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A STUDY OF SOME GENERAL INTEGRALS THAT CONTAINS THE WIDE DENJOY INTEGRAL

Abstract

In this paper, using Thomson's local systems, we introduce some very general integrals, each containing the wide Denjoy integral: the $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integral (of Lusin type); the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integral (of variational type); the $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integral (of Ward type); the $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integral (of Riemann type); We prove that in certain conditions the integrals $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ are equivalent (it is shown that the first integral satisfies a Saks-Henstock type lemma). For the $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integral we only show that it satisfies a quasi Saks Henstock type lemma (see Lemma 7.4). Finally, if $\mathcal{S}_1 = \mathcal{S}_o^+$ and $\mathcal{S}_2 = \mathcal{S}_o^-$ we obtain that the integrals $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{V}]$, $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{W}]$ and $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{D}]$ are equivalent (in fact the $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{D}]$-integral is exactly the wide Denjoy integral). But the equivalence of the three integrals with the $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{R}]$-integral follows only if we assume the additional condition that the primitives of the $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{R}]$-integral are continuous (see Theorem 11.1)

1 Introduction

It is well known that the Denjoy-Perron integral has a Riemann type definition. This was discovered independently by Henstock and Kurzweil, and it is called the Henstock-Kurzweil integral. Also the Denjoy-Perron integral allows characterizations of variational and of Ward type (these characterizations are due to Henstock). A very important fact in the theory of the Henstock-Kurzweil integral is the Saks-Henstock Lemma. Since 1968, Henstock suggested (see [5, p. 222]) that it is possible to obtain a Riemann type definition for the wide Denjoy integral. Starting from this suggestion and from the fact that an explicit theorem wasn't stated in Henstock's book, Lee and Soedijono introduced

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a Riemann type integral, called the AH-integral, about which they claimed that it was equivalent with the β-Ridder integral (note that the β-Ridder integral was also studied by Kubota, but he called it the AD- integral). Indeed, by [12, Theorem 4.1] it follows that the AD integral is contained in the AH integral and the two integrals are equal. But that the converse is also true doesn't seem to follow from their Theorem 4.2. To prove this theorem they need to show the following facts:

1) F is approximately continuous on $[a, b]$;

2) $F'_{ap}(x) = f(x)$ *a.e.* on $[a, b]$;

3) $F \in VBG$ on $[a, b]$;

4) F satisfies Lusin's condition (N) on $[a, b]$;

5) $F \in [VBG]$ on $[a, b]$;

6) $[VBG] \cap (N) = [ACG]$ for approximately continuous functions on $[a, b]$,

where $F(x) = (AH)\int_a^x f(t)\,dt$, $x \in [a, b]$.
To show 1)-4) they use essentially a Saks-Henstock type lemma for the AH integral, claiming that this lemma is easy to prove (I wasn't able to do so).
With or without a Saks-Henstock type lemma, I wasn't able to prove 1).
The items 2), 3) and 4) are true, but with different proofs (for 2) see Lemma 7.5; for 3) see Lemma 7.6; for 4) see Corollary 7.1).
In 4) there is also another error (it seems that the authors used the following statement, that is not true: if $\{[F(a_i), F(b_i)]\}_{i=1}^n$ is a finite set of nonoverlapping intervals then $\{[a_i, b_i]\}_{i=1}^n$ is also a set of nonoverlapping intervals).
The proof of 5) is not clear (because, if a function F satisfies the "strong Lusin condition" then it isn't clear if F is VB on any subset Z with $|Z| = 0$; but it is true that F is VBG on Z, see Theorem 5.1; moreover, if a function is VB on a set A and on a set B, then it is not necessarily VB on $A \cup B$).
Statement 6) is not true. Indeed, Sarkhel and Kar introduced the (PAC) condition that is characterized as follows: *A function F is (PAC) on a closed set E if and only if $F \in [VBG] \cap (N)$ on E* (The generalized Banach-Zarecki theorem [19, Theorem 3.6]). In the same paper the authors constructed *a function $F : [a, b] \to \mathbb{R}$ with the following properties: F is approximately continuous, $F \in (PAC)$, but $F \notin ACG$*. It follows that the function F from above is approximately continuous, $[VBG]$ and (N) on $[a, b]$, but not $[ACG]$. Note however that $VBG \cap [CG] \cap (N) = [ACG]$ (and this follows indeed by the Banach-Zarecki theorem [15, p. 227]).

In this paper, using Thomson's local systems, we introduce some very general integrals, each containing the wide Denjoy integral:

- the $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integral (of Lusin type);
- the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integral (of variational type);
- the $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integral (of Ward type);
- the $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integral (of Riemann type);

We prove that in certain conditions, the integrals $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ are equivalent (it is shown that the first integral satisfies a Saks-Henstock type lemma). For the $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integral we only show that it satisfies a quasi Saks Henstock type lemma (see Lemma 7.4).

Finally, if $\mathcal{S}_1 = \mathcal{S}_o^+$ and $\mathcal{S}_2 = \mathcal{S}_o^-$ we obtain that the integrals $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{V}]$, $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{W}]$ and $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{D}]$ are equivalent (in fact the $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{D}]$-integral is exactly the wide Denjoy integral). But the equivalence of the three integrals with the $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{R}]$-integral follows only if we assume the additional condition, that the primitives of the $[\mathcal{S}_o^+\mathcal{S}_o^-\mathcal{R}]$-integral are continuous (see Theorem 11.1).

2 Preliminaries

We shall use the following well known classes of functions: $\mathcal{C}$ (continuous functions), $\mathcal{D}$ (Darboux functions), $\mathcal{B}_1$ (Baire one functions), AC, VB, $[\underline{ACG}]$, $[ACG]$, ACG (the ACG functions are not supposed to be continuous), VBG, $[CG]$ (or B_1^*) $[VBG]$, (N) (Lusin's condition) , T_2, $N^{-\infty}$ (see for example [15] or [1]). We denote by $\langle x, y\rangle$ the closed interval with the endpoints x and y, where $x, y \in \mathbb{R}$.

Definition 2.1 (Thomson)**.** [21, p. 3] A family $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ is said to be a local system if each $\mathcal{S}(x)$ is a collection of sets with the following properties:

(i) $\{x\} \notin \mathcal{S}(x)$;

(ii) If $\sigma_x \in \mathcal{S}(x)$ then $x \in \sigma_x$;

(iii) If $\sigma_x \in \mathcal{S}(x)$ and $\sigma_x \subset A$ then $A \in \mathcal{S}(x)$;

(iv) If $\sigma_x \in \mathcal{S}(x)$ and $\delta > 0$ then $\sigma_x \cap (x-\delta, x+\delta) \in \mathcal{S}(x)$.

Definition 2.2. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ and $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be local systems and let $x \in \mathbb{R}$, $A \subset \mathbb{R}$.

- (**Thomson**, [21, p. 5]). We define the local system $\mathcal{S}_1 \wedge \mathcal{S}_2 = \{(\mathcal{S}_1 \wedge \mathcal{S}_2)(x)\}_{x\in\mathbb{R}}$ by $(\mathcal{S}_1 \wedge \mathcal{S}_2)(x) = \mathcal{S}_1(x) \cap \mathcal{S}_2(x)$ (it is easy to verify that this is a local system).

- (**Thomson**, [21, p. 37]). $\mathcal{S}_1$ is said to be bilateral at x if σ_x has x as a bilateral accumulation point, whenever $\sigma_x \in \mathcal{S}_1(x)$. $\mathcal{S}_1$ is bilateral on A if it is bilateral at each point of A.

- (**Thomson**, [21, p. 18]). Let $\mathcal{S}_\infty = \{\mathcal{S}_\infty(x) : x \in \mathbb{R}\}$ denote the local system defined at each point x as $\mathcal{S}_\infty(x) = \{\sigma : \sigma$ contains x and has x as an accumulation point $\}$. We can define right and left versions of this, by writing: $\mathcal{S}^+_\infty(x) = \{\sigma : \sigma$ contains x and has x as a right accumulation point $\}$ and $\mathcal{S}^-_\infty(x) = \{\sigma : \sigma$ contains x and has x as a left accumulation point $\}$.

- Let $\mathcal{S}_{\infty,\infty} = \mathcal{S}^+_\infty \wedge \mathcal{S}^-_\infty$. Clearly $\mathcal{S}_{\infty,\infty}(x) = \{\sigma : \sigma$ contains x and has x as a bilateral accumulation point $\}$.

- $\mathcal{S}_1$ is said to be $\mathcal{S}_2$-filtering at x if $\sigma^{'}_x \cap \sigma^{''}_x \in \mathcal{S}_2(x)$ whenever $\sigma^{'}_x, \sigma^{''}_x \in \mathcal{S}_1(x)$. $\mathcal{S}_1$ is said to be $\mathcal{S}_2$-filtering on A if it is so on each point of A.

- (**Thomson**, [21, p. 10]). $\mathcal{S}$ is said to be filtering at x if it is $\mathcal{S}$-filtering at x.

- (**Thomson**, [21, p. 5]). We will write $\mathcal{S}_1 \ll \mathcal{S}_2$ on A, if at every point $x \in A$ we have $\mathcal{S}_1(x) \subseteq \mathcal{S}_2(x)$.

- If $\mathcal{S}_1 \ll \mathcal{S}^+_\infty$ and $\mathcal{S}_2 \ll \mathcal{S}^-_\infty$ then we define the following local system: $(\mathcal{S}_1; \mathcal{S}_2) = \{(\mathcal{S}_1; \mathcal{S}_2)(x)\}_{x\in\mathbb{R}}$, where $(\mathcal{S}_1; \mathcal{S}_2)(x) = \{S : x \in S$ and there exist $\delta > 0$, $A \in \mathcal{S}_1(x)$ and $B \in \mathcal{S}_2(x)$ such that $((x-\delta, x) \cap B) \cup ((x, x+\delta) \cap A) \subset S\}$.

Remark 2.1. If $\mathcal{S}$ is $\mathcal{S}_{\infty,\infty}$-filtering then it is a bilateral local system.

Definition 2.3. Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system. Let $F : [a, b] \to \mathbb{R}$ and $t \in [a, b]$.

- F is said to be $\mathcal{SC}$ ($\mathcal{S}$-continuous) at t if for every $\epsilon > 0$ there exists $\sigma_t \in \mathcal{S}(t)$ such that $|F(x) - F(t)| < \epsilon$, whenever $x \in \sigma_t \cap [a, b]$. F is said to be $\mathcal{SC}$ on a set $A \subset [a, b]$ if it is so at each point $t \in A$.

- F is said to be $\mathcal{S}$-upper (respectively lower) semi-continuous at t if for every $\epsilon > 0$ there exists $\sigma_t \in \mathcal{S}(t)$ such that $F(t) - F(x) < \epsilon$ (respectively $F(t) - F(x) > -\epsilon$), whenever $x \in \sigma_t \cap [a, b]$. F is said to be $\mathcal{S}$-upper (respectively lower) semi-continuous on a set $A \subset [a, b]$ if it is so at each point $t \in A$.

Remark 2.2. With the above notations we have:

(i) If F is $\mathcal{SC}$ at $t \in [a,b]$ then F is both, $\mathcal{S}$-upper semi-continuous and $\mathcal{S}$-lower semi-continuous at t. If $\mathcal{S}$ is a filtering local system the converse is also true.

(ii) Definition 2.3 is a slight modification of Thomson's definitions (31.1) and (31.3) of [21, pp. 70–71].

Definition 2.4. Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a bilateral local system, and let $F : [a,b] \to \mathbb{R}$.

- F is said to be right (respectively left) $\mathcal{SC}$ at a point $x \in [a,b)$ (respectively $x \in (a,b]$), if for every $\epsilon > 0$ there exists $\sigma_x \in \mathcal{S}(x)$ such that $|F(t) - F(x)| < \epsilon$, whenever $t \in \sigma_x \cap [x,b)$ (respectively $t \in \sigma_x \cap (a,x]$). F is said to be right (respectively left) $\mathcal{SC}$ on a set $A \subset [a,b)$ (respectively $A \subset (a,b]$), if it is so at each point $x \in A$. If F is right $\mathcal{SC}$ on $[a,b)$ and left $\mathcal{SC}$ on $(a,b]$, we say that F is bilateral $\mathcal{SC}$ on $[a,b]$.
- F is said to be right (respectively left) $\mathcal{S}$- upper semi-continuous at a point $x \in [a,b)$ (respectively $x \in (a,b]$), if for every $\epsilon > 0$ there exists $\sigma_x \in \mathcal{S}(x)$ such that $F(t) - F(x) < \epsilon$, whenever $t \in \sigma_x \cap [x,b)$ (respectively $t \in \sigma_x \cap (a,x]$). F is said to be right (respectively left) $\mathcal{S}$-upper semi-continuous on a set $A \subset [a,b)$ (respectively $A \subset (a,b]$), if it is so at each point $x \in A$.
- F is said to be right (respectively left) $\mathcal{S}$- lower semi-continuous at a point $x \in [a,b)$ (respectively $x \in (a,b]$), if for every $\epsilon > 0$ there exists $\sigma_x \in \mathcal{S}(x)$ such that $F(t) - F(x) > -\epsilon$, whenever $t \in \sigma_x \cap [x,b)$ (respectively $t \in \sigma_x \cap (a,x]$). F is said to be right (respectively left) $\mathcal{S}$-lower semi-continuous on a set $A \subset [a,b)$ (respectively $A \subset (a,b]$), if it is so at each point $x \in A$.
- F is said to be $\mathcal{SC}_i$ at $t \in [a,b]$ if for every $\epsilon > 0$ there exists $\sigma_t \in \mathcal{S}(t)$ such that $F(x) \leq F(t) + \epsilon$ for $x \in \sigma_t \cap [a,t]$, and $F(t) - \epsilon \leq F(y)$ for $y \in \sigma_t \cap [t,b]$. F is said to be $\mathcal{SC}_d$ at t if $-F$ is $\mathcal{SC}_i$ at t. F is said to be $\mathcal{SC}_i$ (respectively $\mathcal{SC}_d$) on a set A if it is so at each point of A.

Lemma 2.1. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ and $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be local systems such that $\mathcal{S}_1 \ll \mathcal{S}_\infty^+$ on $[a,b)$ and $\mathcal{S}_2 \ll \mathcal{S}_\infty^-$ on $(a,b]$. Let $F : [a,b] \to \mathbb{R}$ and $x \in [a,b]$.*

(i) The following assertions are equivalent:

a) F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ at x;

b) F is left $\mathcal{S}_2\mathcal{C}$ at x if $x \in (a,b]$, and F is right $\mathcal{S}_1\mathcal{C}$ at x if $x \in [a,b)$.

(ii) The following assertions are equivalent:

a) F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$ at x;

b) F is left $\mathcal{S}_2$-upper semi-continuous at x if $x \in (a,b]$, and F is right $\mathcal{S}_1$-lower semi-continuous at x if $x \in [a,b)$.

PROOF. Evident. □

Remark 2.3. Let $F : [a,b] \to \mathbb{R}$.

(i) F is right $\mathcal{S}_\infty^+$ lower semicontinuous at a point $x \in [a,b)$ if and only if F is $\mathcal{S}_\infty^+$ lower semicontinuous at x.

(ii) F is left $\mathcal{S}_\infty^-$ lower semicontinuous at a point $x \in [a,b)$ if and only if F is $\mathcal{S}_\infty^-$ lower semicontinuous at x.

(iii) F is lower internal (this condition is due to Garg, see [1, p. 33]) if and only if F is $(\mathcal{S}_\infty^+;\mathcal{S}_\infty^-)\mathcal{C}_i$ on $[a,b]$. (see (i), (ii) and Lemma 2.1, (ii)).

Lemma 2.2. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ and $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be local systems such that $\mathcal{S}_1 \ll \mathcal{S}_\infty^+$ on $[a,b)$ and $\mathcal{S}_2 \ll \mathcal{S}_\infty^-$ on $(a,b]$. Let $F, G : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Let $F_1 : [a,c] \to \mathbb{R}$, $F_1(x) = F(x)$, and let $F_2 : [c,b] \to \mathbb{R}$, $F_2(x) = F(x)$.*

(i) If $\mathcal{S}_1 = \mathcal{S}_2$ and F is $\mathcal{S}_1\mathcal{C}$ at $x \in [a,b]$ then F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ at x;

(ii) Suppose that $\mathcal{S}_1 = \mathcal{S}_2$ is filtering on $[a,b]$. Then $\mathcal{S}_1\mathcal{C} = (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$.

(iii) $F_1 \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ (resp. $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$) on $[a,c]$ and $F_2 \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ (resp. $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$) on $[c,b]$ if and only if F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ (resp. $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$) on $[a,b]$.

(iv) $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C} \subset (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_d$ on $[a,b]$.

(v) $\mathcal{B}_1 \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i \subset \mathcal{D}_-\mathcal{B}_1$ on $[a,b]$.

(vi) $\mathcal{B}_1 \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C} \ \subset\ \mathcal{B}_1 \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_d \ \subset\ \mathcal{DB}_1$ on $[a,b]$.

(vii) Suppose that $\mathcal{S}_1$ is $\mathcal{S}_\infty^+$-filtering on $[a,b)$ and $\mathcal{S}_2$ is $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Then on $[a,b]$ we have

- $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i + (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i \ \subset\ (\mathcal{S}_\infty^+;\mathcal{S}_\infty^-)\mathcal{C}_i$;
- $(\mathcal{B}_1 \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i) + (\mathcal{B}_1 \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i) \ \subseteq\ \mathcal{B}_1 \cap (\mathcal{S}_\infty^+;\mathcal{S}_\infty^-)\mathcal{C}_i \ =\ \mathcal{D}_-\mathcal{B}_1$;
- $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C} + (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C} \ \subset\ (\mathcal{S}_\infty^+;\mathcal{S}_\infty^-)\mathcal{C}$;

- $(\mathcal{B}_1 \cap (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}) + (\mathcal{B}_1 \cap (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}) \subseteq \mathcal{B}_1 \cap (\mathcal{S}^+_\infty; \mathcal{S}^-_\infty)\mathcal{C} = \mathcal{D}\mathcal{B}_1$.

PROOF. See Theorem 2.5.1, (i), (iv) of [1] and Remark 2.3, (iii). □

Lemma 2.3. *Let $F : [a, b] \to \mathbb{R}$. If $F \in [\underline{ACG}] \cap (\mathcal{S}^+_\infty; \mathcal{S}^-_\infty)\mathcal{C}_i$ on $[a, b]$ and $F'_{ap}(x) \geq 0$ a.e. on $[a, b]$ then F is increasing on $[a, b]$.*

PROOF. We have $[\underline{ACG}] \subset [VBG] \subset T_2$ (see Theorem 2.11.1, (vi) and Theorem 2.18.9 of [1]) and $[\underline{ACG}] \subset N^{-\infty}$ (see for example Theorem 2.20.1 and Lemma 2.21.1 of [1]). Let Q be a perfect subset of $[a, b]$. Since $F \in [VBG]$, there exists a portion $(\alpha, \beta) \cap Q \neq \emptyset$ of Q such that $F \in VB$ on it (see Theorem 1.9.1, (ii) of [1]). Then $F_{/Q}$ is continuous nearly everywhere on $(\alpha, \beta) \cap Q$, hence $F \in \mathcal{B}_1$ on $[a, b]$ (see Theorem 2.2.1 of [1]). By Lemma 2.2, $F \in \mathcal{D}_-\mathcal{B}_1 T_2 \cap N^{-\infty}$ on $[a, b]$. Now Corollary 4.3.1. of [1] completes our proof. □

Corollary 2.1. *Let $F : [a, b] \to \mathbb{R}$. If $F \in [ACG] \cap (\mathcal{S}^+_\infty; \mathcal{S}^-_\infty)\mathcal{C}$ on $[a, b]$ and $F'_{ap}(x) = 0$ a.e. on $[a, b]$ then F is constant on $[a, b]$.*

3 Examples of Local Systems

We recall the following local systems.

