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QUOTIENTS OF DARBOUX FUNCTIONS

Abstract

We prove theorems concerning common divisor for the families of the quotients of Darboux functions with respect to Darboux property.

1 Introduction

The letter $\mathbb{R}$ denotes the real line. The family of all functions from a set X into Y is denoted by Y^X. The word *function* denotes a mapping from $\mathbb{R}$ to $\mathbb{R}$ unless otherwise explicitly stated. We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals. The symbol $\operatorname{card} X$ stands for the cardinality of a set X. We write $\mathfrak{c} = \operatorname{card}\mathbb{R}$. For a cardinal number κ we write $\operatorname{cf}(\kappa)$ for the *cofinality of* κ, and we say that κ is *regular*, if $\kappa = \operatorname{cf}(\kappa)$. The projection of a set $U \subset \mathbb{R}^2$ onto the x-axis is denoted by $\operatorname{dom} U$. We say that a set $A \subset \mathbb{R}$ is *bilaterally* $\mathfrak{c}$*-dense in itself* if $\operatorname{card}(A \cap I) = \mathfrak{c}$ for every nondegenerate interval I with $A \cap I \neq \emptyset$.

Let $f \colon \mathbb{R} \to \mathbb{R}$. For each $y \in \mathbb{R}$ let $[f < y] = \{x \in \mathbb{R} \colon f(x) < y\}$. Similarly we define the symbols $[f > y]$, $[f = y]$, etc.

The symbol $\mathcal{D}$ denotes the class of all *Darboux* functions; i.e., $f \in \mathcal{D}$ iff it has the intermediate value property.

There are several papers concerning theorems on a common summand [2], [1], or factor [6]. In this paper we are concerned with a common divisor for the families of the quotients of Darboux functions with respect to the Darboux property. (We were concerned with a similar problem in [3].) More precisely, we examine the cardinal

$$\mathrm{q}(\mathcal{D}) \stackrel{\mathrm{df}}{=} \min\left\{\operatorname{card}\mathcal{F} : \mathcal{F} \subset {}^{\mathcal{D}}\!/_{\mathcal{D}} \ \&\ \neg\left(\exists_g \forall_{f\in\mathcal{F}}\ f/g \in \mathcal{D}\right)\right\},$$

where

$${}^{\mathcal{D}}\!/_{\mathcal{D}} \stackrel{\mathrm{df}}{=} \left\{f/g : f, g \in \mathcal{D} \ \&\ g(x) \neq 0 \text{ for each } x \in \mathbb{R}\right\}.$$

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In the above definition it is quite natural to restrict ourselves to subfamilies of ${}^{\mathcal{D}}\!/_{\mathcal{D}}$ only. Indeed, if there is a function g such that both f/g and $1/g$ are Darboux, then $f \in {}^{\mathcal{D}}\!/_{\mathcal{D}}$.

2 Main Results

Before we start our examination, recall the following theorem proved by Natkaniec and Orwat [7, Theorem 7].

Theorem 2.1. *Let f be a function. Then $f \in {}^{\mathcal{D}}\!/_{\mathcal{D}}$ iff f satisfies the following conditions*:

D1. *if $a < b$ and $f(a)f(b) < 0$, then $[f = 0] \cap (a,b) \neq \emptyset$;*

D2. *each of the sets $[f > 0]$ and $[f < 0]$ is bilaterally $\mathfrak{c}$-dense in itself.*

Theorem 2.2. $\mathrm{q}(\mathcal{D}) > \mathfrak{c}$.

PROOF. Let $\{f_\alpha : \alpha < \mathfrak{c}\} \subset {}^{\mathcal{D}}\!/_{\mathcal{D}}$. For $\alpha < \mathfrak{c}$ and $i < 2$ define

$$\mathcal{Q}_{\alpha i} = \{[a,b] \cap [(-1)^i f_\alpha > 0] : a, b \in \mathbb{R}\} \setminus \{\emptyset\},$$

and observe that by D2, card $A = \mathfrak{c}$ whenever $A \in \mathcal{Q}_{\alpha i}$. By [4, Lemma 5], there is a family, $\{T_{\alpha i A} : \alpha < \mathfrak{c}, i < 2, A \in \mathcal{Q}_{\alpha i}\}$, composed of pairwise disjoint sets of cardinality $\mathfrak{c}$, such that each $T_{\alpha i A}$ is a subset of A. For each α, i, and A, let $g_{\alpha i A} \colon T_{\alpha i A} \to (0, \infty)$ be an arbitrary surjection. Define the function g by

$$g(x) = \begin{cases} (-1)^i (f_\alpha / g_{\alpha i A})(x) & \text{if } x \in T_{\alpha i A}, \alpha < \mathfrak{c}, i < 2, A \in \mathcal{Q}_{\alpha i}, \\ 1 & \text{otherwise.} \end{cases}$$

