mathcal{F} \subsetneqq \mathcal{G} \subseteq C(X)$.

Observe that even in $L^2([0, 1])$, a maximal orthogonal $\mathcal{F} \subseteq C([0, 1])$ need not be an orthogonal basis for $L^2([0, 1])$. For example, its closed linear span may be the orthogonal complement of a step function (since the continuous functions are dense in the orthogonal complement of any step function). Nevertheless:

Lemma 2.4. *Suppose (X, ν) is a nice measure space, and assume that there is a maximal orthogonal $\mathcal{F} \subseteq C(X)$ which is countable. Then every orthogonal $\mathcal{G} \subseteq C(X)$ is countable.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be any two orthogonal families contained in $C(X)$. For each fixed $f \in \mathcal{F}$, Bessel's Inequality implies that $g \perp f$ for all but countably many $g \in \mathcal{G}$. If $\mathcal{F}$ is maximal then each element of $\mathcal{G}$ fails to be orthogonal to some $f \in \mathcal{F}$; i.e.,

$$\mathcal{G} = \bigcup_{f \in \mathcal{F}} \{g \in \mathcal{G} : \ (g, f) \neq 0\}$$

Thus, if $\mathcal{F}$ is also countable, then so is $\mathcal{G}$. $\square$

Now, the existence of an uncountable orthogonal family contained in $C(X)$ depends on ν, not just X, as the example in Section 3 shows. Nevertheless:

Theorem 2.5. *Suppose that (X,ν) and (X,μ) are nice measure spaces with $\mu \ll \nu$. Suppose that $\mathcal{G} \subseteq C(X)$ is an orthonormal basis for $L^2(X,\nu)$. Then there is an $\mathcal{F} \subseteq C(X)$ which is an orthonormal basis for $L^2(X,\mu)$.*

PROOF. Fix a Baire-measurable $\varphi : X \to [0,\infty)$ such that for all Borel sets E, we have $\mu(E) = \int_E \varphi(x)\, d\nu(x)$ (the Radon-Nikodym Theorem guarantees such a φ exists). Then $\int \varphi\, d\nu = 1$. If φ is bounded, then the closed linear span of $\mathcal{G}$ in $L^2(X,\mu)$ is all of $L^2(X,\mu)$, and the result follows directly by Lemma 2.1. When φ is not bounded, the result still follows by Lemma 2.1, but applied to a different family $\mathcal{E}$ one obtains as follows.

Choose closed G_δ sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ such that $\varphi(x) \le n$ for all $x \in K_n$ and $\nu(X \setminus \bigcup_n K_n) = 0$. For each n, choose $\psi_n \in C(X,[0,1])$ such that $K_n = \psi_n^{-1}\{1\}$, and note that the sequence of functions $(\psi_n)^m$ converges pointwise to χ_{K_n} as $m \to \infty$.

Let $\mathcal{E}$ be the set of all functions of the form $g \cdot (\psi_n)^m$, where $g \in \mathcal{G}$ and $m,n \in \mathbb{N}$. Then $\mathcal{E} \subseteq C(X) \subseteq L^2(X,\mu)$. Let $\mathcal{H}$ be the closed linear span of $\mathcal{E}$ in $L^2(X,\mu)$. Then $\mathcal{H} = L^2(X,\mu)$: To see this, first note that $g \cdot \chi_{K_n} \in \mathcal{H}$ for $g \in \mathcal{G}$. Then, if $h \in C(X)$, each $h \cdot \chi_{K_n} \in \mathcal{H}$ (since φ is bounded on K_n), but this implies that $h \in \mathcal{H}$. Now, use the fact that $C(X)$ is dense in $L^2(X)$.