- $\mathcal{S}^+_o = \{\mathcal{S}^+_o(x)\}_{x\in\mathbb{R}}$, where $S^+_o(x) = \{U : U$ is a right neighborhood of $x\}$.
- $\mathcal{S}^-_o = \{\mathcal{S}^-_o(x)\}_{x\in\mathbb{R}}$, where $S^-_o(x) = \{U : U$ is a left neighborhood of $x\}$.
- $\mathcal{S}^+_{ap} = \{\mathcal{S}^+_{ap}(x)\}_{x\in\mathbb{R}}$, where $S^+_{ap}(x) = \{S : x \in S$ and $\underline{d}^i_+(S; x) = 1\}$.
- $\mathcal{S}^-_{ap} = \{\mathcal{S}^-_{ap}(x)\}_{x\in\mathbb{R}}$, where $S^-_{ap}(x) = \{S : x \in S$ and $\underline{d}^i_-(S; x) = 1\}$.
- For $\alpha \in (0, 1)$ let $\mathcal{S}^+_\alpha = \{\mathcal{S}^+_\alpha(x)\}_{x\in\mathbb{R}}$, where $S^+_\alpha(x) = \{S : x \in S$ and $\underline{d}^i_+(S; x) \geq \alpha\}$.
- For $\alpha \in (0, 1)$ let $\mathcal{S}^-_\alpha = \{\mathcal{S}^-_\alpha(x)\}_{x\in\mathbb{R}}$, where $S^-_\alpha(x) = \{S : x \in S$ and $\underline{d}^i_-(S; x) \geq \alpha\}$.
- $\mathcal{S}^+_{pro,o} = \{\mathcal{S}^+_{pro,o}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{S}^+_{pro,o}(x) = \{A : x \in A$ and there is a measurable set $E \subseteq A$ such that $d^+(E, x) = 1$ and $d_+(E, x) > 0\}$;
- $\mathcal{S}^-_{pro,o} = \{\mathcal{S}^-_{pro,o}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{S}^-_{pro,o}(x) = \{A : x \in A$ and there is a measurable set $E \subseteq A$ such that $d^-(E, x) = 1$ and $d_-(E, x) > 0\}$;

- $\mathcal{S}^+_{pro} = \{\mathcal{S}^+_{pro}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{S}^+_{pro,o}(x) = \{A : x \in A$ and there is a measurable set $E \subset A$ such that $E \cap P \in \mathcal{S}^+_{pro,o}(x)$ whenever P is a measurable set in $\mathcal{S}^+_{pro,o}(x)$;
- $\mathcal{S}^-_{pro} = \{\mathcal{S}^-_{pro}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{S}^-_{pro,o}(x) = \{A : x \in A$ and there is a measurable set $E \subset A$ such that $E \cap P \in \mathcal{S}^-_{pro,o}(x)$ whenever P is a measurable set in $\mathcal{S}^-_{pro,o}(x)$;

(Here $\underline{d}^i_+$ and $\underline{d}^i_-$ are the interior right respectively left densities of the set S at x.)

Remark 3.1. The local systems $\mathcal{S}^+_o$, $\mathcal{S}^-_o$, $\mathcal{S}^+_{ap}$, $\mathcal{S}^-_{ap}$, $\mathcal{S}^+_\alpha$ and $\mathcal{S}^-_\alpha$ were defined by Thomson in [21, pp. 18, 22]. The local systems $\mathcal{S}^+_{pro,o}$, $\mathcal{S}^-_{pro,o}$, $\mathcal{S}^+_{pro}$ and $\mathcal{S}^-_{pro}$ were used by Filipczak in [3] (p. 172; with different names), who gives credit for their introduction to Sarkhel and De. In fact in Sarkhel and De's terminology [18, pp. 30-32], a set $A \in \mathcal{S}^+_{pro}(x)$ if and only if $x \in A$ and $\mathbb{R} \setminus A$ is sparse at x on the right.

Remark 3.2. With the above notations we have:

(i) The local systems $\mathcal{S}^+_o$, $\mathcal{S}^-_o$, $\mathcal{S}^+_{ap}$ and $\mathcal{S}^-_{ap}$ are filtering.

(ii) If $\alpha > \frac{1}{2}$ then $\mathcal{S}^+_\alpha$ is $\mathcal{S}^+_\infty$-filtering, and $\mathcal{S}^-_\alpha$ is $\mathcal{S}^-_\infty$-filtering.

(iii) $(\mathcal{S}^+_o; \mathcal{S}^-_o)\mathcal{C} = \mathcal{C}$;

(iv) $(\mathcal{S}^+_{ap}; \mathcal{S}^-_{ap})\mathcal{C} = \mathcal{C}_{ap}$, where $\mathcal{C}_{ap}$ denotes the class of approximately continuous functions.

(v) $\mathcal{S}^+_{pro}$ and $\mathcal{S}^-_{pro}$ are filtering. (*Indeed. Let P be a measurable set in $\mathcal{S}^+_{pro,o}(x)$, and let $A_1, A_2 \in \mathcal{S}^+_{pro}(x)$, $A = A_1 \cap A_2$. Then $x \in A$ and there exists E_i measurable, $E_i \subset A_i$ such that $E_i \cap P \in \mathcal{S}^+_{pro,o}(x)$, $i = 1, 2$. Let $E = E_1 \cap E_2$. Then $E \cap P = E_1 \cap (E_2 \cap P) \in \mathcal{S}^+_{pro,o}(x)$. Hence $A \in \mathcal{S}^+_{pro}(x)$.*)

(vi) $(\mathcal{S}^+_{pro}; \mathcal{S}^-_{pro})\mathcal{C} = \mathcal{C}_{pro}$, where $\mathcal{C}_{pro}$ is the proximal continuity introduced by Sarkhel and De in [18].

4 A Fundamental Lemma

Lemma 4.1. *Let P be a perfect nowhere dense subset of $[a, b]$, $a, b \in P$, and let $\delta : P \to (0, +\infty)$. Then there exists a finite set $\mathcal{A} = \{([y, z]; x) : x \in \{y, z\} \subset P,\ x$ is a limit point of $[y, z] \cap P$ and $[y, z] \subset (x - \delta(x), x + \delta(x))\}$, such that $\cup_{([y,z],x)\in\mathcal{A}}[y, z] \supseteq P$.*

Proof. Let $\{(a_i, b_i)\}$, $i = \overline{1,\infty}$ be the intervals contiguous to P, and let $\eta : [a,b] \to (0,+\infty)$,

$$\eta(x) \;=\; \begin{cases} \delta(x) & , \;\; if \;\; x \in P \setminus \cup_{i=1}^{\infty}\{a_i, b_i\} \;\; , \\ \min\{\frac{b_i-a_i}{3}, \delta(x)\} & , \;\; if \;\; x \in \{a_i, b_i\} \;\; , \;\; i = \overline{1,\infty} \\ \min\{\frac{x-a_i}{2}, \frac{b_i-x}{2}\} & , \;\; if \;\; x \in (a_i, b_i) \;\; , \;\; i = \overline{1,\infty}\,. \end{cases}$$

Let π be a η-fine partition of $[a,b]$ (i.e., $a = x_o < x_1 < \ldots < x_n = b$ and $t_i \in [x_{i-1}, x_i] \subset (t_i - \eta(t_i), t_i + \eta_i(t_i))$; that such a partition exists follows for example by [1, p. 87]. Let $\pi_1 = \{(I,x) \in \pi : x \in P\}$ and $\pi_2 = \{(I,x) \in \pi : x \notin P\}$. Clearly $\pi = \pi_1 \cup \pi_2$. If $(I,x) \in \pi_2$ then $x \notin P$. Then there exists some i such that $x \in (a_i, b_i)$, hence $x \in I \subset (x - \eta(x), x + \eta(x)) \subset (a_i, b_i)$. It follows that

$$\cup_{(I,x)\in\pi_1} I \supset P\,. \tag{1}$$

Let $(I,x) \in \pi_1$ and let $[y,z]$ be the smallest closed interval that contains $I \cap P$. We have three situations:

1) Suppose that $x \in P \setminus \cup_{i=1}^{\infty}\{a_i, b_i\}$. Then $x, y, z \in P$ and $y \le x \le z$. If $x \in \{y, z\}$ then x is a limit point of $[y,z] \cap P$, and if $x \in (y,z)$ then $[y,x] \cap P$ and $[x,z] \cap P$ have x as a limit point.

2) Suppose that $x = a_i$ for some i. Then $z = a_i$ and $[y, a_i] \cap P$ has a_i as a limit point.

3) Suppose that $x = b_i$ for some i. Then $y = b_i$ and $[b_i, z] \cap P$ has b_i as a limit point.

By 1), 2), 3) and (1) it follows that there exists a finite set $\mathcal{A}$ with the required properties. $\square$

Definition 4.1. Let Z be a real set.

- Let $\mathcal{P}_Z$ be the collection of all sequences $\{Z_n\}_n$ of sets whose union is Z. If in addition each Z_n is closed then we denote this collection by $\overline{\mathcal{P}}_Z$.

- Let $\{Z_n\}_n \in \mathcal{P}_Z$ and $\delta_n : Z_n \to (0,+\infty)$. Let $\beta = \beta(\{Z_n\}, \{\delta_n\})$ denote the collection of all tag intervals $([x,y],t)$, $t \in \{x,y\} \subset Z$, such that $x, y \in Z_n$ and $y - x < \delta_n(t)$ whenever $t \in Z_n$. Let's denote the collection of all β by $\mathcal{B}_Z$. If the collection $\mathcal{P}_Z$ is replaced by $\overline{\mathcal{P}}_Z$ then we denote this collection by $\overline{\mathcal{B}}_Z$. (The collection $\overline{\mathcal{B}}_{\mathbb{R}}$ was defined by Thomson in [20, p. 115], but he called it C).

- Let $\mathcal{B}_o$ denote the collection of all $\mathcal{B}_Z$, with $|Z| = 0$.

- Let $\beta \in \mathcal{B}_Z$ for some real set Z, and let $\pi = \{(I_k, t_k)\}_{k=1}^m$ be a finite subset of β. π is said to be a β-partial partition of Z if the intervals $\{I_k\}_k$ are nonoverlapping.

Definition 4.2. Let P be a real set. We denote by

- $Is^+(P) = \{x \in P \;:\; x$ is a right isolated point of $P\}$;
- $Is^-(P) = \{x \in P \;:\; x$ is a left isolated point of $P\}$;
- $Is(P) = Is^+(P) \cup Is^-(P)$.

Let $\{P_n\}_n$ be a sequence of real sets. We denote by

- $Is^+(\{P_n\}) = \cup_{n=1}^\infty Is^+(P_n)$;
- $Is^-(\{P_n\}) = \cup_{n=1}^\infty Is^-(P_n)$;
- $Is(\{P_n\}) = \cup_{n=1}^\infty Is(P_n)$ (this set is countable, see [15, p. 260]).

Definition 4.3. Let $\{P_n\}_n \in \overline{\mathcal{P}}_{[a,b]}$ and let $\delta_n : P_n \to (0, +\infty)$. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system such that $\mathcal{S}_1 \ll \mathcal{S}_\infty^+$ on $[a,b]$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system such that $\mathcal{S}_2 \ll \mathcal{S}_\infty^-$ on $[a,b]$. For each $x \in Is^+(P_n \cap [a,b))$ let $\sigma_{x,n}^{(1)} \in \mathcal{S}_1(x)$, and for each $x \in Is^-(P_n \cap (a,b])$ let $\sigma_{x,n}^{(2)} \in \mathcal{S}_2(x)$.

(i) Let $\alpha = \alpha(\{P_n\}, \{\delta_n\}, \sigma_{x,n}^{(1)}, \sigma_{x,n}^{(2)})$ denote the collection of all tag intervals $([x,y],t)$, $t \in \{x,y\} \subset [a,b]$ such that:

- For $t = x \in P_n$
  - $y \in (t, t+\delta_n(t)) \cap P_n$ whenever t is a right accumulation point for P_n;
  - $y \in \sigma_{t,n}^{(1)}$ whenever $t \in Is^+(P_n \cap [a,b))$;
- For $t = y \in P_n$
  - $x \in (t-\delta_n(t), t) \cap P_n$ whenever t is a left accumulation point for P_n;
  - $x \in \sigma_{t,n}^{(2)}$ whenever $t \in Is^-(P_n \cap (a,b])$.

(ii) We denote the collection of all α by $\mathcal{A}(\overline{\mathcal{P}}_{[a,b]}; \mathcal{S}_1; \mathcal{S}_2)$

(iii) Let A be a real set such that $A \supset Is(\{P_n\})$. Let $\sigma_t^{(1)} \in \mathcal{S}_1(t)$, with $t \in A \cap [a,b)$, and let $\sigma_t^{(2)} \in \mathcal{S}_2(t)$, with $t \in A \cap (a,b]$. Let $\beta_A = \beta_A(\sigma_t^{(1)}, \sigma_t^{(2)})$ denote the collection of all tag intervals $([x,y],t)$, $t \in \{x,y\} \subset [a,b]$ such that:

- $x = t$ and $y \in \sigma_t^{(1)}$ whenever $t \in A \cap [a,b)$;
- $y = t$ and $x \in \sigma_t^{(2)}$ whenever $t \in A \cap (a,b]$.

(iv) We denote the collection of all β_A by $\mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$.

(v) We denote by $\mathcal{B}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_1,\mathcal{S}_2) = \overline{\mathcal{B}}_{[a,b]} \cup (\cup\{\mathcal{B}_A(\mathcal{S}_1,\mathcal{S}_2) : A$ is a countable subset of $[a,b]$ that contains $Is(\{P_n\})\})$.

(vi) Let $\beta \in \mathcal{B}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_1;\mathcal{S}_2)$ and let $\pi = \{(I_k,t_k)\}_{k=1}^m$ be a finite subset of β. π is said to be a β-partial partition of $[a,b]$ if the intervals $\{I_k\}_k$ are nonoverlapping. If in addition $\cup_{k=1}^m I_k = [a,b]$ then π is said to be a β-partition of $[a,b]$.

Remark 4.1. With the notations of Definition 4.3 we have:

(i) $\mathcal{B}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_1,\mathcal{S}_2) \supset \mathcal{A}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_1,\mathcal{S}_2)$.

(ii) $\overline{\mathcal{B}}_{[a,b]} \cup \mathcal{B}_{Is(\{P_n\})}(\mathcal{S}_o^+;\mathcal{S}_o^-)$ is the family PC, introduced by Henstock [21, p. 115].

(iii) $\mathcal{A}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_{ap}^+;\mathcal{S}_{ap}^-)$ was defined in [12]

Lemma 4.2 (Fundamental lemma)**.** *For each $\beta \in \mathcal{A}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_1;\mathcal{S}_2)$, there exists a β-partition of $[a,b]$ (see Definition 4.3). Particularly, the assertion is true for every $\beta \in \mathcal{B}(\overline{\mathcal{P}}_{[a,b]};\mathcal{S}_1;\mathcal{S}_2)$*

Proof. We shall use the Romanovski Lemma (see for example [1, p. 10]). Let $\mathcal{A} = \{(p,q) \subseteq (a,b) : [p_1,q_1]$ admits a β-partition whenever $(p_1,q_1) \subseteq (p,q)\}$.

(i) If $(p,q) \in \mathcal{A}$ and $(q,r) \in \mathcal{A}$ then clearly $(p,r) \in \mathcal{A}$.

(ii) If $(p,q) \in \mathcal{A}$ and $(p_1,q_1) \subset (p,q)$ then $(p_1,q_1) \in \mathcal{A}$ (see the definition of $\mathcal{A}$.

(iii) Let $(c,d) \subseteq (a,b)$ such that $(p,q) \in \mathcal{A}$ whenever $[p,q] \subset (c,d)$. We show that $(c,d) \in \mathcal{A}$. Let $c \in P_n$. Let $c_1 \in (c, c+\delta_n(c)) \cap P_n \cap (c,(c+d)/2)$ if c is a right accumulation point for P_n, and let $c_1 \in \sigma_c^{(1)} \cap (c,(c+d)/2)$ if c is right isolated in $[a,b) \cap P_n$. Then $([c,c_1],c) \in \beta$. Similarly we find $d_1 \in ((c+d)/2, d)$ such that $([d_1,d],d) \in \beta$. But $(c_1,d_1) \in \mathcal{A}$ and $[c,d] = [c,c_1] \cup [c_1,d_1] \cup [d_1,d]$. Therefore $[c,d]$ admits a β-partition. Analogously we obtain that $[c_2,d_2]$ admits a β-partition, whenever $(c_2,d_2) \subset (c,d)$. Hence $(c,d) \in \mathcal{A}$.

(iv) Let $E \subset [a,b]$ be a perfect set such that all intervals contiguous to E are contained in $\mathcal{A}$. We show that there exists $(p,q) \in \mathcal{A}$ such that $E \cap (p,q) \neq \emptyset$. Since $E = \cup_{n=1}^{\infty}(E \cap P_n)$, by the Baire Category Theorem (see for example [1, p. 10]) it follows that there exists a positive integer n and an interval (p,q)

such that $\emptyset \neq (p,q) \cap E = (E \cap P_n) \cap (p,q)$. We may suppose without loss of generality that $p, q \in E$ and $[p,q] \cap E$ is perfect. Applying Lemma 4.1 to $[p,q] \cap E$ and δ_n, there exists a finite set $\pi = \{([x,y],t) : t \in \{x,y\} \subset E$, t is a limit point of $[x,y] \cap E$, and $\{[x,y]\}$ are nonoverlapping intervals $\}$ and $\cup_{([x,y],t)\in\pi}[x,y] \supseteq E$. Clearly π is a β- partial partition of $[a,b]$. Since $[p,q] \setminus \cup_{([x,y],t)\in\pi}[x,y]$ consists of a finite number of intervals contiguous to E, it follows that $[p,q]$ admits a β- partition. Similarly it follows that each $[p_1,q_1]$ admits such a partition, whenever $(p_1,q_1) \subset (p,q)$. Therefore $(p,q) \in \mathcal{A}$.

By (i)-(iv) and the Romanovski Lemma, it follows that $(a,b) \in \mathcal{A}$.

The second part follows by Remark 4.1, (i). $\square$

Remark 4.2. Lemma 4.2 generalizes a result of Henstock [7, p. 56] as well as Theorem 3.1 of Lee and Soedijono [12, p. 265].

5 A Characterization of ACG $\cap\,\mathcal{C}$ on a Real Compact Set

Definition 5.1. Let $F : [a,b] \to \mathbb{R}$ and $P \subseteq [a,b]$. F is said to be $N_{\mathcal{B}_o}$ on P if it has the following property: for every $\epsilon > 0$ and every $Z \subset P$, $|Z| = 0$, there exists a $\beta = \beta(\{Z_i\},\{\delta_i\}) \in \mathcal{B}_Z \in \mathcal{B}_o$, such that $\sum_{([x,y],t)\in\pi} |F(y) - F(x)| < \epsilon$, whenever π is a β-partial partition of Z.