Evidently g is positive. We will show that each function f_α/g is Darboux.

Let $\alpha < \mathfrak{c}$, $a < b$, and assume that $(f_\alpha/g)(a) < (f_\alpha/g)(b)$. (The other case is analogous.) Fix a $y \in \big((f_\alpha/g)(a), (f_\alpha/g(b))\big)$. We consider three cases.

If $y = 0$, then $f_\alpha(a) < 0 < f_\alpha(b)$. So by D1, $(f_\alpha/g)(x) = f_\alpha(x) = 0$ for some $x \in (a,b)$.

If $y > 0$, then $f_\alpha(b) > 0$, so $A = [a,b] \cap [f_\alpha > 0] \neq \emptyset$. Thus $A \in \mathcal{Q}_{\alpha 0}$. Consequently, there is an $x \in T_{\alpha 0 A} \subset [a,b]$ such that $(f_\alpha/g)(x) = g_{\alpha 0 A}(x) = y$.

We proceed similarly if $y < 0$. □

To prove the next theorem we need the following definition.

$$\mathrm{a}(\mathcal{D}) \stackrel{\text{df}}{=} \min\big\{\operatorname{card} \mathcal{F} : \mathcal{F} \subset \mathbb{R}^{\mathbb{R}} \;\&\; \neg\big(\exists_g \forall_{f \in \mathcal{F}}\; f + g \in \mathcal{D}\big)\big\}.$$

This cardinal was defined by Natkaniec [5] and was thoroughly examined by Ciesielski and Miller [1]. It is well-known that $\mathfrak{c} < \mathrm{a}(\mathcal{D}) \leq 2^{\mathfrak{c}}$ [2]. Ciesielski and Miller generalized this result by showing that $\mathrm{cf}(\mathrm{a}(\mathcal{D})) > \mathfrak{c}$. They also proved that it is pretty much all that can be said about $\mathrm{a}(D)$ in ZFC, by showing that $\mathrm{a}(\mathcal{D})$ can be equal to any regular cardinal between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$, and that it can be equal to $2^{\mathfrak{c}}$ independently of the cofinality of $2^{\mathfrak{c}}$ [1]. (Actually, Ciesielski and Miller showed these results for the family of functions almost continuous in the sense of Stallings [8].)

Theorem 2.3. $\mathrm{a}(\mathcal{D}) = \min\big\{\mathrm{card}\,\mathcal{F} : \mathcal{F} \subset (0,\infty)^{\mathbb{R}} \ \&\ \neg\big(\exists_g \forall_{f\in\mathcal{F}}\ f/g \in \mathcal{D}\big)\big\}$.

PROOF. First we will prove that $\mathrm{a}(\mathcal{D})$ is not smaller than the right-hand side of the above equality. Pick a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ such that $\mathrm{card}\,\mathcal{F} = \mathrm{a}(\mathcal{D})$ and

$$\forall_{g\in\mathbb{R}^{\mathbb{R}}}\ \exists_{f\in\mathcal{F}}\ f + g \notin \mathcal{D}. \tag{1}$$

Let $\mathcal{F}^* = \{\exp\circ f : f \in \mathcal{F}\}$. Then $\mathcal{F}^* \subset (0,\infty)^{\mathbb{R}}$ and $\mathrm{card}\,\mathcal{F}^* = \mathrm{card}\,\mathcal{F}$. We will show that for each $g\colon \mathbb{R} \to \mathbb{R}\setminus\{0\}$ there is an $f^* \in \mathcal{F}^*$ such that $f^*/g \notin \mathcal{D}$. Let $g\colon \mathbb{R} \to \mathbb{R}\setminus\{0\}$. By (1), $f - \ln\circ|g| \notin \mathcal{D}$ for some $f \in \mathcal{F}$. Hence

$$\exp\circ(f - \ln\circ|g|) = (\exp\circ f)/|g| \notin \mathcal{D},$$

and consequently, $(\exp\circ f)/g \notin \mathcal{D}$.