The result will now follow by Lemma 2.1 if we can verify, for each $f \in \mathcal{G}$ and each $m,n,p,q \in \mathbb{N}$, $\{g \in \mathcal{G} : (g(\psi_n)^m, f(\psi_p)^q)_\mu \ne 0\}$ is countable. Now for each $r \in \mathbb{N}$, Bessel's Inequality (applied in $L^2(X,\nu)$) implies that $\int g(\psi_n)^m \overline{f}(\psi_p)^q \chi_{K_r} \varphi\, d\nu = 0$ for all but countably many $g \in \mathcal{G}$, since the function $(\psi_n)^m f(\psi_p)^q \chi_{K_r} \varphi$ is in $L^2(X,\nu)$. It follows that $(g(\psi_n)^m, f(\psi_p)^q)_\mu = \int g(\psi_n)^m \overline{f}(\psi_p)^q\ \varphi\, d\nu = 0$ for all but countably many $g \in \mathcal{G}$. $\square$

3 Small Orthogonal Families

We shall build a large nice (X,ν) in which every orthogonal family of continuous functions is countable. In order to do this, we apply Lemma 2.4; it is enough to obtain some countable maximal $\mathcal{F} \subseteq C(X)$. Again, we shall, for definiteness, assume that the scalar field is $\mathbb{C}$. $\mathcal{F}$ will be obtained by projecting X onto a small space M, for which we use the following notation.

Definition 3.1. (X,ν,Γ,M) is a *nice quadruple* iff (X,ν) is a nice measure space and Γ is a continuous map onto the compact Hausdorff space M. In this case, let $\mu = \nu\Gamma^{-1}$ be the induced measure on M. We regard $L^2(M,\mu)$ as contained in $L^2(X,\nu)$ via the inclusion Γ^* (where $\Gamma^*(g) = g \circ \Gamma$). Let Π_Γ

be the orthogonal projection from $L^2(X,\nu)$ onto $L^2(M,\mu)$. If $f \in L^2(X,\nu)$, we say $f \perp L^2(M,\mu)$ iff $\Pi_\Gamma(f) = 0$.

Lemma 3.2. *In the notation of Definition 3.1, if $f \in L^2(X,\nu)$ then the following are equivalent:*

1. $f \perp L^2(M,\mu)$.
2. $\int_{\Gamma^{-1}K} f(x)\, d\nu(x) = 0$ *for all closed* $K \subseteq M$.

Definition 3.3. The nice quadruple (X,ν,Γ,M) is *injective* iff Π_Γ is 1-1 on $C(X)$.

Note that this is the same as saying that a nice quadruple is injective iff for all $f \in C(X)$, if $f \perp L^2(M,\mu)$, then $f \equiv 0$.

Lemma 3.4. *Let (X,ν) be a nice measure space. Then the following are equivalent:*

1. *Every orthogonal subfamily of $C(X)$ is countable.*
2. *There is a continuous map Γ onto a compact second countable space M such that (X,ν,Γ,M) is injective.*

Proof. (2) $\to$ (1): Assuming (2), let $\mathcal{F} \subseteq C(M)$ be an orthonormal basis for $L^2(M)$. Then $\Gamma^*(\mathcal{F}) \cup \{0\} \subseteq C(X)$, and is maximal orthogonal, so (1) follows by Lemma 2.4.

(1) $\to$ (2): Again by Lemma 2.4, let $\{f_n : n \in \mathbb{N}\} \subseteq C(X)$ be maximal orthogonal. Let $\Gamma : X \to \mathbb{C}^\mathbb{N}$ be the product map: $(\Gamma(x))_n = f_n(x)$. Let M be the range of Γ. Observe that a non-zero $g \in C(X)$ with $\Pi_\Gamma(g) = 0$ would contradict maximality. $\square$

The next lemma explains how we obtain the situation of Lemma 3.4.2:

Lemma 3.5. *Let (X,ν,Γ,M) be a nice quadruple. Assume, for some fixed $\epsilon > 0$, we have: Whenever $W \subseteq X$ is open and non-empty, there is a closed $K \subseteq M$ such that $\mu(K) > 0$ and $\nu(\Gamma^{-1}(K) \cap W) \geq (\frac{1}{2} + \epsilon)\mu(K)$. Then (X,ν,Γ,M) is injective.*

Proof. Suppose $f \in C(X)$ is non-zero and satisfies $f \perp L^2(M,\mu)$. We may assume that $\|f\|_{\sup} = 1$, and that some $f(x) = 1$. For any $\delta > 0$, we may choose a non-empty open $W \subseteq X$ such that $|f(x) - 1| \leq \delta$ for all $x \in W$, and then choose K as above. Applying $f \perp L^2(M,\mu)$ to the characteristic function of K, we have $\int_{\Gamma^{-1}K} f(x)\, d\nu(x) = 0$, so that $|\int_{\Gamma^{-1}K \cap W} f| = |\int_{\Gamma^{-1}K \setminus W} f|$.