Lemma 5.1. *Let $F_1, F_2 : [a,b] \to \mathbb{R}$ and $P \subseteq [a,b]$. If $F_1, F_2 \in N_{\mathcal{B}_o}$ on P and $\alpha_1, \alpha_2 \in \mathbb{R}$ then $\alpha_1 F_1 + \alpha_2 F_2 \in N_{\mathcal{B}_o}$ on P.*

PROOF. Clearly $\alpha_1 F_1$ and $\alpha_2 F_2$ are $N_{\mathcal{B}_o}$ on P, so it is sufficient to prove that $F_1 + F_2 \in N_{\mathcal{B}_o}$ on P. Let $\epsilon > 0$ and $Z \subset P$, $|Z| = 0$. Since $F_k \in N_{\mathcal{B}_o}$, $k = 1, 2$, there exists $\beta_k = \beta_k(\{Z_i^{(k)}\},\{\delta_i^{(k)}\}) \in \mathcal{B}_Z$ such that $\sum_{([x,y],t)\in\pi} |F_k(y) - F_k(x)| < \epsilon/2$, whenever π is a β_k-partial partition of Z. Let $Z_{ij} = Z_i^{(1)} \cap Z_j^{(2)}$. Then $\{Z_{ij}\}_{i,j} \in \mathcal{P}_Z$. Let $\delta_{ij} : Z_{ij} \to (0,\infty)$, $\delta_{ij}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$. Then $\beta = \beta(\{Z_{ij}\},\{\delta_{ij}\}) \in \mathcal{B}_Z$. Let π be a β-partial partition of Z. Clearly π is also a β_1 and a β_2-partial partition of Z. It follows that $\sum_{([x,y],t)\in\pi} |(F_1 + F_2)(y) - (F_1 + F_2)(x)| \leq \sum_{([x,y],t)\in\pi} |F_1(y) - F_1(x)| + \sum_{([x,y],t)\in\pi} |F_2(y) - F_2(x)| < \epsilon$. Therefore $F_1 + F_2 \in N_{\mathcal{B}_o}$ on P. $\square$

Lemma 5.2. *Let $F : [a,b] \to \mathbb{R}$ and $Z \subset [a,b]$ such that $|Z| = 0$. If $F \in VBG$ on Z and $|F(Z)| > 0$ then there exists $Z_o \subset Z$ such that F is bounded and strictly monotone on Z_o, and $|F(Z_o)| > 0$.*

PROOF. Since $|F(Z)| > 0$ and $F \in VBG$ on Z it follows that $Z = \cup_{i=1}^{\infty} A_i$, such that $F \in VB$ on each A_i and $|F(A_i)| > 0$ for at least one A_i. Therefore we may suppose without loss of generality that $F \in VB$ on Z. Let

$A = \{x \in Z \,:\, (F_{/Z})^{'}(x)$ exists and is finite$\}$;

$B = \{x \in Z \,:\, (F_{/Z})^{'}(x)$ does not exist, finite or infinite$\}$;

$C = \{x \in Z \,:\, (F_{/Z})^{'}(x) = \pm\infty\}$.

Clearly $Z = A \cup B \cup C$. Let $\tilde{F} : [a,b] \to \mathbb{R}$ such that $\tilde{F} \in VB$ on $[a,b]$ and $\tilde{F} = F$ on Z (this is possible, see for example [1, p. 42]). Then $\tilde{F}^{'}(x)$ does not exist on B. By Theorem 7.2 of [15, p. 230], $|F(B)| = 0$. (That $|F(B)| = 0$ follows also directly from Theorem 4.4 of [15, p. 223].)
We show that F is LG on A (a function F is said to be LG on a set A if $A = \cup_n A_n$ and F is Lipschitz on each A_n, see [1]). Let $A_n = \{x \in A \,:\, |(F_{/Z})^{'}(x)| < n\}$. Then $A = \cup_{n=1}^{\infty} A_n$. For $x \in A_n$ there exists $\delta(x) > 0$ such that $|(F(y) - F(x))/(y-x)| < n$, whenever $y \neq x$, $y \in Z \cap (x - \delta(x), x + \delta(x))$. Let $A_{n,j} = \{x \in A_n \,:\, \delta(x) > 1/j\}$ and $A_{n,j,k} = A_{n,j} \cap [a + (k-1)/j, a + k/j]$, $j = 1, 2, \ldots$, $k = 0, \pm 1, \pm 2, \ldots$. Then $A_n = \cup_j (\cup_k A_{n,j,k})$. If $x < y$, $x, y \in A_{n,j,k}$ then $0 < y - x < 1/j < \min\{\delta(x), \delta(y)\}$. It follows that $|F(y) - F(x)| < n \cdot |y - x|$, hence F is Lipschitz on $A_{n,j,k}$. Therefore F is LG on A. Since $LG \subset (N)$ (see for example Corollary 2.32.1, (iv) of [1]) and $|A| = 0$, it follows that $|F(A)| = 0$.
But $F(Z) = F(A) \cup F(B) \cup F(C)$, hence $|F(C)| = |F(Z)| > 0$. We may suppose without loss of generality that $C = \{x \in Z \,:\, (F_{/Z})^{'} = +\infty\}$. Let $\delta : C \to (0, +\infty)$ such that $(F(y) - F(x))/(y - x) > 1$, whenever $y \neq x$ and $y \in Z \cap (x - \delta(x), x + \delta(x))$. Let

$C_i = \{x \in C \,:\, \delta(x) > 1/i\}$, $i = 1, 2, \ldots$;

$C_{ij} = C_i \cap [a + (j-1)/i, a + j/i]$, $i = 1, 2, \ldots$, $j = 0, \pm 1, \pm 2, \ldots$.

Then $C = \cup_i C_i = \cup_i \cup_j C_{ij}$. Let $x, y \in C_{i,j}$, $x < y$. Then $y - x \leq 1/i < \min\{\delta(x), \delta(y)\}$. It follows that $F(y) - F(x) > y - x > 0$, hence F is strictly increasing on C_{ij}. Since $F(C) = \cup_{i=1}^{\infty} \cup_{j=1}^{i} F(C_{ij})$ and $|F(C)| > 0$, it follows that there exists some C_{ij} such that $|F(C_{ij})| > 0$. Let's denote this C_{ij} by Z_o. Since F is VB on $Z_o \subset Z$, F is bounded on Z_o. Thus Z_o has the required properties. □

Theorem 5.1. *Let $F : [a,b] \to \mathbb{R}$ and $P \subseteq [a,b]$. If F is $N_{\mathcal{B}_o}$ on P and $Z \subset P$, $|Z| = 0$ then*

(i) F is VBG on Z;

(ii) $|F(Z)| = 0$, hence $N_{\mathcal{B}_o} \subset (N)$.

PROOF. (i) For $\epsilon = 1$ let $\beta = \beta(\{Z_i\}, \{\delta_i\}) \in \mathcal{B}_Z$ be such that $\sum_{([x,y],t)\in\pi} |F(y) - F(x)| < 1$ whenever π is a β-partial partition of Z. Let $Z_{i,j} = \{x \in Z_i : \delta_i(x) > 1/j\}$ and $Z_{i,j,k} = Z_{i,j} \cap [a + (k-1)/j, a + k/j]$, $j = 1, 2, \ldots$, $k = 0, \pm 1, \pm 2, \ldots$. Then $Z_i = \cup_j Z_{i,j} = \cup_j (\cup_k Z_{i,j,k})$. Let $\{[\alpha_n, \beta_n]\}$, $n = 1, 2, \ldots, p$ be a finite set of nonoverlapping closed intervals, with endpoints in $Z_{i,j,k}$. Since $0 < \beta_n - \alpha_n < 1/j < \min\{\delta_i(\alpha_n), \delta_i(\beta_n)\}$, it follows that $([\alpha_n, \beta_n], \alpha_n) \in \beta$, hence $\sum_{n=1}^{p} |F(\beta_n) - F(\alpha_n)| < 1$. Therefore F is VBG on the set Z.

(ii) Suppose on the contrary that $|F(Z)| > 0$. By (i) and Lemma 5.2, it follows that there exists $Z_o \subset Z$ such that F is strictly increasing (for example) and bounded on Z_o, and $|F(Z_o)| > 0$. Let $\epsilon = |F(Z_o)|$. For $\epsilon/4$ there exists a $\beta = \beta(\{Z_i\}, \{\delta_i\}) \in \mathcal{B}_{Z_o}$ such that $\sum_{([x,y],t)\in\pi} |F(y) - F(x)| < \epsilon/4$, whenever π is a β-partial partition of Z_o. Since F is VB on Z_o, it follows that $F_{/Z_o}$ is continuous nearly everywhere on Z_o. We may suppose without loss of generality that $F_{/Z_o}$ is continuous on Z_o (because $|F(Z_o)| > 0$). Since each Z_i contains countable many isolated points of Z_i (see [15, p. 260]), we may suppose without loss of generality that Z_i contains no isolated points of Z_i. Let $Y_1 = Z_1$, $Y_1 = Z_i \setminus (\cup_{j=1}^{i-1} Z_j)$, $i \geq 2$. Then $Z_o = \cup_{i=1}^{\infty} Y_i$ and $Y_{i_1} \cap Y_{i_2} = \emptyset$ for $i_1 \neq i_2$. Let $t \in Z_o$. Then there exists an unique i such that $t \in Y_i$. Let

$$\mathcal{A}_i = \big\{\langle F(t), F(x)\rangle\big\}_{\substack{(\langle t,x\rangle,t)\in\beta(Z_i,\delta_i)\\ t\in Y_i}}$$

and $\mathcal{A} = \big\{\langle F(x), F(y)\rangle\big\}_{(\langle x,y\rangle,x)\in\beta}$. Then $\mathcal{A}$ is a Vitali cover for $F(Z_o)$ (indeed: if $x_o \in Z_o$ then $x \in Z_i$ for some i, and for each

$$y \in \Big(\big(x - \delta_i(x), x + \delta_i(x)\big) \cap Z_i\Big) \setminus \{x\} \neq \emptyset$$

we have $\langle F(x), F(y)\rangle \in \mathcal{A}$, $F(x) \neq F(y)$; if $y \to x$ then $F(y) \to F(x)$). Let $\pi \subset \beta$ be a finite subset such that

$$\sum_{(\langle x,y\rangle,x)\in\pi} |F(y) - F(x)| > \frac{3}{4}|F(Z_o)|$$

and $\big\{\langle F(x), F(y)\rangle\big\}_{(\langle x,y\rangle,x)\in\pi}$ contains only pairwise disjoint closed intervals (by Vitali's Covering Theorem – see for example [1, p. 10]). Since F is strictly increasing, π is a partition, so

$$\frac{3}{4}|F(Z_o)| < \sum_{(\langle x,y\rangle,x)\in\pi} |F(y) - F(x)| < \frac{\epsilon}{4} = \frac{1}{4}|F(Z_o)|\,,$$

a contradiction. □

Lemma 5.3. ([1, p. 12]) *Let P be a closed subset of $[a,b]$ and let $F \in \mathcal{C}$ on P. The following assertions are equivalent:*

(i) $F \in VBG$ on P;

(ii) For every closed subset S of P there exists $(\alpha,\beta) \cap S \neq \emptyset$ such that F is VB on (α,β).

(iii) $F \in VBG$ on Z whenever $Z \subset P$ and $|Z| = 0$.

Remark 5.1. (i) $\Leftrightarrow$ (ii) in Lemma 5.3 follows also by [15, p. 223].

Lemma 5.4. *Let $F : [a,b] \to \mathbb{R}$ and $P \subset [a,b]$. If $F \in ACG$ on P then $F \in N_{\mathcal{B}_o}$ on P.*

Proof. Let $\epsilon > 0$ and $Z \subset P$, $|Z| = 0$. Since F is ACG on P, F is ACG on Z too. So there exists a sequence of sets $\{Z_i\}_i$ such that $Z = \cup_{i=1}^{\infty} Z_i$ and F is AC on each Z_i. For $\epsilon/2^i$ let $\eta_i > 0$ be given by the fact that F is AC on Z_i, and let G_i be an open set such that $Z_i \subset G_i$ and $|G_i| < \eta_i$. Further, let $\delta_i : Z_i \to (0,+\infty)$ such that $(x - \delta_i(x), x + \delta_i(x)) \subset G_i$, for every $x \in Z_i$; let $\beta = \beta(\{Z_i\},\{\delta_i\})$ and let π be a β-partial partition of Z. We denote by $\pi_i = \{([x,y],t) \in \pi : t \in Z_i\}$. Clearly $x, y \in Z_i$ and $[x,y] \subset G_i$, whenever $([x,y],t) \in \pi_i$. It follows that

$$\sum_{([x,y],t)\in\pi} |F(y) - F(x)| = \sum_{i=1}^{\infty} \sum_{([x,y],t)\in\pi_i} |F(y) - F(x)| < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon\,,$$

hence $F \in N_{\mathcal{B}_o}$ on P. □

Theorem 5.2. *Let P be a closed subset of $[a,b]$ and let $F \in \mathcal{C}$ on P. The following assertions are equivalent:*

(i) $F \in ACG$ on P;

(ii) $F \in N_{\mathcal{B}_o}$ on P.

Proof. (i) $\Rightarrow$ (ii) See Lemma 5.4

(ii) $\Rightarrow$ (i) By Theorem 5.1, F is (N) on P, and F is VBG on every $Z \subset P$, with $|Z| = 0$. By Lemma 5.3, it follows that F is VBG on P. Therefore $F \in VBG \cap (N) \cap \mathcal{C} = ACG \cap \mathcal{C}$ (see for example [1, p. 75]). □

6 The Lusin Type $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$ Integral

Definition 6.1. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^+_\infty$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^-_\infty$-filtering on $(a,b]$. Let $f : [a,b] \to \overline{\mathbb{R}}$. f is said to be $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$, if there exists $F : [a,b] \to \mathbb{R}$ such that F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$, $F \in [ACG]$ on $[a,b]$ and $F'_{ap}(x) = f(x)\, a.e.$ on $[a,b]$. We write $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^x f(t)dt = F(x) - F(a)$.

Lemma 6.1. *The $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integral is well defined.*

PROOF. Let $F,G : [a,b] \to \mathbb{R}$, $F,G \in [ACG] \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$ such that $F'_{ap}(x) = G'_{ap}(x) = f(x)$ *a.e.* on $[a,b]$. Then $(F-G)'_{ap}(x) = 0$ *a.e.* on $[a,b]$ and $F - G \in [ACG] \cap (\mathcal{S}^+_\infty;\mathcal{S}^-_\infty)\mathcal{C}$ on $[a,b]$. By Corollary 2.1, $F - G$ is constant on $[a,b]$. □

Remark 6.1. We have the following special cases for the $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integral:

- The $[\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{D}]$-integral is in fact the wide Denjoy integral $\mathcal{D}$. Therefore any $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integral contains the $\mathcal{D}$-integral.
- The $[\mathcal{S}^+_{ap}\mathcal{S}^-_{ap}\mathcal{D}]$-integral is in fact the β-Ridder integral (that is also called by Kubota the AD integral, see [14], [8]).
- For $\alpha,\beta \in (1/2,1)$ we obtain the $[\mathcal{S}^+_\alpha\mathcal{S}^-_\beta\mathcal{D}]$-integral, that seems to be new.
- The $[\mathcal{S}^+_{pro}\mathcal{S}^-_{pro}\mathcal{D}]$-integral is strictly contained in many of the integrals studied by Sarkhel, De and Kar in [18], [16], [17], [19].

Lemma 6.2. *Let $f : [a,b] \to \overline{\mathbb{R}}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$, and let $c \in (a,b)$. Then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on both $[a,c]$ and $[c,b]$, and we have*

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f(t)\,dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^c f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_c^b f(t)\,dt \tag{2}$$

PROOF. Let $F(x) = [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^x f(t)\,dt$ and let

$F_1 : [a,c] \to \mathbb{R}$, $F_1(x) = F(x)$ if $x \in [a,c]$;

$F_2 : [c,b] \to \mathbb{R}$, $F_2(x) = F(x)$ if $x \in [c,b]$.

By Lemma 2.2, $F_1, F_2 \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,c]$ respectively $[c,b]$. Then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,c]$ and on $[c,b]$, and we have

$[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^c f(t)\,dt = F_1(c) - F_1(a) = F(c) - F(a)$;

$[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_c^b f(t)\,dt = F_2(b) - F_2(c) = F(b) - F(c)$.

Now (2) follows immediately. □

Lemma 6.3. *Let $f : [a,b] \to \overline{\mathbb{R}}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,c]$ and on $[c,b]$, where $c \in (a,b)$. Then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$ and*

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f(t)\,dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^c f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_c^b f(t)\,dt \qquad (3)$$

Proof. Let $F_1 : [a,c] \to \mathbb{R}$, $F_1 \in [ACG] \cap (\mathcal{S}_1\mathcal{S}_2)\mathcal{C}$ such that $(F_1)'_{ap}(x) = f(x)$ *a.e.* on $[a,c]$. Let $F_2 : [c,b] \to \mathbb{R}$, $F_2 \in [ACG] \cap (\mathcal{S}_1\mathcal{S}_2)\mathcal{C}$ such that $(F_2)'_{ap}(x) = f(x)$ *a.e.* on $[c,b]$. Let $F : [a,b] \to \mathbb{R}$,

$$F(x) = \begin{cases} F_1(x) & , \quad x \in [a,c] \\ F_2(x) + F_1(c) - F_2(c) & , \quad x \in [c,b] \end{cases}$$

Then $F \in [ACG] \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$ (see Lemma 2.2) and $F'_{ap}(x) = f(x)$ *a.e.* on $[a,b]$. Hence f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$ and we have (3). □

Lemma 6.4. *Let $f_1, f_2 : [a,b] \to \overline{\mathbb{R}}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$, and let $\alpha_1, \alpha_2 \in \mathbb{R}$. If $\mathcal{S}_1$ is filtering on $[a,b)$ and $\mathcal{S}_2$ is filtering on $(a,b]$, then $\alpha_1 f_1 + \alpha_2 f_2$ is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b(\alpha_1 f_1 + \alpha_2 f_2)(t)\,dt = \alpha_1 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f_1(t)\,dt + \alpha_2 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f_2(t)\,dt$.*

Proof. Since f_1 and f_2 are $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$, there exist $F_1, F_2 : [a,b] \to \mathbb{R}$, belonging to $[ACG] \cap (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$, such that $(F_1)'_{ap}(x) = f_1(x)$ and $(F_2)'_{ap}(x) = f_2(x)$ *a.e.* on $[a,b]$. Clearly $(\alpha_1 F_1 + \alpha_2 F_2)'_{ap}(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$ *a.e.* on $[a,b]$, and $\alpha_1 F_1 + \alpha_2 F_2 \in [ACG]$ on $[a,b]$. Then $\alpha_1 F_1 + \alpha_2 F_2$ is left $\mathcal{S}_1\mathcal{C}$ on $(a,b]$ (since $\mathcal{S}_1$ is filtering on $(a,b]$) and $\alpha_1 F_1 + \alpha_2 F_2$ is right $\mathcal{S}_2\mathcal{C}$ on $(a,b]$ (since $\mathcal{S}_2$ is filtering on $[a,b)$). By Lemma 2.1 it follows that $\alpha_1 F_1 + \alpha_2 F_2$ is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$. Thus $\alpha_1 f_1 + \alpha_2 f_2$ is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b(\alpha_1 f_1 + \alpha_2 f_2)(t)\,dt = (\alpha_1 F_1 + \alpha_2 F_2)(b) - (\alpha_1 F_1 + \alpha_2 F_2)(a) = \alpha_1 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f_1(t)\,dt + \alpha_2 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f_2(t)\,dt$. □

7 The Riemann Type $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$ Integral

Definition 7.1. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^+_\infty$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^-_\infty$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$. f is said to be $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$ integrable on $[a,b]$, if there is a real number

I with the following property: for every $\epsilon > 0$ there exists $\beta = \beta(\{X_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ such that for every countable set A with $A \supset Is(\{X_i\})$ there exists $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1, \mathcal{S}_2)$ so that $|s(f,\pi) - I| < \epsilon/2$ whenever π is a $(\beta \cup \beta_A)$-partition of $[a,b]$.