The proof of the opposite inequality is analogous. $\square$

By Theorem 2.1, we have $(0,\infty)^{\mathbb{R}} \subset {}^{\mathcal{D}}\!/_{\mathcal{D}}$. So, we obtain the following corollary.

Corollary 2.4. $\mathrm{q}(\mathcal{D}) \leq \mathrm{a}(\mathcal{D})$.

For a partially ordered set $(\mathbb{P}, \leq)$, we say that $G \subset \mathbb{P}$ is a $\mathbb{P}$-*filter*, if

- for all $p, q \in G$, there exists $r \in G$ with $r \leq p$ and $r \leq q$, and
- for all $p, q \in \mathbb{P}$, if $p \in G$ and $p \leq q$, then $q \in G$.

Define $D \subset \mathbb{P}$ to be *dense*, if for every $p \in \mathbb{P}$ there exists $q \in D$ with $q \leq p$.

For a cardinal κ and a poset $\mathbb{P}$, define the following statements (*Martin's Axiom for* $\mathbb{P}$ and *Lusin's Axiom for* $\mathbb{P}$):

$\mathrm{MA}_\kappa(\mathbb{P})$: for any family $\mathfrak{D}$ of dense subsets of $\mathbb{P}$ with $\mathrm{card}\,\mathfrak{D} < \kappa$, there exists a $\mathbb{P}$-filter G such that $D \cap G \neq \emptyset$ for every $D \in \mathfrak{D}$.

$\mathrm{Lus}_\kappa(\mathbb{P})$: there exists a sequence $\langle G_\alpha : \alpha < \kappa \rangle$ of $\mathbb{P}$-filters, called a κ-Lusin sequence, such that for every dense set $D \subset \mathbb{P}$

$$\mathrm{card}\big\{\alpha < \kappa : G_\alpha \cap D = \emptyset\big\} < \kappa.$$

From now on, let

$$\mathbb{P} = \big\{p \in (0,\infty)^X : X \subset \mathbb{R} \ \& \ \mathrm{card}\, X < \mathfrak{c}\big\}.$$

Define $p \leq q$ if $q \subset p$, i.e., if p extends q as a partial function.

The proof of the next theorem is actually a repetition of argument used by Ciesielski and Miller [1, Lemma 3.1].

Theorem 2.5. $\mathrm{MA}_\kappa(\mathbb{P})$ *implies* $\mathrm{q}(\mathcal{D}) \geq \kappa$.

PROOF. Assume $\mathrm{MA}_\kappa(\mathbb{P})$. By Theorem 2.2, we may assume that $\kappa > \mathfrak{c}$.

First observe that for every $x \in \mathbb{R}$, the set $D_x = \{p \in \mathbb{P} : x \in \mathrm{dom}\, p\}$ is dense in $\mathbb{P}$. Indeed, let $x \in \mathbb{R}$ and $p \in \mathbb{P}$. If $x \in \mathrm{dom}\, p$, then put $q = p$; otherwise let $q = p \cup \{(x, 1)\}$. Clearly $q \in D_x$ and $q \leq p$.

Now we will show that for any $f \in {}^{\mathcal{D}}\!/\!{}_{\mathcal{D}}$, $y \neq 0$, and $a < b$, if the set $[a,b] \cap [f/y > 0]$ is nonempty, then the set

$$D_{fyab} = \big\{p \in \mathbb{P} : \exists_{x \in [a,b] \cap \mathrm{dom}\, p}\ p(x) = f(x)/y\big\}$$

is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$, $f \in {}^{\mathcal{D}}\!/\!{}_{\mathcal{D}}$, $y > 0$, $a < b$, and assume that $[a,b] \cap [f/y > 0] \neq \emptyset$. Since f satisfies D2, $\mathrm{card}\big([a,b] \cap [f/y > 0]\big) = \mathfrak{c}$. We have $\mathrm{card}\,\mathrm{dom}\, p < \mathfrak{c}$; so there is an $x \in (a,b) \cap [f/y > 0] \setminus \mathrm{dom}\, p$. Then the function $q = p \cup \{(x, f(x)/y)\}$ satisfies $q \in D_{fyab}$ and $q \leq p$.