Note that $\mu(K) = \nu(\Gamma^{-1}K)$, so that $\nu(\Gamma^{-1}K \setminus W) \leq (\frac{1}{2} - \epsilon)\mu(K)$. So, we have:

$$
\begin{aligned}
|\int_{\Gamma^{-1}K \cap W} f| &\geq \nu(\Gamma^{-1}K \cap W)(1-\delta) \geq (\frac{1}{2} + \epsilon)\mu(K)(1-\delta) \\
|\int_{\Gamma^{-1}K \setminus W} f| &\leq \nu(\Gamma^{-1}K \setminus W) \leq (\frac{1}{2} - \epsilon)\mu(K)
\end{aligned}
$$

So, $(\frac{1}{2} + \epsilon)(1-\delta) \leq (\frac{1}{2} - \epsilon)$. Letting $\delta \searrow 0$, we have a contradiction. □

Note that if $\epsilon = 0$, the lemma could fail; consider $X = M \times 2$, with the product measure.

In general, the *Maharam dimension* of a measure ν is the cardinality of an orthonormal basis for $L^2(\nu)$; ν is called *Maharam-homogeneous* iff there is no set K of positive measure such that the dimension of ν restricted to K is less than the dimension of ν. As usual, $\mathfrak{c} = 2^{\aleph_0}$.

Theorem 3.6. *There is a strictly positive regular Borel probability measure ν on $2^{\mathfrak{c}}$ (i.e., $\{0,1\}^{\mathfrak{c}}$, with the usual product topology) such that*

1. *ν is Maharam-homogeneous of dimension $\mathfrak{c}$.*
2. *$L^2(2^{\mathfrak{c}}, \nu)$ contains no uncountable orthogonal family of continuous functions.*

PROOF. Let $M = 2^{\mathbb{N}}$, with μ the usual product measure. Let $X = M \times 2^{\mathfrak{c}}$, and let $\Gamma : X \to M$ be projection. We shall build ν on X, which is homeomorphic to $2^{\mathfrak{c}}$.

Let $\{d_m : m \in \mathbb{N}\}$ be dense in $(0,1)^{\mathfrak{c}}$. For each m, let λ_m be the product measure on $2^{\mathfrak{c}}$ obtained by flipping unfair coins with bias d_m. That is, let $d^1_m(\alpha) = d_m(\alpha)$ and $d^0_m(\alpha) = 1 - d_m(\alpha)$. If

$$B = \{v \in 2^{\mathfrak{c}} : v(\alpha_1) = \ell_1 \& \cdots \& v(\alpha_r) = \ell_r\} \tag{1}$$

is a basic clopen set, then $\lambda_m(B) = \prod_{j=1}^r d^{\ell_r}_m(\alpha_j)$.

List all non-empty clopen subsets of M as $\{U_n : n \in \mathbb{N}\}$. Then, choose closed nowhere dense $K_{m,n} \subseteq U_n$ so that the $K_{m,n}$ for $m, n \in \mathbb{N}$ are all disjoint, each $\mu(K_{m,n}) > 0$, and $\sum_{m,n} \mu(K_{m,n}) = 1$. ¿Finally, let ν on $M \times 2^{\mathfrak{c}}$ be the sum of the product measures $(\mu \restriction K_{m,n}) \times \lambda_m$, so that for Borel $E \subseteq M \times 2^{\mathfrak{c}}$,

$$\nu(E) = \sum_{m,n} \int_{K_{m,n}} \lambda_m(E_x) \, d\mu(x) \quad .$$

We are now done if we can verify the hypotheses of Lemma 3.5 We actually show that whenever $W \subseteq X$ is open and non-empty and $\epsilon > 0$, there is a closed