Theorem 7.1. *The number I in Definition 7.1 is unique, and we denote it by $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f(t)dt$.*

PROOF. Suppose that there exist two numbers I_1 and I_2 as in Definition 7.1. For $\epsilon > 0$ and I_k, $k = 1, 2$ let $\beta_k = \beta_k(\{X_i^{(k)}\}, \{\delta_i^{(k)}\})$ be given by Definition 7.1. For $A = Is(\{X_i^{(1)}\}) \cup Is(\{X_i^{(2)}\})$ let

$$\beta_A^{(k)} = \beta_A^{(k)}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_A(\mathcal{S}_1; \mathcal{S}_2)\,,$$

$k = 1, 2$ be given by Definition 7.1. We define

- $\sigma_x^{(1)} = \sigma_x^{(1,1)} \cap \sigma_x^{(2,1)} \in \mathcal{S}_\infty^+(x)$;
- $\sigma_x^{(2)} = \sigma_x^{(1,2)} \cap \sigma_x^{(2,2)} \in \mathcal{S}_\infty^-(x)$;
- $\delta_{i,j} : X_i \cap X_j \to (0, +\infty)$, $\delta_{i,j}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$;
- $\beta_3 = \beta_3(\{X_i^{(1)} \cap X_j^{(2)}\}, \{\delta_{i,j}\}) \in \overline{\mathcal{B}}_{[a,b]}$;
- $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}(\mathcal{S}_\infty^+; \mathcal{S}_\infty^-)$;
- $\beta = \beta_3 \cup \beta_A$;

By Lemma 4.2 there exists π, an β-partition of $[a,b]$. Clearly π is also a $(\beta_k \cup \beta_A^{(k)})$-partition of $[a,b]$, $k = 1, 2$. It follows that $|s(f,\pi) - I_k| < \epsilon$, $k = 1, 2$, hence $|I_1 - I_2| < 2\epsilon$. Since ϵ is arbitrary we obtain that $I_1 = I_2$. □

Lemma 7.1. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$, and let $\alpha_1, \alpha_2 \in \mathbb{R}$. If $\mathcal{S}_1$ is filtering on $[a,b)$ and $\mathcal{S}_2$ is filtering on $(a,b]$, then $\alpha_1 f_1 + \alpha_2 f_2$ is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$- integrable on $[a,b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b (\alpha_1 f_1 + \alpha_2 f_2)(t)dt = \alpha_1 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f_1(t)dt + \alpha_2 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f_2(t)dt$.*

PROOF. Let $I = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f_1(t)dt$ and $J = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f_2(t)dt$. Suppose that $\alpha_1, \alpha_2 \neq 0$ and let $\epsilon > 0$.

For $\epsilon/(2|\alpha_2|)$ let $\beta_1 = \beta_1(\{X_i^{(1)}\}, \{\delta_i^{(1)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by the fact that f_1 is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$.

For $\epsilon/(2|\alpha_1|)$ let $\beta_2 = \beta_1(\{X_i^{(2)}\}, \{\delta_i^{(2)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by the fact that f_2 is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$.

Let $X_{ij} = X_i^{(1)} \cap X_j^{(2)}$. Then $\{X_{ij}\}_{i,j} \in \overline{\mathcal{P}}_{[a,b]}$.

Let $\delta_{ij} : X_{ij} \to (0, +\infty)$, $\delta_{ij}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$.

Let $\beta = \beta(\{X_{ij}\}, \{\delta_{ij}\}) \in \overline{\mathcal{B}}_{[a,b]}$.

Clearly $Is(\{X_{ij}\})$ contains both, $Is(\{X_i^{(1)}\})$ and $Is(\{X_j^{(2)}\})$.

Let A be a countable subset of $[a,b]$ that contains $Is(\{X_{ij}\})$.

For $\epsilon/(2|\alpha_2|)$, let $\beta_A^{(1)} = \beta_A^{(1)}(\sigma_x^{(1,1)}, \sigma_x^{(1,2)}) \in \mathcal{B}_A(\mathcal{S}_1; \mathcal{S}_2)$ be given by the fact that f_1 is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$.

For $\epsilon/(2|\alpha_1|)$, let $\beta_A^{(2)} = \beta_A^{(2)}(\sigma_x^{(2,1)}, \sigma_x^{(2,2)}) \in \mathcal{B}_A(\mathcal{S}_1; \mathcal{S}_2)$ be given by the fact that f_2 is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$.

Let $\sigma_x^{(1)} = \sigma_x^{(1,1)} \cap \sigma_x^{(2,1)}$. Since $\mathcal{S}_1$ is filtering, $\sigma_x^{(1)} \in \mathcal{S}_1(x)$.

Let $\sigma_x^{(2)} = \sigma_x^{(1,2)} \cap \sigma_x^{(2,2)}$. Since $\mathcal{S}_2$ is filtering, $\sigma_x^{(2)} \in \mathcal{S}_2(x)$.

Let $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1; \mathcal{S}_2)$.

Let π be a $(\beta \cup \beta_A)$-partition of $[a,b]$. Then π is also a $(\beta_1 \cup \beta_A^{(1)})$-partition and a $(\beta_2 \cup \beta_A^{(2)})$-partition of $[a,b]$. Then $|s(\alpha_1 f_1 + \alpha_2 f_2; \pi) - (\alpha_1 I + \alpha_2 J)| \leq |\alpha_1| \cdot |s(f_1; \pi) - I| + |\alpha_2| \cdot |s(f_2; \pi) - J| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore $\alpha_1 I + \alpha_2 J = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}] \int_a^b (\alpha_1 f_1 + \alpha_2 f_2)(t)dt$. $\square$

Lemma 7.2 (A Cauchy criterion)**.** *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$. The following assertions are equivalent:*

(i) *f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$.*

(ii) *For $\epsilon > 0$ there exists $\beta = \beta(\{X_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ (depending on ϵ) and for every pair of countable subsets A_1 and A_2 of $[a,b]$ there exist $\beta_{A_k} = \beta_{A_k}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A_k}(\mathcal{S}_1; \mathcal{S}_2)$ (depending only on ϵ), such that $|s(f, \pi_1) - s(f, \pi_2)| < \epsilon$, whenever π_k is a $(\beta \cup \beta_{A_k})$-partition of $[a,b]$, $k = 1, 2$.*

PROOF. (i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) For $\epsilon_k = 1/k$, $k = \overline{2, +\infty}$, let $\beta_k = \beta_k(\{X_i^{(k)}\}, \{\delta_i^{(k)}\})$ be given by (ii). Let $A_k = \mathrm{Is}(\{X_i^{(k)}\})$. For $n > m > 2$ let $A_{m,n} = A_m \cup A_n \cup \mathrm{Is}(\{X_i^{(m)} \cap X_j^{(n)}\})$. By (ii), for ϵ_m and the pair of sets A_m and $A_{m,n}$, there exist

$$\beta_{A_m} = \beta_{A_m}(\sigma_x^{(m,1)}, \sigma_x^{(m,2)}) \in \mathcal{B}_{A_m}(\mathcal{S}_1; \mathcal{S}_2)$$

and

$$\beta^{(1)}_{A_{m,n}} = \beta^{(1)}_{A_{m,n}}(\sigma_x^{(m,n,1,1)}, \sigma_x^{(m,n,1,2)}) \in \mathcal{B}_{A_{m,n}}(\mathcal{S}_1; \mathcal{S}_2)$$

such that $|s(f, \pi_1) - s(f, \pi_2)| < \epsilon_m$ whenever π_1 is a $(\beta_m \cup \beta_{A_m})$-partition for $[a, b]$ and π_2 is a $(\beta_m \cup \beta^{(1)}_{A_{m,n}})$-partition of $[a, b]$.

Again by (ii), for ϵ_n and the pair of sets A_n and $A_{m,n}$, there exist $\beta_{A_n} = \beta_{A_n}(\sigma_x^{(n,1)}, \sigma_x^{(n,2)}) \in \mathcal{B}_{A_n}(\mathcal{S}_1; \mathcal{S}_2)$ and $\beta^{(2)}_{A_{m,n}} = \beta^{(2)}_{A_{m,n}}(\sigma_x^{(m,n,2,1)}, \sigma_x^{(m,n,2,2)}) \in \mathcal{B}_{A_{m,n}}(\mathcal{S}_1; \mathcal{S}_2)$ such that $|s(f, \pi_1) - s(f, \pi_2)| < \epsilon_n$ whenever π_1 is a $(\beta_n \cup \beta_{A_n})$-partition for $[a, b]$ and π_2 is a $(\beta_n \cup \beta^{(2)}_{A_{m,n}})$-partition of $[a, b]$. Let

$$\delta_{i,j}^{(m,n)} : X_i^{(m)} \cap X_j^{(n)} \to (0, +\infty),\ \delta_{i,j}^{(m,n)}(x) = \min\{\delta_i^{(m)}(x), \delta_j^{(n)}(x)\};$$

$$\beta_{m,n} = \beta_{m,n}(\{X_i^{(m)} \cap X_j^{(n)}\}, \{\delta_{i,j}^{(m,n)}\}) \in \overline{\mathcal{B}}_{[a,b]};$$

$$\sigma_x^{(m,n,1)} = \sigma_x^{(m,n,1,1)} \cap \sigma_x^{(m,n,2,1)};$$

$$\sigma_x^{(m,n,2)} = \sigma_x^{(m,n,1,2)} \cap \sigma_x^{(m,n,2,2)};$$

$$\beta^{(3)}_{A_{m,n}} = \beta^{(3)}_{A_{m,n}}(\sigma_x^{(m,n,1)}, \sigma_x^{(m,n,2)}) \in \mathcal{B}_{A_{m,n}}(\mathcal{S}^+_\infty; \mathcal{S}^-_\infty);$$

$$\beta = \beta_{m,n} \cup \beta^{(3)}_{A_{m,n}}.$$

By Lemma 4.2 there exists π a β-partition of $[a, b]$. But π is also a $\beta_m \cup \beta^{(1)}_{A_{m,n}}$-partition and a $(\beta_n \cup \beta^{(2)}_{A_{m,n}})$-partition of $[a, b]$. Let π_m be a $(\beta_m \cup \beta_{A_m})$-partition of $[a, b]$ and let π_n be a $(\beta_n \cup \beta_{A_n})$-partition of $[a, b]$. Then $|s(f, \pi_m) - s(f, \pi)| < \epsilon_m$ and $|s(f, \pi_n) - s(f, \pi)| < \epsilon_n$. It follows that $|s(f, \pi_m) - s(f, \pi_n)| < 1/m + 1/n < 2/m = 2\epsilon_m$. Therefore $\{s(f, \pi_m)\}_m$ is a Cauchy sequence. Let's denote its limit by I. Then $|s(f, \pi_m) - I| \leq 2\epsilon_m$. Let $\epsilon > 0$ and $m > 2$ such that $3/m < \epsilon$. Let A be a countable subset of $[a, b]$. Then there exists $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1; \mathcal{S}_2)$ such that $|s(f, \pi) - s(f, \pi_m)| < 1/m$ whenever π is a $(\beta_m \cup \beta_A)$-partition of $[a, b]$. It follows that $|s(f, \pi) - I| < 1/m + 2/m < \epsilon$. Therefore f is $(\mathcal{S}_1\mathcal{S}_2\mathcal{R})$- integrable on $[a, b]$. $\square$

Lemma 7.3. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^+_\infty$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^-_\infty$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$.*

(i) If $a < c < b$ and f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,c]$ and on $[c,b]$ then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$ and we have

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^c f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_c^b f(t)\,dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f(t)\,dt.$$

(ii) If $a \le c < d \le b$ and f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$ then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[c,d]$.

Proof. (i) Let $I^{(1)} = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^c f(t)\,dt$ and $I^{(2)} = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_c^b f(t)\,dt$. Consider $\epsilon > 0$.

For $\frac{\epsilon}{2}$ and $I^{(1)}$ let $\beta^{(1)} = \beta^{(1)}(\{X_i^{(1)}\}, \{\delta_i^{(1)}\}) \in \overline{\mathcal{B}}_{[a,c]}$ be given by Def. 7.1.

For $\frac{\epsilon}{2}$ and $I^{(2)}$ let $\beta^{(2)} = \beta^{(2)}(\{X_i^{(2)}\}, \{\delta_i^{(2)}\}) \in \overline{\mathcal{B}}_{[c,b]}$ be given by Def. 7.1.

Let $A^{(1)} = \mathrm{Is}(\{X_i^{(1)}\})$ and $A^{(2)} = \mathrm{Is}(\{X_i^{(2)}\})$.

Note that $c \in A^{(1)} \cap A^{(2)}$, $\{X_i^{(1)}, X_j^{(2)}\}_{i,j} \in \overline{\mathcal{P}}_{[a,b]}$ and $\mathrm{Is}(\{X_i^{(1)}\} \cup \{X_j^{(2)}\}) = A^{(1)} \cup A^{(2)}$.

Let A be a countable subset of $[a,b]$.

Let $A_1 = A^{(1)} \cup (A \cap [a,c])$ and $A_2 = A^{(2)} \cup (A \cap [c,b])$.

Clearly $A^{(1)} \subseteq A_1$ and $A^{(2)} \subseteq A_2$. By Definition 7.1, for $\epsilon/2$ and A_k, $k = 1,2$ there exists $\beta_{A_k} = \beta_{A_k}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A_k}(\mathcal{S}_1;\mathcal{S}_2)$ such that $|s(f,\pi) - I^{(1)}| < \epsilon/2$ whenever π is a $(\beta^{(1)} \cup \beta_{A_1})$-partition of $[a,c]$, and $|s(f,\pi) - I^{(2)}| < \epsilon/2$ whenever π is a $(\beta^{(2)} \cup \beta_{A_2})$-partition of $[c,b]$. Let $\delta_i^{(1)\star} : X_i^{(1)} \to (0,+\infty)$ and $\delta_j^{(2)\star} : X_j^{(2)} \to (0,+\infty)$ be defined as follows:

$$\delta_i^{(1)\star}(x) = \begin{cases} \min\{\delta_i^{(1)}(x), c-x\} & \text{if } x < c \\ \min\{\delta_i^{(1)}(c), c-a\} & \text{if } x = c \end{cases}$$

$$\delta_j^{(2)\star}(x) = \begin{cases} \min\{\delta_j^{(2)}(x), c-x\} & \text{if } x > c \\ \min\{\delta_j^{(1)}(c), b-c\} & \text{if } x = c \end{cases}$$

Let

$$\sigma_x^{(1)} = \begin{cases} \sigma_x^{(1,1)} \cap (2x-c, c) \in \mathcal{S}_1(x) & \text{if } x \in [a,c) \cap A_1 \\ \sigma_x^{(2,1)} \in \mathcal{S}_1(x) & \text{if } x \in [c,b) \cap A_2 \end{cases}$$

$$\sigma_x^{(2)} = \begin{cases} \sigma_x^{(1,2)} \in \mathcal{S}_2(x) & \text{if } \in (a,c] \cap A_1 \\ \sigma_x^{(2,2)} \cap (c, 2x-c) \in \mathcal{S}_2(x) & \text{if } \in (c,b] \cap A_2 \end{cases}$$

Let $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \subset \beta_{A_1} \cup \beta_{A_2}$;

Let $\{X_k'\}_k$ be a relabeling of the set $\{X_i^{(1)}\}_i \cup \{X_j^{(2)}\}_j$;

Let $\delta_k' : X_k' \to (0, +\infty)$, $\delta_k' = \delta_i^{(1)\star}$ if $X_k' = X_i^{(1)}$, and $\delta_k' = \delta_j^{(2)\star}$ if $X_k' = X_j^{(2)}$

$\beta_o = \beta_o(\{X_k'\}, \{\delta_k'\}) \subset \beta^{(1)} \cup \beta^{(2)}$.

$\beta = \beta_o \cup \beta_A$.

Let π be a β-partition of $[a,b]$. Let $([x,y],t) \in \pi$. If $t < c$ then $y < c$, and if $t > c$ then $x > c$. It follows that

$$C = \cup_{\substack{([x,y],t)\in\pi \\ t<c}} [x,y] \subset [a,c) \quad \text{and} \quad D = \cup_{\substack{([x,y],t)\in\pi \\ t>c}} [x,y] \subset (c,b]\,.$$

Let $x_c = \sup C$ and $y_c = \inf D$. We observe that $([x_c, c], c)$ and $([c, y_c], c)$ belong to β. Let $\pi_1 = \{(I,t) \in \pi : t \le c\}$ and $\pi_2 = \{(I,t) \in \pi : t \ge c\}$. Then π_1 is a $(\beta^{(1)} \cup \beta_{A_1})$-partition of $[a,c]$ and π_2 is a $(\beta^{(2)} \cup \beta_{A_2})$-partition of $[c,b]$. We have $|s(f,\pi) - I^{(1)} - I^{(2)}| = |s(f,\pi_1) - I^{(1)} + s(f,\pi_2) - I^{(2)}| < |s(f,\pi_1) - I^{(1)}| + |s(f,\pi_2) - I^{(2)}| < \epsilon$. It follows that f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^b f(t)\,dt = I^{(1)} + I^{(2)}$.

(ii) Let $a < c < d < b$.

For $\epsilon > 0$ let $\beta = \beta(\{X_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by Lemma 7.2.

Let $\beta_1 = \beta_1(\{X_i \cap [c,d]\}, \{\delta_{i/X_i\cap[c,d]}\})$.

Let $A^{(1)}$, $A^{(2)}$ be a pair of countable subsets of $[c,d]$.

Let $\beta_3 = \beta_3(\{X_i \cap [a,c]\}, \{\delta_{i/X_i\cap[a,c]}\})$.

Let A_3 be a countable subset of $[a,c]$ such that $A_3 \supset \mathrm{Is}(\{X_i \cap [a,c]\})$.

Let $\beta_4 = \beta_4(\{X_i \cap [d,b]\}, \{\delta_{i/X_i\cap[d,b]}\})$.

Clearly

$$\beta \supseteq \beta_1 \cup \beta_3 \cup \beta_4\,. \tag{4}$$

Let A_4 be a countable subset of $[d,b]$ such that $A_4 \supset \mathrm{Is}(\{X_i \cap [d,b]\})$.