To show that $\mathrm{q}(\mathcal{D}) \geq \kappa$ pick a family of functions $\mathcal{F} \subset {}^{\mathcal{D}}\!/\!{}_{\mathcal{D}}$ with $\mathrm{card}\,\mathcal{F} < \kappa$. Define

$$\mathfrak{D} = \big\{D_x : x \in \mathbb{R}\big\} \cup \big\{D_{fyab} : f \in \mathcal{F}, y \neq 0, a < b, [a,b] \cap [f/y > 0] \neq \emptyset\big\}.$$

Then $\mathfrak{D}$ is a family of dense subsets of $\mathbb{P}$ and $\mathrm{card}\,\mathfrak{D} < \kappa$. Applying $\mathrm{MA}_\kappa(\mathbb{P})$ we can find a $\mathbb{P}$-filter G which meets every $D \in \mathfrak{D}$.

Let $g = \bigcup G$. Evidently g is a function and g is positive. For every $x \in \mathbb{R}$, we have $D_x \cap G \neq \emptyset$; so $\mathrm{dom}\, g = \mathbb{R}$. We will show that each f/g is Darboux. Let $f \in \mathcal{F}$, $a < b$, and assume that $(f/g)(a) < (f/g)(b)$. (The other case is analogous.) Fix a $y \in \big((f/g)(a), (f/g)(b)\big)$. We consider three cases. If $y = 0$, then $f(a) < 0 < f(b)$. So by D1, $(f/g)(x) = f(x) = 0$ for some $x \in (a,b)$. If $y > 0$, then $f(b) > 0$, so $[a,b] \cap [f/y > 0] \neq \emptyset$. Since $D_{fyab} \cap G \neq \emptyset$, there are a $p \in G$ and an $x \in [a,b] \cap \mathrm{dom}\, p$ such that $p(x) = f(x)/y$. Then $(f/g)(x) = (f/p)(x) = y$.

We proceed similarly if $y < 0$. $\square$

To prove the next theorem we will use two other posets. Let

$$\mathbb{P}' = \{p \in \mathbb{R}^X : X \subset \mathbb{R} \ \& \ \operatorname{card} X < \mathfrak{c}\},$$

and $p \leq q$ iff $q \subset p$. Moreover let

$$\mathbb{P}^* = \{(p, \mathcal{E}) : p \in \mathbb{P}', \ \mathcal{E} \subset \mathbb{R}^{\mathbb{R}} \ \& \ \operatorname{card} \mathcal{E} < \mathfrak{c}\},$$

and define $(p, \mathcal{E}) \leq (q, \mathcal{F})$ iff

$$q \subset p,\ \mathcal{E} \supset \mathcal{F}, \text{ and } p(x) \neq f(x) \text{ for all } x \in \operatorname{dom} p \setminus \operatorname{dom} q \text{ and } f \in \mathcal{F}.$$

Theorem 2.6. *Suppose that $\kappa > \mathfrak{c}$, κ is regular, and $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ holds. Then $\mathrm{q}(\mathcal{D}) = \mathrm{a}(\mathcal{D}) = \kappa$.*

PROOF. The inequality $\mathrm{q}(\mathcal{D}) \leq \mathrm{a}(\mathcal{D})$ follows by Corollary 2.4. The inequality $\mathrm{a}(\mathcal{D}) \leq \kappa$ follows by [1, Lemma 3.2 and Theorem 2.1]. By [1, Lemma 3.3], $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ implies $\mathrm{MA}_\kappa(\mathbb{P}')$. But the posets $\mathbb{P}$ and $\mathbb{P}'$ are order isomorphic; so $\mathrm{MA}_\kappa(\mathbb{P})$ holds. Now the inequality $\mathrm{q}(\mathcal{D}) \geq \kappa$ follows by Theorem 2.5. □

Ciesielski and Miller proved that the assumptions of Theorem 2.6 are independent of ZFC [1]. So, we have the following problem.

Problem. Can the equality $\mathrm{q}(\mathcal{D}) = \mathrm{a}(\mathcal{D})$ be proved in ZFC?

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