$K \subseteq M$ such that $\mu(K) > 0$ and $\nu(\Gamma^{-1}(K) \cap W) \geq (1-\epsilon)\mu(K)$. To do this, we may assume that $W = U_n \times B$, where B is as in (1) above. K will be $K_{m,n}$ for a suitable m. Then $\nu(\Gamma^{-1}(K) \cap W) = \nu(K_{m,n} \times B) = \mu(K_{m,n}) \prod_{j=1}^{r} d_m^{\ell_r}(\alpha_j)$. We thus only need choose m so that $\prod_{j=1}^{r} d_m^{\ell_r}(\alpha_j) \geq (1-\epsilon)$, which is certainly possible since $\{d_m : m \in \mathbb{N}\}$ is dense in $(0,1)^{\mathfrak{c}}$. □

Finally, we remark that this example is as large as possible, since whenever $|C(X)| > \mathfrak{c}$, there is an uncountable orthogonal family, by Lemma 3.4. (Note that if X is an infinite compact Hausdorff space, then $|C(X)| = w(X)^{\aleph_0}$, where $w(X)$ is the weight of X (the least size of a base for the topology)). One can, however, construct arbitrarily large examples with no continuous orthonormal bases by applying:

Theorem 3.7. *Suppose that (X, ν) and (Y, ρ) are both nice measure spaces, and there is an orthonormal basis for $L^2(X \times Y, \nu \times \rho)$ contained in $C(X \times Y)$. Then there are orthonormal bases for $L^2(X, \nu), L^2(Y, \rho)$ contained in $C(X), C(Y)$, respectively.*

Proof. Let $\mathcal{G} \subseteq C(X \times Y)$ be an orthonormal basis for $L^2(X \times Y, \nu \times \rho)$. To produce a basis for $L^2(X, \nu)$, let $\Gamma : X \times Y \to X$ be projection, and apply Lemma 2.1, with $\mathcal{E} = \Pi_\Gamma(\mathcal{G}) \subseteq \mathcal{H} = L^2(X, \nu)$ (regarding $L^2(X)$ as contained in $L^2(X \times Y)$, as in Definition 3.1).

First, note that the closed linear span of $\mathcal{E}$ will be all of $L^2(X)$, because the closed linear span of $\mathcal{G}$ is $L^2(X \times Y)$ and Π_Γ is orthogonal projection.

Next, observe that for each $G \in \mathcal{G}$, $\Pi_\Gamma(G) = g$, where $g(x) = \int G(x, y)\, dy$. To see this, note that since G is continuous, $g \in C(X) \subseteq L^2(X)$. Also, for each $f \in L^2(X)$,

$$(g, f) = \int g(x)\overline{f}(x)\, dx = \int \int G(x, y)\overline{f}(x)\, dxdy = (G, f) \quad .$$

So $\Pi_\Gamma(G) = g$ follows from the uniqueness of orthogonal projections.

In particular, $\mathcal{E} \subseteq C(X)$, so that Lemma 2.1 will produce an orthonormal base contained in $C(X)$.

Finally, countability of $\mathcal{E}_f = \{g \in \mathcal{E} : (g, f) \neq 0\}$, for any $f \in \mathcal{E}$, follows from Bessel's inequality: For each $g = \Pi_\Gamma(G) \in \mathcal{E}$, since $(g, f) = (G, f)$, we have $\sum\{|(G, f)|^2 : G \in \mathcal{G}\} \leq \|f\|^2$. □

For example, let κ be any infinite cardinal such that $\kappa^{\aleph_0} = \kappa$. We may then obtain a nice (Z, μ) such that $|C(Z)| = \kappa$ and there is no orthonormal basis for $L^2(Z, \mu)$ contained in $C(Z)$; we just start with an X as in Theorem 3.6, and then $Z = X \times Y$ for a suitable Y (applying Theorem 3.7). However, assuming also that $2^\lambda < \kappa$ for all $\lambda < \kappa$ (for example, κ could be $\beth_{\omega_1}$, or κ could be strongly inaccessible), every maximal orthogonal family $\mathcal{F} \subseteq C(Z)$ must have size κ: If $|\mathcal{F}| = \lambda < \kappa$, we could always find distinct $g, h \in C(Z)$ such that $(g, f) = (h, f)$ for all $f \in \mathcal{F}$ (since there are only $2^\lambda < \kappa = |C(Z)|$ possibilities for $\langle (g, f) : f \in \mathcal{F} \rangle$). Then $(g - h) \perp \mathcal{F}$, so $\mathcal{F}$ cannot be maximal.

References

[1] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.