Let $A_1 = A^{(1)} \cup A_3 \cup A_4 \cup \mathrm{Is}(\{X_i \cap [c,d]\})$ and $A_2 = A^{(2)} \cup A_3 \cup A_4 \cup \mathrm{Is}(\{X_i \cap [c,d]\})$ (both contain $\mathrm{Is}(\{X_i\})$).

For A_k, $k = 1,2$ let $\sigma_x^{(k,1)} \in \mathcal{S}_1(x)$, $x \in [a,b) \cap A_k$ and $\sigma_x^{(k,2)} \in \mathcal{S}_2(x)$, $x \in (a,b] \cap A_k$ be given by Lemma 7.2.

Let $\beta_{A_k} = \beta_{A_k}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A_k}(\mathcal{S}_1, \mathcal{S}_2)$, $k = 1,2$.

Let $\beta_{A_3}^{(k)} = \beta_{A_3}^{(k)}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A_3}(\mathcal{S}_1; \mathcal{S}_2)$, $k = 1,2$.

Let $\beta_{A_4}^{(k)} = \beta_{A_4}^{(k)}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A_4}(\mathcal{S}_1; \mathcal{S}_2)$, $k = 1,2$.

Let $\sigma_x^{(1)} = \sigma_x^{(1,1)} \cap \sigma_x^{(2,1)} \in \mathcal{S}_\infty^+$, if $x \in (A_3 \cup A_4) \setminus \{b\}$.

Let $\sigma_x^{(2)} = \sigma_x^{(1,2)} \cap \sigma_x^{(2,2)} \in \mathcal{S}_\infty^-$, if $x \in (A_3 \cup A_4) \setminus \{a\}$.

Let $\beta_{A_3} = \beta_{A_3}(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_{A_3}(\mathcal{S}_\infty^+; \mathcal{S}_\infty^-)$.

Let $\beta_{A_4} = \beta_{A_4}(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_{A_4}(\mathcal{S}_\infty^+; \mathcal{S}_\infty^-)$.

Let $\beta_{A^{(k)}} = \beta_{A^{(k)}}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A^{(k)}}(\mathcal{S}_1; \mathcal{S}_2)$, $k = 1,2$.

Clearly $\beta_{A_k} \supseteq \beta_{A^{(k)}} \cup \beta_{A_3}^{(k)} \cup \beta_{A_4}^{(k)}$, $k = 1,2$.

Let π_1 be a $(\beta_1 \cup \beta_{A^{(1)}})$-partition of $[c,d]$.

Let π_2 be a $(\beta_1 \cup \beta_{A^{(2)}})$-partition of $[c,d]$.

By Lemma 4.2 there exists π_3, a $(\beta_3 \cup \beta_{A_3})$-partition of $[a,c]$. Clearly π_3 is both, a $(\beta_3 \cup \beta_{A_3}^{(1)})$ partition and a $(\beta_3 \cup \beta_{A_3}^{(2)})$ partition of $[a,c]$.

By Lemma 4.2 there exists π_4, a $(\beta_4 \cup \beta_{A_4})$-partition of $[d,b]$. Clearly π_4 is both, a $(\beta_4 \cup \beta_{A_4}^{(1)})$-partition and a $(\beta_4 \cup \beta_{A_4}^{(2)})$ partition of $[d,b]$.

Let $\pi^{(k)} = \pi_k \cup \pi_3 \cup \pi_4$, $k = 1,2$. Then $\pi^{(1)}$ and $\pi^{(2)}$ are $(\beta \cup \beta_{A_k})$-partitions of $[a,b]$ (see (4)).

We have

$$|s(f,\pi^{(1)}) - s(f,\pi^{(2)})| < \epsilon\,, \quad \text{hence} \quad |s(f,\pi_1) - s(f,\pi_2)| < \epsilon\,.$$

By Lemma 7.2 it follows that f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[c,d]$. □

Lemma 7.4 (A quasi Saks-Henstock lemma)**.** *Let* $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ *be a local system* $\mathcal{S}_\infty^+$*-filtering on* $[a,b)$*, and let* $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ *be a local system* $\mathcal{S}_\infty^-$*-filtering on* $(a,b]$*. Let* $f : [a,b] \to \mathbb{R}$ *be* $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$*-integrable on* $[a,b]$*. Let* $F : [a,b] \to \mathbb{R}$*,* $F(a) = 0$*,* $F(x) = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}] \int_a^x f(t)\,dt$*,* $x \in (a,b]$*. For every* $\epsilon > 0$ *there exists* $\beta = \beta(\{X_i\},\{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$*, such that* *(i)* $|s(f,\pi)-S(F,\pi)|<\epsilon$, *whenever* π *is a* β*-partial partition of* $[a,b]$.

(ii) $\sum_{([x,y],t)\in\pi}|f(t)(y-x)-(F(y)-F(x))|<2\epsilon$ *whenever* π *is a* β*-partial partition of* $[a,b]$.

PROOF. Let $\epsilon>0$. For $\frac{\epsilon}{2}$ and $F(b)$ let $\beta=\beta(\{X_i\},\{\delta_i\})$ be given by Definition 7.1. Let π be a β-partial partition of $[a,b]$.

(i) Let (c_k,d_k), $k=1,2,\dots,n$ be the components of the open set $(a,b)\setminus(\cup_{(I,t)\in\pi}I)$. By Lemma 7.3 we have $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_{c_k}^{d_k}f(t)\,dt=F(d_k)-F(c_k)$. For each $k=1,2,\dots,n$ let $\beta^{(k)}=\beta^{(k)}(\{Y_j^{(k)}\},\{\delta_j^{(k)}\})\in\overline{\mathcal{B}}_{[c_k,d_k]}$ be given by Definition 7.1. Let $A^{(k)}=\mathrm{Is}(\{Y_j^{(k)}\})$, $k=1,2,\dots n$ and let $A_o=\cup_{k=1}^n A^{(k)}\cup\mathrm{Is}(\{X_i\})$. By Definition 7.1 it follows that for $\epsilon/(2n)$, $A^{(k)}$ and $\beta^{(k)}$ there exists $\beta_{A^{(k)}}=\beta_{A^{(k)}}(\sigma_x^{(k,1)},\sigma_x^{(k,2)})\in\mathcal{B}_{A^{(k)}}(\mathcal{S}_1;\mathcal{S}_2)$ such that $|s(f,\pi')-(F(d_k)-F(c_k))|<\epsilon/(2n)$ whenever π' is a β_k partition of $[c_k,d_k]$, where $\beta_k=\beta^{(k)}\cup\beta_{A^{(k)}}$. For $\epsilon/2$, A_o and β there exists $\beta_{A_o}=\beta_{A_o}(\sigma_x^{(0,1)},\sigma_x^{(0,2)})\in\mathcal{B}_{A_o}(\mathcal{S}_1;\mathcal{S}_2)$ such that $|s(f,\pi')-F(b)|<\epsilon/2$ whenever π' is a β_o-partition of $[a,b]$, where $\beta_o=\beta\cup\beta_{A_o}$. Let

$$\sigma_x^{(k,+)}=\sigma_x^{(0,1)}\cap\sigma_x^{(k,1)}\in\mathcal{S}_\infty^+(x),\text{ if }x\in A^{(k)}\cap[c_k,d_k);$$

$$\sigma_x^{(k,-)}=\sigma_x^{(0,2)}\cap\sigma_x^{(k,2)}\in\mathcal{S}_\infty^-(x),\text{ if }x\in A^{(k)}\cap(c_k,d_k].$$

Clearly $\{X_i\cap Y_j^{(k)}\}_{(i,j)}\in\overline{\mathcal{P}}_{[c_k,d_k]}$. Let

$$\delta_{i,j}^{(k)}:X_i\cap Y_j^{(k)}\to(0,+\infty),\ \delta_{i,j}(x)<\min\{\delta_i(x),\delta_j^{(k)}(x)\};$$

$$\alpha_k=\alpha_k(\{X_i\cap Y_j^{(k)}\},\delta_{i,j}^{(k)},\sigma_x^{(k,+)},\sigma_x^{(k,-)})\in\mathcal{A}(\overline{\mathcal{P}}_{[c_k,d_k]};\mathcal{S}_\infty^+;\mathcal{S}_\infty^-).$$

Let $\pi^{(k)}$ be an α_k-partition of $[c_k,d_k]$, $k=1,2,\dots,n$ (see Lemma 4.2). Then $\pi^{(k)}$ is also a β_k- partition and a β_o-partition of $[c_k,d_k]$. Then $\pi_o=\pi\cup(\cup_{k=1}^n\pi^{(k)})$ is a β_o- partition of $[a,b]$. Since $F(b)=S(F,\pi_o)$, it follows that $|s(f,\pi_o)-S(F,\pi_o)|<\epsilon/2$. We have $|s(f,\pi)-S(F,\pi)+\sum_{k=1}^n(F(d_k)-F(c_k)-s(f,\pi^{(k)}))|<\epsilon/2$, hence $|s(f,\pi)-S(F,\pi)|<\epsilon$.

(ii) Let $\pi_1=\{([x,y],t)\in\pi\ :\ f(t)(y-x)>F(y)-F(x)\}$ and $\pi_2=\{([x,y],t)\in\pi\ :\ f(t)(y-x)\le F(y)-F(x)\}$. Then $\pi=\pi_1\cup\pi_2$. By (i) we have $|s(f,\pi_1)-S(F,\pi_1)|<\epsilon$ and $|s(f,\pi_2)-S(F,\pi_2)|<\epsilon$. Thus $\sum_{([x,y],t)\in\pi}|f(t)(y-x)-(F(y)-F(x))|<\epsilon+\epsilon=2\epsilon$. □

Corollary 7.1. *Let* $\mathcal{S}_1=\{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ *be a local system* $\mathcal{S}_\infty^+$*-filtering on* $[a,b)$*, and let* $\mathcal{S}_2=\{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ *be a local system* $\mathcal{S}_\infty^-$*-filtering on* $(a,b]$*. Let* $f:[a,b]\to\mathbb{R}$ *be* $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$*-integrable on* $[a,b]$*. Let* $F:[a,b]\to\mathbb{R}$*,* $F(a)=0$*,*

$F(x) = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^x f(t)\,dt$, $x \in (a,b]$. *Then* $F \in N_{\mathcal{B}_o}$ *on* $[a,b]$*, hence* $F \in (N)$ *on* $[a,b]$.

Proof. Let $\epsilon > 0$ and $Z \subset [a,b]$, $|Z| = 0$. By Lemma 5.4.5 of [1], there exists $\delta : Z \to (0,+\infty)$ such that $|s(f;\pi)| < \epsilon$, whenever π is a McShane δ-fine partial partition (i.e., if $([x,y],t) \in \pi$ then $[x,y] \subset (t-\delta(t), t+\delta(t))$) of $[a,b]$, with all tags in Z. By Lemma 7.4, (i), there exists $\beta = \beta(\{X_i\}_i, \{\delta_i\}_i) \in \overline{\mathcal{B}}_{[a,b]}$, such that $|s(f,\pi) - S(F,\pi)| < \epsilon$, whenever π is a β-partial partition of $[a,b]$. Let $Z_i = Z \cap X_i$ and let $\delta_i^\star : Z_i \to (0,+\infty)$, $\delta_i^\star(x) = \min\{\delta(x), \delta_i(x)\}$. Let $\beta_o = \beta_o(\{Z_i\}, \{\delta_i^\star\}) \in \mathcal{B}_Z$ and let π_o be a β_o-partial partition of Z. Then π_o is also a β-partial partition of $[a,b]$. It follows that $|S(F,\pi_o)| \le |s(f,\pi_o) - S(F,\pi_o)| + |s(f,\pi_o)| < 2\epsilon$ (see Lemma 7.4). □

Lemma 7.5. *Let* $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ *be a local system* $\mathcal{S}_\infty^+$*-filtering on* $[a,b)$*, and let* $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ *be a local system* $\mathcal{S}_\infty^-$*-filtering on* $(a,b]$*. Let* $f : [a,b] \to \mathbb{R}$ *be* $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$*-integrable on* $[a,b]$*. Let* $F : [a,b] \to \mathbb{R}$*,* $F(a) = 0$*,* $F(x) = [\mathcal{S}\mathcal{S}_2\mathcal{R}]\int_a^x f(t)\,dt$*,* $x \in (a,b]$*. Then* $F'_{ap}(x) = f(x)$ *a.e. on* $[a,b]$*.*

Proof. Let $A = \{x \in (a,b)$: there exists $\alpha(x) > 0$ with the following property: for every $\eta(x) > 0$, with $(x-\eta(x), x+\eta(x)) \subset (a,b)$, and for every D_x with $\underline{d}^i(D_x,x) = 1$, there exists $y \in D_x \cap (x-\eta(x), x+\eta(x))$ such that $|F(y) - F(x) - f(x)(y-x)| > \alpha(x)(y-x)\}$. Let $A_n = \{x \in (a,b) : \alpha(x) \ge 1/n\}$. Then $A = \cup_{n=1}^\infty A_n$. By Lemma 7.4 (ii), for $\epsilon > 0$, there exists $\beta = \beta(\{X_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ such that $\sum_{([x,y],t)\in\pi} |F(y) - F(x) - f(t)(y-x)| < \epsilon$, whenever π is a β-partial partition of $[a,b]$. Let $A_{n,i} = \{x \in A_n \cap X_i : d(X_i,x) < 1\}$. By the Lebesgue Density Theorem (see for example [1, p. 10]) it follows that $|A_{n,i}| = 0$. Let $B_n = A_n \setminus (\cup_{i=1}^\infty A_{n,i})$. Clearly $|A_n| = |B_n|$. If $x \in B_n$ and $x \in X_i$ for some i then $d(X_i,x) = 1$ (indeed, if $d(X_i,x) \ne 1$ then $x \in A_{n,i}$, a contradiction). For $\delta_i(x) > \eta(x) > 0$ with $(x-\eta(x), x+\eta(x)) \subset (a,b)$, there exists $y \in X_i \cap (x-\eta(x), x+\eta(x))$ such that $|F(y) - F(x) - f(x)(y-x)| \ge (1/n)|y-x|$. If $y > x$ then $([x,y],x) \in \beta$ and if $y < x$ then $([y,x],x) \in \beta$. Let $\mathcal{A} = \{[x,y] : z \in B_n \cap \{x,y\}$ and $([x,y],z) \in \beta\}$. Then $\mathcal{A}$ is a Vitali cover of B_n, hence by Vitali's Covering Theorem (see for example [1, p. 11]) there exists π, a β-partial partition of $[a,b]$ such that $|B_n| \le \epsilon + \sum_{([x,y],z)\in\pi}(y-x) \le \epsilon + n \cdot \sum_{([x,y],z)\in\pi} |F(y) - F(x) - f(z)(y-x)| < \epsilon + n\epsilon$. Since ϵ is arbitrary, it follows that $|B_n| = 0$, hence $|A| = 0$. Therefore $F'_{ap}(x) = f(x)$ on $(a,b) \setminus A$. Thus $F'_{ap} = f$ *a.e.* on $[a,b]$. □

Lemma 7.6. *Let* $f : [a,b] \to \mathbb{R}$*,* $f \in [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$*-integrable on* $[a,b]$*, and let* $F(x) = [\mathcal{S}_1\mathcal{S}_2\mathcal{R}]\int_a^x f(t)\,dt$*,* $x \in [a,b]$*. Then* $F \in VBG$ *on* $[a,b]$*.*

Proof. For $\epsilon = 1$ let $\beta = \beta(\{X_i\},\{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by Definition 7.1. Clearly $\cup_{i=1}^{\infty} X_i = [a,b]$. Let

$$X_{ij} = \{x \in X_i \,:\, \delta_i(x) > (b-a)/j\},\ j = 1,2,\ldots;$$

$$X_{ijk} = X_{ij} \cap [a + (k-1)\tfrac{b-a}{j},\ a + k\tfrac{b-a}{j}],\ k = 1,2,\ldots,j;$$

$$X_{ijkm} = \{x \in X_{ijk} \,:\, |f(x)| < m\},\ m = 1,2,\ldots.$$

Clearly $[a,b] = \cup_{i,j,k,m} X_{ijkm}$. Let $\{[a_n,b_n]\}$, $n = 1,2,\ldots,p$ be a finite set of nonoverlapping intervals with endpoints in X_{ijkm}. Then $([a_n,b_n],a_n) \in \beta$. By Lemma 7.4, $\sum_{n=1}^{p} |f(a_n)(b_n-a_n) - (F(b_n)-F(a_n))| < 2$. It follows that $\sum_{n=1}^{p} |F(b_n)-F(a_n)| \le \sum_{n=1}^{p} |f(a_n)(b_n-a_n) - (F(b_n)-F(a_n))| + \sum_{n=1}^{p} |f(a_n)|(b_n-a_n) < 2 + m(b-a)$, hence $F \in VB$ on X_{ijkm}. Therefore F is also VBG on $[a,b]$. □

Remark 7.1. To prove Lemma 7.5 and Lemma 7.6 we have followed the technique used by Lee and Soedijono in Theorem 4.2 of [12], and our proof is based on a quasi Saks- Henstock type lemma, i.e., Lemma 7.4 (see Section 1). But we do not know if either of the two integrals, the AH integral and the $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integral, satisfy a Saks-Henstock type lemma.

8 The Ward Type $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ Integral

Definition 8.1. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$.

- We define the following classes of majorants: $[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b]) = \{M : [a,b] \to \mathbb{R} \,:\, M(a) = 0;\ M \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$ on $[a,b]$; there exists a $\beta = \beta(\{X_i\},\{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ such that $M(y) - M(x) \ge f(t)(y-x)$, whenever $([x,y],t) \in \beta$.

- We define the following class of minorants: $[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b]) = \{m : [a,b] \to \mathbb{R} \,:\, -m \in [\overline{\mathcal{S}_1\mathcal{S}_{(2)}\mathcal{W}}](-f;[a,b])\}$.

- If $[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b]) \ne \emptyset$ then we denote by $\overline{J}_f(x)$ (or simply $\overline{J}(x)$), $x \in [a,b]$ the lower bound of all $M(x)$, $M \in [\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])$.

- If $[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b]) \ne \emptyset$ then we denote by $\underline{J}_f(x)$ (or simply $\underline{J}(x)$), $x \in [a,b]$ the upper bound of all $m(x)$, $m \in [\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])$.

- If $[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b]) \times [\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b]) \neq \emptyset$ and $\overline{J}(b) = \underline{J}(b)$ then f is said to be $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a,b]$. In this case we write $\overline{J}(b) = \underline{J}(b) = [\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^b f(t)\,dt$.

Lemma 8.1. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ and $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be local systems such that $\mathcal{S}_1 \ll \mathcal{S}_\infty^+$ on $[a,b)$ and $\mathcal{S}_2 \ll \mathcal{S}_\infty^-$ on $(a,b]$. Let $F, f : [a,b] \to \mathbb{R}$ and let $A = \{a_1, a_2, a-3, \ldots\}$ be a countable subset of (a,b). If $F \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$ on $[a,b]$ then for $\epsilon > 0$, there exist $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$ and $H : [a,b] \to \mathbb{R}$ such that:*

(i) $H(a) = 0$; $H(b) < \epsilon$;

(ii) H *is increasing on* $[a,b]$;

(iii) $(F+H)(x)-(F+H)(t) \geq f(t)(x-t)$ *whenever* $x \in [t,b)\cap\sigma_t^{(1)}$, $t \in \{a\}\cup A$ *and* $(F+H)(t) - (F+H)(x) \geq f(t)(t-x)$ *whenever* $x \in (a,t] \cap \sigma_t^{(2)}$, $t \in \{b\} \cup A$.

(iv) $F + H \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$ *on* $[a,b]$.

Moreover, if $F \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$ then $|F(y) - F(x) - f(t)(y-x)| \leq H(y) - H(x)$, whenever$([x,y],t) \in \beta_A(\sigma_x^{1)}, \sigma_x^{(2)})$.

Proof. Let $\epsilon > 0$. Since $F \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$ on $[a,b]$ it follows that there exists $S_a^{(1)} \in \mathcal{S}_1(a)$ such that

$$F(x) - F(a) > -\frac{\epsilon}{2^3} \quad \text{whenever} \quad x \in S_a^{(1)}, \;\; x > a\,,$$

and for every a_i there exists $S_{a_i}^{(1)} \in \mathcal{S}_1(a_i)$ such that

$$F(x) - F(a_i) > -\frac{\epsilon}{2^{i+3}} \quad \text{whenever} \quad x \in S_{a_i}^{(1)}, \;\; x > a - i\,.$$

Let $H_1 : [a,b] \to \mathbb{R}$, $H_1(a) = 0$,

$$H_1(x) = \frac{\epsilon}{2^2} + \sum_{a_i < x} \frac{\epsilon}{2^{i+3}}\,.$$

Clearly H_1 is increasing and $H_1(b) = \epsilon/2$. Let $x > a$. Then $H_1(x) - H_1(a) \geq \epsilon/2^2$, so

$$(F + H_1)(x) - (F + H_1)(a) > \frac{\epsilon}{2^2} - \frac{\epsilon}{2^3} = \frac{\epsilon}{2^3} > |f(a)|(x-a)\,,$$

whenever

$$x \in S_a^{(1)} \cap \left(a - \frac{\epsilon}{(|f(a)|+1)\cdot 2^3}, a + \frac{\epsilon}{(|f(a)|+1)\cdot 2^3} \right) =: \sigma_a^{(1)} \quad \text{and } x > a.$$

Let $x > a_j$. Then $H_1(x) - H_1(a_j) \geq \epsilon/2^{j+2}$, so

$$(F+H_1)(x) - (F+H_1)(a_j) > \frac{\epsilon}{2^{j+2}} - \frac{\epsilon}{2^{j+3}} = \frac{\epsilon}{2^{j+3}} > |f(a_j)|(x-a_j)\,,$$

whenever

$$x \in S_{a_j}^{(1)} \cap \left(a_j - \frac{\epsilon}{(|f(a_j)|+1)\cdot 2^{j+3}}, a_j + \frac{\epsilon}{(|f(a_j)|+1)\cdot 2^{j+3}} \right) =: \sigma_{a_j}^{(1)}$$

and $x > a_j$. Similarly, there exists $S_b^{(2)} \in \mathcal{S}_2(b)$ such that

$$F(y) - F(b) < \frac{\epsilon}{2^3} \quad \text{whenever} \quad y \in S_b^{(2)}, \quad y < b\,,$$

and for every a_i there exists $S_{a_i}^{(2)} \in \mathcal{S}_2(a_i)$ such that

$$F(y) - F(a_i) < \frac{\epsilon}{2^{i+3}} \quad \text{whenever} \quad y \in S_{a_i}^{(2)}, \quad y < a_i\,.$$

Let $H_2 : [a,b] \to \mathbb{R}$, $H_2(a) = 0$,

$$H_2(x) = \sum_{a_i \leq x} \frac{\epsilon}{2^{i+2}}\,.$$

Clearly H_2 is increasing and $H_2(b) = \epsilon/2$. Let

$$\sigma_b^{(2)} := S_b^{(2)} \cap \left(b - \frac{\epsilon}{(|f(b)|+1)\cdot 2^3}, b + \frac{\epsilon}{(|f(b)|+1)\cdot 2^3} \right)$$

and

$$\sigma_{a_i}^{(2)} := S_{a_i}^{(2)} \cap \left(a_i - \frac{\epsilon}{(|f(a_i)|+1)\cdot 2^{i+3}}, a_i + \frac{\epsilon}{(|f(a_i)|+1)\cdot 2^{i+3}} \right).$$

Then $(F+H_2)(t) - (F+H_2)(x) \geq f(t)(t-x)$ whenever $x \in (a,t] \cap \sigma_t^{(2)}$, $t \in \{b\} \cup A$.

The function $H = H_1 + H_2$ satisfies the required properties.

If $F \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$ then, for example, $S_{a_i}^{(1)}$ may be chosen such that

$$|F(x) - F(a_i)| < \frac{\epsilon}{2^3} \quad \text{whenever} \quad x \in S_{a_i}^{(1)}, \quad x > a_i\,.$$

Then for $x \in \sigma_{a_i}^{(1)}$, $x > a_i$ it follows that

$$|F(x) - F(a_i) - f(a_i)(x - a_i)| < |F(x) - F(a_i)| + |f(a_i)|(x - a_i) <$$

$$< \frac{\epsilon}{2^3} + \frac{\epsilon}{2^3} = \frac{\epsilon}{2^2} \leq H_1(x) - H_1(a_i) \leq H(x) - H(a_i)\,.$$

□

Lemma 8.2. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$. If $\overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b]) \times \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b]) \neq \emptyset$ and $(M,m) \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b]) \times \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b])$ then we have*

(i) $M - m$ is positive and increasing on $[a,b]$, hence $M(b) \geq m(b)$;

(ii) $M - \underline{J}$ is positive and increasing on $[a,b]$, hence $M(b) \geq \underline{J}(b)$;

(iii) $\overline{J} - m$ is positive and increasing on $[a,b]$, hence $\overline{J}(b) \geq m(b)$;

(iv) $\overline{J} - \underline{J}$ is positive and increasing on $[a,b]$, hence $\overline{J}(b) \geq \underline{J}(b)$.

Proof. (i) Let $a \leq c < d \leq b$. For M there exists a $\beta^{(1)} = \beta^{(1)}(\{X_i^{(1)}\},\{\delta_i^{(1)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ such that $M(y) - M(x) \geq f(t)(y-x)$, whenever $([x,y],t) \in \beta^{(1)}$. For m there exists a $\beta^{(2)} = \beta^{(2)}(\{X_i^{(2)}\},\{\delta_i^{(2)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ such that $m(y) - m(x) \leq f(t)(y-x)$, whenever $([x,y],t) \in \beta^{(2)}$.

Let $A = \mathrm{Is}(\{X_i^{(1)} \cap X_j^{(2)} \cap [c,d]\}_{i,j})$ and let $\epsilon > 0$. By Lemma 8.1, there exist $\beta_A^{(k)} = \beta_A^{(k)}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$ and $H_k : [c,d] \to \mathbb{R}$, $k = 1,2$, $H_k(c) = 0$, $H_k(d) \leq \epsilon$, H_k increasing, such that $M(y) - M(x) + H_1(y) - H_1(x) \geq f(t)(y-x)$, whenever $([x,y],t) \in \beta_A^{(1)}$ and $m(y) - m(x) - (H_2(y) - H_2(x)) \leq f(t)(y-x)$, whenever $([x,y],t) \in \beta_A^{(2)}$. Let

$\delta_{i,j} : X_i^{(1)} \cap X_j^{(2)} \cap [c,d] \to (0,+\infty)$, $\delta_{i,j}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$.

$\beta = \beta(\{X_i^{(1)} \cap X_j^{(2)} \cap [c,d]\},\{\delta_{i,j}\}) \in \overline{\mathcal{B}}_{[c,d]}$.

$\sigma_x^{(1)} = \sigma_x^{(1,1)} \cap \sigma_x^{(2,1)}$, for $x \in [c,d) \cap A$.

$\sigma_x^{(2)} = \sigma_x^{(1,2)} \cap \sigma_x^{(2,2)}$, for $x \in (c,d] \cap A$.

$\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_\infty^+, \mathcal{S}_\infty^-)$.

By Lemma 4.2 there exists π, a $\beta\cup\beta_A$-partition of $[c,d]$. If $([x,y],t)\in\beta\cup\beta_A$ then $([x,y],t)\in\beta^{(k)}\cup\beta_A^{(k)}$, $k=1,2$. It follows that $(M-m)(d)-(M-m)(c)+2\epsilon\geq(H_1+M)(d)-(H_1+M)(c)-((m-H_2)(d)-(m-H_2)(c))=\sum_{([x,y],t)\in\pi}((H_1+M)(y)-(H_1+M)(x))-\sum_{([x,y],t)\in\pi}((m-H_2)(y)-(m-H_2)(x))\geq\sum_{([x,y],t)\in\pi}(f(t)(y-x)-f(t)(y-x))=0$. If $\epsilon\to 0$ then we obtain that $M-m$ is increasing on $[a,b]$. Since $M(a)=m(a)=0$, $M-m$ is positive on $[a,b]$. Clearly $M(b)\geq m(b)$.

(ii) By (i) we have $M(d)-M(c)\geq m(d)-m(c)\geq m(d)-\underline{J}(c)$, hence $M(d)-M(c)\geq\underline{J}(d)-\underline{J}(c)$ whenever $a\leq c<d\leq b$. It follows that $M(d)-\underline{J}(d)\geq M(c)-\underline{J}(c)$. Thus $M-\underline{J}$ is increasing and positive (since $M(a)=\underline{J}(a)=0$) on $[a,b]$. Clearly $M(b)\geq\underline{J}(b)$.

(iii) We have $M(d)-\overline{J}(c)\geq M(d)-M(c)\geq m(d)-m(c)$ (see (i)), hence $\overline{J}(d)-\overline{J}(c)\geq m(d)-m(c)$ whenever $a\leq c<d\leq b$. Therefore $\overline{J}-m$ is increasing and positive (since $\overline{J}(a)=m(a)=0$). Clearly $\overline{J}(b)\geq m(b)$.

(iv) We have $M(d)-\overline{J}(c)\geq M(d)-M(c)\geq\underline{J}(d)-\underline{J}(c)$ (see (ii)), hence $\overline{J}(d)-\overline{J}(c)\geq\underline{J}(d)-\underline{J}(c)$ whenever $a\leq c<d\leq b$. Therefore $\overline{J}-\underline{J}$ is increasing and positive (since $\overline{J}(a)=\underline{J}(a)=0$). Clearly $\overline{J}(b)\geq\underline{J}(b)$. □

Lemma 8.3. *Let $\mathcal{S}_1=\{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2=\{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f:[a,b]\to\mathbb{R}$. The following assertions are equivalent:*

(i) f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ integrable on $[a,b]$.

(ii) $[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])\neq\emptyset$ and for every $\epsilon>0$ there exists $(M,m)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])]\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])$ such that $M(b)-m(b)<\epsilon$.

PROOF. (i) $\Rightarrow$ (ii) For $\epsilon>0$ there exists a pair $(M,m)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])$ such that $M(b)-\epsilon/2<\overline{J}(b)=\underline{J}(b)<m(b)+\epsilon/2$ (see the definitions of $\overline{J}(b)$ and $\underline{J}(b)$), hence $M(b)-m(b)<\epsilon$.

(ii) $\Rightarrow$ (i) By Lemma 8.2, (iv) we have $0\leq\overline{J}(b)-\underline{J}(b)\leq M(b)-m(b)<\epsilon$. Since ϵ is arbitrary, it follows that $\overline{J}(b)=\underline{J}(b)$. □

Lemma 8.4. *Let $\mathcal{S}_1=\{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2=\{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f:[a,b]\to\mathbb{R}$ and $c\in(a,b)$. If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ integrable on $[a,b]$ then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ integrable on $[a,c]$ and on $[c,b]$, and*

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^b f(t)\,dt=[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^c f(t)\,dt+[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_c^b f(t)\,dt\,. \qquad (5)$$

PROOF. By Lemma 8.3 it follows that $[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])\neq\emptyset$ and for every $\epsilon>0$ there exists $(M,m)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,b])$ such that $M(b)-m(b)<\epsilon$.

Let $M_1=M_{/[a,c]}$ and $m_1=m_{/[a,c]}$. Let $M_2=M_{/[c,b]}-M(c)$ and $m_2=m_{/[c,b]}-m(c)$. Then

$$(M_1,m_1)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,c])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,c])\neq\emptyset\,. \tag{6}$$

and

$$(M_2,m_2)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[c,b])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[c,b])\neq\emptyset\,. \tag{7}$$

We prove for example (6) ((7) follows similarly). Clearly $M\in(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}_i$ on $[a,c]$. For M there exists $\beta=\beta(\{X_i\},\{\delta_i\})\in\overline{\mathcal{B}}_{[a,b]}$ such that $M(y)-M(x)\geq f(t)(y-x)$, whenever $([x,y],t)\in\beta$. Then $\beta_o=\beta_o(\{X_i\cap[a,c]\},\{(\delta_i)_{/X_i\cap[a,c]}\})\in\mathcal{B}_{[a,c]}$ and $\beta_o\subset\beta$. Then $M_1(y)-M_1(x)\geq f(t)(y-x)$, whenever $([x,y],t)\in\beta_o$. Therefore $M_1\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,c])$. Similarly, $m_1\in[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,c])$. Thus we obtain that (6) is true.

By Lemma 8.2, (i) we have that $M_1(c)-m_1(c)<\epsilon$ and $M_2(c)-m_2(c)<\epsilon$. Therefore, by Lemma 8.3, it follows that f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a,c]$ and $[c,b]$.

We also have $m_1(c)=m(c)<[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^c f(t)\,dt<M(c)=M_1(c)$ and $m_2(b)=m(b)-m(c)\leq[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_c^b f(t)\,dt\leq M(b)-M(c)=M_2(b)$. It follows that $m(b)\leq[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^c f(t)\,dt+[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_c^b f(t)\,dt\leq M(b)$. But $m(b)\leq[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_c^b f(t)\,dt\leq M(b)$ and $M(b)-m(b)<\epsilon$. Since ϵ is arbitrary we obtain (5). □

Lemma 8.5. *Let $\mathcal{S}_1=\{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2=\{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f:[a,b]\to\mathbb{R}$ and $c\in(a,b)$. If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a,c]$ and on $[c,b]$, then f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a,b]$ and*

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^b f(t)\,dt=[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_a^c f(t)\,dt+[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]\int_c^b f(t)\,dt \tag{8}$$

PROOF. For $\epsilon>0$, let $(M_1,m_1)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,c])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[a,c])\neq\emptyset$ with $M_1(c)-m_1(c)<\epsilon$, and $(M_2,m_2)\in[\overline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[c,b])\times[\underline{\mathcal{S}_1\mathcal{S}_2\mathcal{W}}](f;[c,b])\neq\emptyset$ with $M_2(c)-m_2(c)<\epsilon$ (see Lemma 8.3). Let

$$M(x)=\begin{cases} M_1(x) & ,\quad x\in[a,c]\\ M_1(c)+M_2(x) & ,\quad x\in[c,b]\end{cases}$$

and

$$m(x) = \begin{cases} m_1(x) & , \quad x \in [a, c] \\ m_1(c) + m_2(x) & , \quad x \in [c, b]\,. \end{cases}$$

For M_1 let $\beta_1 = \beta_1(\{X_i^{(1)}\}, \{\delta_i^{(1)}\}) \in \overline{\mathcal{B}}_{[a,c]}$ be given by Definition 8.1, and for M_2 let $\beta_2 = \beta_2(\{X_i^{(2)}\}, \{\delta_i^{(2)}\}) \in \overline{\mathcal{B}}_{[c,b]}$ be given by the same definition. Then $\{X_i^{(1)}\}_i \cup \{X_i^{(2)}\}_i \in \overline{\mathcal{P}}_{[a,b]}$. Let $\beta = \beta_1 \cup \beta_2$. If $([x, y], t) \in \beta$ then either $([x, y], t) \in \beta_1$ or $([x, y], t) \in \beta_2$. In both cases we have $M(y) - M(x) \geq f(t)(y - x)$. By Lemma 2.2, (iii), $M \in (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}_i$ on $[a, b]$. Therefore $M \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f; [a, b])$. Similarly we can show that $m \in \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f; [a, b])$. Since $M(b) - m(b) < 2\epsilon$, by Lemma 8.3, it follows that $f \in [\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ on $[a, b]$. We also have $m_1(c) = m(c) \leq [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^c f(t)\,dt \leq M(c) = M_1(c)$ and $m_2(b) = m(b) - m(c) \leq [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_c^b f(t)\,dt \leq M_2(b) = M(b) - M(c)$. It follows that $m(b) \leq [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^c f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_c^b f(t)\,dt \leq M(b)$. But $m(b) \leq [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f(t)\,dt \leq M(b)$ and $M(b) - m(b) < 2\epsilon$. Since ϵ is arbitrary we obtain (8). □

Lemma 8.6. *Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a, b]$, and let $\alpha_1, \alpha_2 \in \mathbb{R}$. If $\mathcal{S}_1$ is filtering on $[a, b)$ and $\mathcal{S}_2$ is filtering on $(a, b]$, then $\alpha_1 f_1 + \alpha_2 f_2$ is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$- integrable on $[a, b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b (\alpha_1 f_1 + \alpha_2 f_2)(t)\,dt = \alpha_1 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f_1(t)\,dt + \alpha_2 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f_2(t)\,dt$.*

PROOF. Let $(M, m) \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f_1; [a, b]) \times \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f_1; [a, b]) \neq \emptyset$. If $\alpha > 0$ then $(\alpha M, \alpha m) \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(\alpha f_1; [a, b]) \times \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(\alpha f_1; [a, b]) \neq \emptyset$. Hence

$$\alpha f_1 \text{ is } [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] - \text{integrable and}$$

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b \alpha f_1(t)\,dt = \alpha \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f_1(t)\,dt \tag{9}$$

If $\alpha < 0$ then $(\alpha m, \alpha M) \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(\alpha f_1; [a, b]) \times \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(\alpha f_1; [a, b]) \neq \emptyset$ and (9) is valid.
It remains to prove that the lemma is true for $\alpha_1 = \beta_1 = 1$.

Let $M_k \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f_k; [a, b]) \neq \emptyset$, $k = 1, 2$

For M_k let $\beta^{(k)} = \beta^{(k)}(\{X_i^{(k)}\}, \{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by Definition 8.1, $k = 1, 2$.

Let $X_{ij} = X_i^{(1)} \cap X_j^{(2)}$. Then $\{X_{ij}\}_{i,j} \in \overline{\mathcal{P}}_{[a,b]}$.

Let $\delta_{ij} : X_{ij} \to (0, +\infty)$, $\delta_{ij}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$.

Let $\beta = \beta(\{X_{ij}\}, \{\delta_{ij}\}) \in \overline{\mathcal{B}}_{[a,b]}$.

Let $([x, y], t) \in \beta$. Clearly $([x, y], t)$ also belongs to $\beta^{(k)}$, $k = 1, 2$, so $M_k(y) - M_k(x) \geq f_k(t)(y - x)$.

It follows that $(M_1 + M_2)(y) - (M_1 + M_2)(x) \geq (f_1(t) + f_2(t))(y - x)$. Since $\mathcal{S}_1$ is filtering on $[a, b)$ and $\mathcal{S}_2$ is filtering on $(a, b]$, it follows that $M_1 + M_2 \in \underline{(\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}_i}$ on $[a, b]$. Hence $M_1 + M_2 \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f_1 + f_2; [a, b])$ and $\overline{J}_{f_1+f_2}(b) \leq \overline{J}_{f_1}(b) + \overline{J}_{f_2}(b)$. Similarly we obtain that $\underline{J}_{f_1}(b) + \underline{J}_{f_2}(b) \leq \underline{J}_{f_1+f_2}(b)$. By Lemma 8.2, (iv) we obtain that $f_1 + f_2 \in [\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b (f_1 + f_2)(t)\, dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f_1(t)\, dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f_2(t)\, dt$. □

9 The Variational Type $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-Integral

Definition 9.1. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a, b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a, b]$. Let $f : [a, b] \to \mathbb{R}$. f is said to be $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a, b]$, if there exists $H : [a, b] \to \mathbb{R}$, $H \in (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}$, with the following property: for every $\epsilon > 0$ there exist $\beta = \beta(\{X_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ and $G : [a, b] \to \mathbb{R}$, such that $G(a) = 0$, $G(b) \leq \epsilon$, G is increasing on $[a, b]$, and $|H(y) - H(x) - f(t)(y - x)| < G(y) - G(x)$, whenever $([x, y], t) \in \beta$. H is called the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ indefinite integral of f on $[a, b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_a^b f(t)\, dt = H(b) - H(a)$.

Lemma 9.1. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^+$-filtering on $[a, b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_\infty^-$-filtering on $(a, b]$. Let $f : [a, b] \to \mathbb{R}$. If $H_1, H_2 : [a, b] \to \mathbb{R}$ are $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ indefinite integrals of f on $[a, b]$ then $H_1(b) - H_1(a) = H_2(b) - H_2(a)$. Therefore the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ integral is well defined.*

Proof. Let $H_k : [a, b] \to \mathbb{R}$, $H_k \in (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}$, $k = 1, 2$, satisfying the following property: for $\epsilon > 0$ there exist $\beta^{(k)} = \beta^{(k)}(\{X_i^{(k)}\}, \{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ and $G^{(k)} : [a, b] \to \mathbb{R}$ such that $G^{(k)}(a) = 0$, $G^{(k)}(b) \leq \epsilon$, $G^{(k)}$ is increasing on $[a, b]$ and $|H^{(k)}(y) - H^{(k)}(x) - f(t)(y - x)| < G^{(k)}(y) - G^{(k)}(x)$, whenever $([x, y], t) \in \beta^{(k)}$, $k = 1, 2$. Let $X_{m,n} = X_m^{(1)} \cap X_n^{(2)}$. Then $\{X_{m,n}\}_{m,n} \in \overline{\mathcal{P}}_{[a,b]}$. Let $A = \mathrm{Is}(\{X_{m,n}\})$. By Lemma 8.1, there exist $\beta_A^{(k)} = \beta_A^{(k)}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_A(\mathcal{S}_1; \mathcal{S}_2)$ and $h^{(k)} : [a, b] \to \mathbb{R}$, $h^{(k)}(a) = 0$, $h^{(k)}(b) \leq \epsilon$, $h^{(k)}$ increasing, such that $|H^{(k)}(y) - H^{(k)}(x) - f(t)(y - x)| \leq h^{(k)}(y) - h^{(k)}(x)$, whenever $([x, y], t) \in \beta^{(k)}$, $k = 1, 2$. Let $\sigma_x^+ = \sigma_x^{(1,1)} \cap \sigma_x^{(2,1)}$ and $\sigma_x^- = \sigma_x^{(1,2)} \cap \sigma_x^{(2,2)}$. Let $\delta_{m,n} : X_{m,n} \to (0, +\infty)$, $\delta_{m,n}(x) = \min\{\delta_m^{(1)}(x), \delta_n^{(2)}(x)\}$. Let $\beta =$

$\beta(\{X_{m,n}\},\{\delta_{m,n}\}) \in \overline{\mathcal{B}}_{[a,b]}$ and $\beta_A = \beta_A(\sigma_x^+,\sigma_x^-) \in \mathcal{B}_A(\mathcal{S}_\infty^+,\mathcal{S}_\infty^-)$. Then, by Lemma 4.2 there exists π, a $\beta\cup\beta_A$-partition of $[a,b]$. But π is also a $\beta^{(1)}\cup\beta_A^{(1)}$- and a $(\beta^{(2)}\cup\beta_A^{(2)})$-partition of $[a,b]$. Therefore $|(H_1-H_2)(b)-(H_1-H_2)(a)| = |\sum_{([x,y],t)\in\pi}(H_1-H_2)(y)-(H_1-H_2)(x)| = |\sum_{([x,y],t)\in\pi}(H_1(y)-H_1(x)-f(t)(y-x)-(H_2(y)-H_2(x)-f(t)(y-x))| \le \sum_{([x,y],t)\in\pi}|H_1(y)-H_1(x)-f(t)(y-x)|+\sum_{([x,y],t)\in\pi}|H_2(y)-H_2(x)-f(t)(y-x)| \le \sum_{([x,y],t)\in\pi\cap\beta}|G^{(1)}(y)-G^{(1)}(x)|+\sum_{([x,y],t)\in\pi\cap\beta}|G^{(2)}(y)-G^{2)}(x)|+\sum_{([x,y],t)\in\pi\cap\beta_A^{(1)}}|H_1(y)-H_1(x)-f(t)(y-x)|+\sum_{([x,y],t)\in\pi\cap\beta_A^{(2)}}|H_2(y)-H_2(x)-f(t)(y-x)| \le G^{(1)}(b)+G^{(2)}(b)+h^{(1)}(b)+h^{(2)}(b) \le 4\epsilon$. Since ϵ is arbitrary, it follows that $H_1(b)-H_1(a) = H_2(b)-H_2(a)$. □

Lemma 9.2. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f : [a,b]\to\mathbb{R}$ and $a<c<b$. If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$ then f is also $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,c]$ and on $[c,b]$, and*

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^c f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_c^b f(t)\,dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f(t)\,dt \qquad (10)$$

PROOF. Since f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ integrable on $[a,b]$, there exists a function $H : [a,b]\to\mathbb{R}$, $H\in(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ satisfying the following property: for every $\epsilon>0$, there exist $\beta=\beta(\{X_i\},\{\delta_i\})\in\overline{\mathcal{B}}_{[a,b]}$ and $G:[a,b]\to\mathbb{R}$, such that $G(a)=0$, $G(b)\le\epsilon$, G is increasing on $[a,b]$, and $|H(y)-H(x)-f(t)(y-x)|<G(y)-G(x)$, whenever $([x,y],t)\in\beta$.
We define

$X_i^{(1)} = X_i\cap[a,c]$ and $X_i^{(2)} = X_i\cap[c,b]$; Clearly $\{X_i^{(1)}\}_i\in\overline{\mathcal{P}}_{[a,c]}$ and $\{X_i^{(2)}\}_i\in\overline{\mathcal{P}}_{[c,b]}$;

$\beta^{(1)} = \beta^{(1)}(\{X_i^{(1)}\},\{\delta_i\})\in\overline{\mathcal{B}}_{[a,c]}$;

$\beta^{(2)} = \beta^{(2)}(\{X_i^{(2)}\},\{\delta_i\})\in\overline{\mathcal{B}}_{[c,b]}$;

$H_1, G_1 : [a,c]\to\mathbb{R}$, $H_1(x)=H(x)$, $G_1(x)=G(x)$.

$H_2, G_2 : [c,b]\to\mathbb{R}$, $H_2(x)=H(x)$, $G_2(x)=G(x)-G(c)$.

By Lemma 2.2, (iii), $H_1,H_2\in(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,c]$ respectively $[c,b]$. If $([x,y],t)\in\beta^{(1)}$ then $([x,y],t)\in\beta$, hence f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,c]$, and we have $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^c f(t)\,dt = H(c)-H(a)$. Similarly it follows that f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[c,b]$, and $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_c^b f(t)\,dt = H(b)-H(c)$.
Since $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f(t)\,dt = H(b)-H(a)$, we obtain (10). □

Lemma 9.3. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}^+_\infty$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}^-_\infty$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$ and $a < c < b$. If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,c]$ and $[c,b]$ then f is also $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$ and*

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^c f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_c^b f(t)\,dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f(t)\,dt$$

Proof. Let $I_1 = [a,c]$ and $I_2 = [c,b]$. Let $H_k : I_k \to \mathbb{R}$, $H_k \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$, such that for every $\epsilon > 0$, there exist $\beta_k = \beta_k(\{X_i^{(k)}\},\{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{I_k}$, $k = 1,2$, and $G_k : I_k \to (0,+\infty)$ such that G_k is increasing, $G_1(a) = 0$, $G_1(c) < \epsilon$, $G_2(c) = 0$, $G_2(b) < \epsilon$, and $|f(t)(y-x) - (H_k(y) - H_k(x))| < G_k(y) - G_k(x)$, whenever $([x,y],t) \in \beta_k$, $k = 1,2$.
Let $H : [a,b] \to \mathbb{R}$,

$$H(x) = \begin{cases} H_1(x) & , \quad if \quad x \in [a,c] \\ H_1(c) + H_2(x) - H_2(c) & , \quad if \quad x \in [c,b]\,. \end{cases}$$

Let $G : [a,b] \to (0,+\infty)$,

$$G(x) = \begin{cases} G_1(x) & , \quad if \quad x \in [a,c] \\ G_1(c) + G_2(x) & , \quad if \quad x \in [c,b]\,. \end{cases}$$

Let $\beta = \beta_1 \cup \beta_2$. If $([x,y],t) \in \beta$ then either $([x,y],t) \in \beta_1$ or $([x,y],t) \in \beta_2$, and clearly in both cases we have $|f(t)(y-x) - (H(y) - H(x))| < G(y) - G(x)$. Therefore f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f(t)\,dt = H(b) - H(a) = H(b) - H(c) + H(c) - H(a) = H_2(b) - H_2(c) + H_1(c) - H_1(a) = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_c^b f(t)\,dt + [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^c f(t)\,dt$. □

Lemma 9.4. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$, and let $\alpha_1, \alpha_2 \in \mathbb{R}$. If $\mathcal{S}_1$ is filtering on $[a,b)$ and $\mathcal{S}_2$ is filtering on $(a,b]$, then $\alpha_1 f_1 + \alpha_2 f_2$ is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$- integrable on $[a,b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b(\alpha_1 f_1 + \alpha_2 f_2)(t)\,dt = \alpha_1 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f_1(t)\,dt + \alpha_2 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f_2(t)\,dt\,.$*

Proof. Let $\alpha_k \neq 0$.

For f_k let $H_k : [a,b] \to \mathbb{R}$, $k = 1,2$, $H_k \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$, with the following property: for every $\epsilon > 0$ there exist $\beta^{(k)} = \beta^{(k)}(\{X_i^{(k)}\},\{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$, and $G_k : [a,b] \to \mathbb{R}$, such that $G_1(a) = 0$, $G_1(b) \le \frac{\epsilon}{2|\alpha_2|}$, $G_2(a) = 0$, $G_2(b) < \frac{\epsilon}{2|\alpha_1|}$, G_k is increasing on $[a,b]$, and $|H_k(y) - H_k(x) - f(t)(y - x)| < G_k(y) - G_k(x)$, whenever $([x,y],t) \in \beta^{(k)}$.

Let $X_{ij} = X_i^{(1)} \cap X_j^{(2)}$. Then $\{X_{ij}\}_{i,j} \in \overline{\mathcal{P}}_{[a,b]}$.

Let $\delta_{ij} : X_{ij} \to (0, +\infty)$, $\delta_{ij}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$.

Let $\beta = \beta(\{X_{ij}\}, \{\delta_{ij}\}) \in \overline{\mathcal{B}}_{[a,b]}$.

Let $G(x) = |\alpha_1|G_1(x) + |\alpha_2|G_2 x)$. Clearly $G(a) = 0$, $G(b) < \epsilon$ and G is increasing on $[a, b]$.

Let $([x, y], t) \in \beta$. Clearly $([x, y], t)$ also belongs to $\beta^{(k)}$, $k = 1, 2$.

Then we have $|(\alpha_1 H_1 + \alpha_2 H_2)(y) - (\alpha_1 H_1 + \alpha_2 H_2)(x) - (\alpha_1 f_1 + \alpha_2 f_2)(t)(y - x)| \leq |\alpha_1| \cdot |H_1(y) - H_1(x) - f_1(t)(y-x)| + |\alpha_2| \cdot |H_2(y) - H_2(x) - f_2(t)(y-x)| < |\alpha_1|(G_1(y) - G_1(x)) + |\alpha_2|(G_2(y) - G_2(x)) = G(y) - G(x)$. But $\alpha_1 H_1 + \alpha_2 H_2 \in (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}$ on $[a, b]$. It follows that $\alpha_1 H_1 + \alpha_2 H_2$ is the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-indefinite integral of $\alpha_1 f_1 + \alpha_2 f_2$ on $[a, b]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_a^b (\alpha_1 f_1 + \alpha_2 f_2)(t)\, dt = \alpha_1 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_a^b f_1(t)\, dt + \alpha_2 \cdot [\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_a^b f_2(t)\, dt$. □

10 The Relations Between the $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-, the $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$- and the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-Integrals

Definition 10.1. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^+$-filtering on $[a, b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^-$-filtering on $(a, b]$. A function $f : [a, b] \to \mathbb{R}$ is said to be $bi[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a, b]$ if there exists $H : [a, b] \to \mathbb{R}$, $H \in (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}_i \cap (\mathcal{S}_1; \mathcal{S}_2)\mathcal{C}_d$, satisfying the following property: for every $\epsilon > 0$ there exists $\beta^{(k)} = \beta^{(k)}(\{X_i^{(k)}\}, \{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$, $k = 1, 2$, and $G_k : [a, b] \to [0, +\infty)$, with $G_k(a) = 0$, $G_k(b) \leq \epsilon$, G_k increasing on $[a, b]$, $k = 1, 2$ such that $H(y) - H(x) - f(t)(y - x) < G_1(y) - G_1(x)$, whenever $([x, y], t) \in \beta^{(1)}$ and $-H(y) + H(x) + f(t)(y - x) < G_2(y) - G_2(x)$ whenever $([x, y], t) \in \beta^{(2)}$. H is called the $bi[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-indefinite integral of f on $[a, b]$ and we write $bi[\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_a^b f(t)\, dt = H(b) - H(a)$.

Lemma 10.1. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^+$-filtering on $[a, b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^-$-filtering on $(a, b]$. Let $f : [a, b] \to \mathbb{R}$. Then we have:*

(i) If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$- integrable on $[a, b]$ then f is also $bi[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a, b]$, and the two integrals are equal. Moreover, if $\mathcal{S}_1$ is filtering on $[a, b)$ and $\mathcal{S}_2$ is filtering on $(a, b]$ then the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integral and the $bi[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integral are equivalent.

(ii) The following assertions are equivalent:

a) f is bi$[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$;

b) f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a,b]$,

and the two integrals are equal.

(iii) If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$ then f is also $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$-integrable on $[a,b]$ and the two integrals are equal.

PROOF. (i) The first part is obvious. We show the second part. For $\epsilon > 0$ let $\beta_k = \beta_k(\{X_i^{(k)}\}, \{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by Definition 10.1.

Let $X_{ij} = X_i^{(1)} \cap X_j^{(2)}$; then $\{X_{ij}\}_{i,j} \in \overline{\mathcal{P}}_{[a,b]}$.

Let $\delta_{ij} : X_{ij} \to (0,+\infty)$, $\delta_{ij}(x) = \min\{\delta_i^{(1)}(x), \delta_j^{(2)}(x)\}$.

Let $\beta = \beta(\{X_{ij}\}, \{\delta_{ij}\}) \in \overline{\mathcal{B}}_{[a,b]}$.

Let A be a countable subset of $[a,b]$ that contains $\mathrm{Is}(\{X_{ij}\})$; then $\mathrm{Is}(\{X_{ij}\}) \supseteq \mathrm{Is}(\{X_i^{(1)}\}) \cup \mathrm{Is}(\{X_j^{(2)}\})$.

Let $\beta_A^{(k)} = \beta_A^{(k)}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$ and $G_k : [a,b] \to (0,+\infty)$ be given by Definition 10.1, $k = 1,2$.

Let $\sigma_x^{(1)} = \sigma_x^{(1,1)} \cap \sigma_x^{(2,1)} \in \mathcal{S}_1(x)$.

Let $\sigma_x^{(2)} = \sigma_x^{(1,2)} \cap \sigma_x^{(2,2)} \in \mathcal{S}_2(x)$.

Let $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$.

Let $([x,y],t) \in \beta \cup \beta_A$. Then $([x,y],t) \in \beta^{(k)} \cup \beta_A^{(k)}$, $k = 1,2$. It follows that $H(y) - H(x) - f(t)(y-x) < G_1(y) - G_1(a)$ and $-H(y) + H(x) + f(t)(y-x) < G_2(y) - G_2(a)$. Let $G = G_1 + G_2$. Then $|H(y) - H(x) - f(t)(y-x)| < G(y) - G(x)$.

(ii) a) $\Rightarrow$ b) Let $\epsilon > 0$. By Definition 10.1, there exist $H : [a,b] \to \mathbb{R}$ and $\beta^{(k)} = \beta^{(k)}(\{X_i^{(k)}\}, \{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$ with the following property: for every countable subset $A^{(k)}$ of $[a,b]$ that contains $\mathrm{Is}(\{X_i^{(k)}\})$, $k = 1,2$ there exist $\beta_{A^{(k)}} = \beta_{A^{(k)}}(\sigma_x^{(k,1)}, \sigma_x^{(k,2)}) \in \mathcal{B}_{A^{(k)}}(\mathcal{S}_1;\mathcal{S}_2)$, $k = 1,2$ and $G_k : [a,b] \to [0,+\infty)$, with $G_k(a) = 0$, $G_k(b) < \epsilon$, G_k increasing on $[a,b]$, $k = 1,2$, such that $H(y) - H(x) - f(t)(y-x) < G_1(y) - G_1(x)$, whenever $([x,y],t) \in \beta^{(1)} \cup \beta_{A^{(1)}}$ and $-H(y) + H(x) + f(t)(y-x) < G_2(y) - G_2(x)$, whenever $([x,y],t) \in \beta^{(2)} \cup \beta_{A^{(2)}}$. Let $M = H + G_2$ and $m = H - G_1$. Then $(M,m) \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b]) \times$

$\underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b])$. It follows that $\overline{J}(b) \le H(b)$ and $\underline{J}(b) \ge H(b)$. By Lemma 8.2, (iv), we obtain that $\overline{J}(b) \ge \underline{J}(b)$, hence $H(b) = [\mathcal{S}_1\mathcal{S}_2\mathcal{W}] \int_a^b f(t)\,dt$.

b) $\Rightarrow$ a) Since f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$- integrable on $[a,b]$, we have $\overline{J}(b) = \underline{J}(b)$ (see Definition 8.1). By Lemma 8.2, (iv) it follows that $\overline{J} - \underline{J} = 0$ on $[a,b]$. Let $H(x) = \overline{J}(x) = \underline{J}(x)$, $x \in [a,b]$. Clearly $H(a) = 0$. For $\epsilon > 0$ let $(M,m) \in \overline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b]) \times \underline{[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]}(f;[a,b]) \ne \emptyset$ such that $H(b) - \epsilon/2 < m(b)$ and $M(b) < H(b) + \epsilon/2$. By Definition 8.1, there exists a $\beta^{(k)} = \eta^{(k)}(\{X_i^{(k)}\},\{\delta_i^{(k)}\}) \in \overline{\mathcal{B}}_{[a,b]}$, $k = 1,2$ with the following property: for every countable subset $A^{(k)}$ of $[a,b]$ that contains $\text{Is}(\{X_i^{(k)}\})$, $k = 1,2$, there is a $\beta_{A^{(k)}} = \beta_{A^{(k)}}(\sigma_x^{(k,1)},\sigma_x^{(k,2)}) \in \mathcal{B}_{A^{(k)}}(\mathcal{S}_1;\mathcal{S}_2)$, $k = 1,2$, such that $M(y)-M(x) \ge f(t)(y-x)$ whenever $([x,y],t) \in \beta^{(1)} \cup \beta_{A^{(1)}}$ and $m(y) - m(x) \le f(t)(y-x)$ whenever $([x,y],t) \in \beta^{(2)} \cup \beta_{A^{(2)}}$. Let $G_1 = H - m$ and $G_2 = M - H$ on $[a,b]$. Then $H(y) - H(x) - f(t)(y-x) \le G_1(y) - G_1(x)$ whenever $([x,y],t) \in \beta^{(1)} \cup \beta_{A^{(1)}}$ and $f(t)(y-x) - (H(y) - H(x)) \le G_2(y) - G_2(x)$, whenever $([x,y],t) \in \beta^{(2)} \cup \beta_{A^{(2)}}$.

(iii) See (i) and (ii). $\square$

Definition 10.2. Let $F : [a,b] \to \mathbb{R}$ and let P be a closed subset of $[a,b]$, $c = \inf(P)$, $d = \sup(P)$. Let $\{(c_k,d_k)\}_k$ be the intervals contiguous to P. We define the function $F_P : [c,d] \to \mathbb{R}$ such that $F_P(x) = F(x)$, $x \in P$ and F_P is linear on each $[c_k,d_k]$.

Lemma 10.2. *Let $F : [a,b] \to \mathbb{R}$, let P be a closed subset of $[a,b]$ and let A be a measurable subset of P such that $F \in VB$ on P. Then F_P is derivable a.e. on $(\inf P, \sup P)$, F is approximately derivable a.e. on A and $F_P'(x) = F_{ap}'(x)$ a.e. on A.*

PROOF. F_P is VB on $[\inf P, \sup P]$ (see for example [1, p. 44]), hence F_P is derivable *a.e.* on $(\inf P, \sup P)$. Let $P_o = \{x \in P : d(P,x) = 1\}$. By Lebesgue's Density Theorem, P_o is measurable and $|P_o| = |P|$. It follows that $F_P'(x) = F_{ap}'(x)$ *a.e.* on P_o, hence $F_P'(x) = F_{ap}'(x)$ *a.e.* on A. $\square$

Theorem 10.1. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^+$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system, $\mathcal{S}_\infty^-$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$. If f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$- integrable on $[a,b]$ then f is also $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$, and the two integrals are equal.*

PROOF. Since f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$-integrable on $[a,b]$, there exists a function $F : [a,b] \to \mathbb{R}$ such that $F \in [ACG]$ and $F \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$. Let $\{P_i\}_i \in \overline{\mathcal{P}}_{[a,b]}$ such that $F \in AC$ on each P_i. Let $\epsilon > 0$ and let A be a countable subset of $[a,b]$ that contains $\text{Is}(\{P_i\})$. Suppose that $A = \{a_1, a_2, \ldots, a_i, \ldots\}$. Since $F \in$

$(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$, it follows that there exists $\beta_A = \beta_A(\sigma_x^{(1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$ such that $|F(x) - F(a_i)| < \epsilon/2^i$ whenever $x \in (\sigma_{a_i}^{(2)} \cap (a, a_i]) \cup (\sigma_{a_i}^{(1)} \cap [a_i, b))$. Let $G_1, G_2 : [a,b] \to \mathbb{R}$, $G_1(a) = G_2(a) = 0$,

$$G_1(x) = \sum_{a_i < x} \mathcal{O}(F; [a_i, b) \cap \sigma_{a_i}^{(1)}) \text{ and } G_2(x) = \sum_{a_i \le x} \mathcal{O}(F; (a, a_i]) \cap \sigma_{a_i}^{(2)}) \,.$$

(Here $\mathcal{O}(F;X)$ denotes the oscillation of the function F on the set X.) Clearly G_1 and G_2 are increasing on $[a,b]$, $G_1(b) < \epsilon$ and $G_2(b) < \epsilon$. Fix some a_i. Then

$$|F(x) - F(a_i)| \le \mathcal{O}(F; [a_i; x] \cap \sigma_{a_i}^{(1)}) < G_1(x) - G_1(a_i) \,, \tag{11}$$

whenever $x \in [a_i, b) \cap \sigma_{a_i}^{(1)}$, and

$$|F(x) - F(a_i)| \le \mathcal{O}(F; [x, a_i] \cap \sigma_{a_i}^{(2)}) < G_2(a_i) - G_2(x) \,, \tag{12}$$

whenever $x \in (a, a_i] \cap \sigma_{a_i}^{(2)}$. Let $S_i = \{x \in P_i \,:\, F'_{P_i}(x) = f(x)\}$. Then S_i is measurable and $|S_i| = |P_i|$ (see Lemma 10.2). Let $\delta_i : S_i \to (0, +\infty)$ such that

$$|F(y) - F(x) - f(x)(y-x)| < \frac{\epsilon}{2(b-a)}|y - x| \,,$$

whenever $y \in P_i \cap (x - \delta_i(x), x + \delta_i(x))$. Let $G_3(x) = \epsilon(x-a)/(2(b-a))$. For $\epsilon/2^i$ let $\eta_i > 0$ be given by the fact that $F \in AC$ on P_i. Let $B_i = P_i \setminus S_i$. Then $|B_i| = 0$, hence there exists an open set G_i such that $B_i \subset G_i$ and $|G_i| < \eta_i$. Let $C = A \cup (\cup_{i=1}^{\infty} B_i)$. Then $|C| = 0$. By the Tolstoff- Zahorski Theorem (see for example Theorem 2.14.6 of [1]), there exists $G_4 : [a,b] \to (0,+\infty)$ such that G_4 is increasing on $[a,b]$, $G_4(a) = 0$, $G_4(b) < \epsilon$ and $G'_4(x) = +\infty$ whenever $x \in C$. Let $\delta : C \to (0,+\infty)$ such that $(G_4(y) - G_4(x))/(y-x) > |f(x)|$ whenever $y \in (x - \delta(x), x + \delta(x))$. Let $\sigma_x^{(k)\star} = \sigma_x^{(k)} \cap (x - \delta(x), x + \delta(x))$, $k = 1, 2$, $x \in A$. By (11) we obtain that $|F(x) - F(a_i) - f(a_i)(x - a_i)| \le G_1(x) - G_1(a_i) + G_4(x) - G_4(a_i)$, whenever $x \in [a_i, b) \cap \sigma_x^{(1)\star}$ and $|F(x) - F(a_i) - f(a_i)(x - a_i)| \le G_2(x) - G_2(a_i) + G_4(x) - G_4(a_i)$, whenever $x \in (a, a_i] \cap \sigma_x^{(2)\star}$. Let $\delta_i : B_i \to (0,+\infty)$ such that $\delta_i(x) < \delta(x)$ and $(x - \delta_i(x), x + \delta_i(x)) \subset G_i)$. Let $G_5(x) = \sum_{i=1}^{\infty} V(F; B_i \cap [a,x])$. Then $G_5(a) = 0$, $G_5(b) < \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon$. We have $|F(y) - F(x) - f(x)(y-x)| \le V(F; B_i \cap [x,y]) + G_4(y) - G_4(x) \le G_5(y) - G_5(x) + G_4(y) - G_4(x)$, whenever $x \in B_i$ and $y \in P_i \cap (x - \delta_i(x), x + \delta_i(x))$. Let $G = \sum_{i=1}^{5} G_i$ and let $\beta = \beta(\{P_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$. It follows that $|F(y) - F(x) - f(t)(y-x)| < G(y) - G(x)$ whenever $([x,y], t) \in \beta \cup \beta_A$. Therefore F is the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-indefinite integral of f on $[a,b]$. Hence $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]\int_a^b f(t)\,dt = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^b f(t)\,dt\,.$ $\square$

Lemma 10.3. *Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^+_\infty$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^-_\infty$-filtering on $(a,b]$. Let $f : [a,b] \to \mathbb{R}$ be $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$.*

*(i) (**Saks-Henstock type lemma**) For $\epsilon > 0$ there is a $\beta = \beta(\{X_i\},\{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ with the following property: for every countable subset A of $[a,b]$ that contains Is$(\{X_i\})$ there is a $\beta_A = \beta_A(\sigma_x^{1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1\mathcal{S}_2)$ such that*

$$\sum_{([x,y],t)\in\pi} \left| [\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_x^y f(t)\,dt - f(t)(y-x) \right| < \epsilon\,,$$

whenever π is a $(\beta \cup \beta_A)$-partial partition of $[a,b]$.

(ii) f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$, and the $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$ and $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$ integrals are equal.

(iii) If $F(x) = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_a^x f(t)\,dt$ then F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$.

PROOF. (i) Since f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$-integrable on $[a,b]$, there exists $H : [a,b] \to \mathbb{R}$ such that for every $\epsilon > 0$ there exists a $\beta = \beta(\{X_i\},\{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ with the following property: for every countable subset A of $[a,b]$ that contains Is$(\{X_i\})$ there is a $\beta_A = \beta_A(\sigma_x^{1)}, \sigma_x^{(2)}) \in \mathcal{B}_A(\mathcal{S}_1;\mathcal{S}_2)$ and there exists an increasing function $G : [a,b] \to [0,+\infty)$, such that $G(a) = 0$, $G(b) < \epsilon$ and $|H(y) - H(x) - f(t)(y-x)| < G(y) - G(x)$, whenever $([x,y],t) \in \beta \cup \beta_A$. Let π be a $(\beta\cup\beta_A)$-partial partition of $[a,b]$. By Lemma 9.2, $H(y) - H(x) = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}]\int_x^y f(t)\,dt$, hence

$$\sum_{([x,y],t)\in\pi} \left| f(t)(y-x) - [\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_x^y f(t)\,dt \right| =$$

$$\sum_{([x,y],t)\in\pi} |f(t)(y-x) - (H(y)-H(x))| \le \sum_{([x,y],t)\in\pi} (G(y)-G(x)) \le G(b) < \epsilon\,.$$

(ii) With the notations of (i), let π be a $(\beta\cup\beta_A)$-partition of $[a,b]$. Then

$$|s(f;\pi) - (H(b)-H(a))| = \left| \sum_{([x,y],t)\in\pi} (f(t)(y-x) - H(y) + H(x)) \right|$$

$$\le \sum_{([x,y],t)\in\pi} |f(t)(y-x) - H(y) + H(x)| \le \sum_{([x,y],t)\in\pi} (G(y)-G(x)) < \epsilon\,.$$

It follows that f is $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$-integrable on $[a,b]$, and then

$$[\mathcal{S}_1\mathcal{S}_2\mathcal{R}] \int_a^b f(t)\,dt = H(b) - H(a) = [\mathcal{S}_1\mathcal{S}_2\mathcal{V}] \int_a^b f(t)\,dt\,.$$

(iii) Let $x_o \in [a,b)$ and $\epsilon > 0$. Let $\delta > 0$ such that $|f(x_o)| \cdot \delta < \epsilon$. For ϵ let $\beta = \beta(\{X_i\}, \{\delta_i\}) \in \overline{\mathcal{B}}_{[a,b]}$ be given by (i). Let $A = \{x_o\} \cup \mathrm{Is}(\{X_i\}$. Then for every $x \in \sigma^{(1)}_{x_o} \in \mathcal{S}_1(x_o)$, $x > x_o$ we have $|F(x) - F(x_o)| \le |F(x) - F(x_o) - f(x_o)(x - x_o)| + |f(x_o)|(x - x_o) < 2\epsilon$. It follows that F is right $\mathcal{S}_1$-continuous on $[a,b)$. Similarly we obtain that F is left $\mathcal{S}_2$-continuous on $(a,b]$. By Lemma 2.1, (i), F is $(\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$ on $[a,b]$. □

11 The Characterization of the $\mathcal{D}$-Integral

Theorem 11.1. *Let $f : [a,b] \to \mathbb{R}$. The following assertions are equivalent:*

(i) f is $[\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{D}]$-integrable (i.e., $\mathcal{D}$-integrable) on $[a,b]$;

(ii) f is $[\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{V}]$- integrable on $[a,b]$;

(iii) f is $[\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{W}]$-integrable on $[a,b]$.

(iv) f is $[\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{R}]$-integrable on $[a,b]$ and F is continuous on $[a,b]$, where $F(x) = [\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{R}]\int_a^x f(t)\,dt$.

Moreover, all the integrals are equal.

Proof. (i) ⇒ (ii) and the equality of the integrals follow by Theorem 10.1.

(ii) ⇔ (iii) and the equality of the integrals follow by Lemma 10.1.

(ii) ⇒ (iv) and the equality of the integrals follow by Lemma 10.3.

(iv) ⇒ (i) By hypothesis $F(x) = [\mathcal{S}^+_o\mathcal{S}^-_o\mathcal{R}]\int_a^x f(t)\,dt$ is continuous on $[a,b]$. By Corollary 7.1, $F \in (N)$ on $[a,b]$, and by Lemma 7.6, F is VBG on $[a,b]$. Therefore $F \in \mathcal{C} \cap VBG \cap (N) = \mathcal{C} \cap ACG$ on $[a,b]$ (see for example [1, p. 75]). Now, by Lemma 7.5, $F'_{ap}(x) = f(x)$ *a.e.* on $[a,b]$. It follows that f is $\mathcal{D}$-integrable on $[a,b]$, and the two integrals are equal. □

Remark 11.1. The fact that $F \in ACG$ in the proof of Theorem 11.1, (iv) ⇒ (i), can also be obtained as follows: by Corollary 7.1, $F \in N_{\mathcal{B}_o}$ on $[a,b]$; and by Theorem 5.2, $F \in ACG$ on $[a,b]$.

12 Query

Definition 12.1. Let $\mathcal{S}_1 = \{\mathcal{S}_1(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^+_\infty$-filtering on $[a,b)$, and let $\mathcal{S}_2 = \{\mathcal{S}_2(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}^-_\infty$-filtering on $(a,b]$. A function $f : [a,b] \to \mathbb{R}$ is said to be strong $[\mathcal{S}_1\mathcal{S}_1\mathcal{D}]$ integrable on $[a,b]$ if there exists a function $F : [a,b] \to \mathbb{R}$ with the following properties:

(i) $F \in (\mathcal{S}_1;\mathcal{S}_2)\mathcal{C}$;

(ii) $F \in [VBG] \cap (N)$;

(iii) $F'_{ap}(x) = f(x)$ *a.e.* on $[a, b]$.

We write strong $[\mathcal{S}_1\mathcal{S}_1\mathcal{D}] \int_a^b f(t)\, dt = F(b) - F(a)$.

Remark 12.1. Note that Lemma 7.2 remains true if condition $[\underline{ACG}]$ is replaced by $[VBG] \cap (N)$, and Corollary 2.1 remains true if $[ACG]$ is replaced by $[VBG] \cap (N)$. But $[VBG] \cap (N)$ is a real linear space on $[a, b]$ (see Corollary 3.1.1 and Theorem 3.6 of [19]). It follows that the above integral is well defined.

Question. *How can the definitions of* $[\mathcal{S}_1\mathcal{S}_2\mathcal{V}]$, $[\mathcal{S}_1\mathcal{S}_2\mathcal{W}]$ *and* $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$ *be modified such that each of them contain the strong* $[\mathcal{S}_1\mathcal{S}_2\mathcal{D}]$*- integral?*

Remark 12.2. Definition 12.1 can be extended by replacing condition (ii) with "$F \in VBG \cap (N) \cap \mathcal{B}_1$". This is so because Lemma 7.2 still remains true if $[\underline{ACG}]$ is replaced by $VBG \cap (N) \cap \mathcal{B}_1$, Corollary 2.1 still remains true if $[ACG]$ is replaced by $VBG \cap (N) \cap \mathcal{B}_1$, and $VBG \cap (N) \cap \mathcal{B}_1$ is a real linear space (the proof of the latter is not easy, and it is shown by the author in [2]).

A special case of this new definition is an integral defined by Gordon in [4] (Definition 3): (i) is replaced by "$F \in \mathcal{C}_{ap}$", and (ii) by "$F \in VBG \cap (N)$". However his argumentation about the integral being well defined is incomplete, because he doesn't take in consideration whether $VBG \cap (N) \cap \mathcal{C}_{ap}$ is a linear space or not.

13 Appendix

* After this paper has been accepted for publication in the present journal, the author withdrew it, having in mind a lot of revisions. The present paper contains a lot of them, but his main intention was to give up the two local systems in the definitions of the general integrals, and use instead only one. He had started to do so but never finished his work. In what follows the revised statements (with proofs) will be given as well as the changed definitions. The modified results will be labelled with the old numbers and a 'prime' sign.

Lemma 2.1′. *Let* $\mathcal{S} = \{\mathcal{S}(x)\}_{x \in \mathbb{R}}$ *be a bilateral local system satisfying the following property for each* $x \in \mathbb{R}$:

$$\text{If } \sigma'_x, \sigma''_x \in \mathcal{S}(x) \text{ then } \big(\sigma'_x \cap (-\infty, x]\big) \cup \big(\sigma''_x \cap [x, +\infty)\big) \in \mathcal{S}(x)\,. \qquad (*)$$

Let $F : [a, b] \to \mathbb{R}$*. Then we have:*

(i) F *is* $\mathcal{SC}$ *on* $[a, b]$ *if and only if it is bilaterally* $\mathcal{SC}$ *on* $[a, b]$.

(ii) F *is* $\mathcal{SC}_i$ *on* $[a, b]$ *if and only if* F *is simultaneously right* $\mathcal{S}$*-lower semi-continuous on* $[a, b)$ *and left* $\mathcal{S}$*-upper semi-continuous on* $(a, b]$.

PROOF. Evident. □

*Extracted by Gabriela Ene from the author's notes.

4′ A Fundamental Lemma

Following the notations in [21] (pp. 5,6), we shall denote by $\mathcal{I}$ the collection of all nondegenerate, real compact intervals. A subset β of the product $\mathcal{I} \times \mathbb{R}$ is said to be a covering relation if $x \in I$ whenever $(I, x) \in \beta$ (see [22, p. 5]). If β is a covering relation and E a real set then $\beta(E)$, $\beta[E]$ and $\sigma(\beta)$ denote the following sets:

- $\beta(E) = \{(I, x) \in \beta \,:\, I \subset E\}$;
- $\beta[E] = \{(I, x) \in \beta \,:\, x \in E\}$;
- $\sigma(\beta) = \cup_{(I,x)\in\beta} I$.

A packing is a covering relation β with the property that for distinct pairs (I_1, x_1) and (I_2, x_2) the intervals I_1 and I_2 do not overlap. Evidently a packing is either finite or countable infinite. Using the language of Henstock we call a finite packing β a division (a β-division) of an interval $[a, b]$ if $\sigma(\beta) = [a, b]$.

Definition 13.1. Let E be a real set and $\delta : E \to (0, +\infty)$. We denote by $\beta(E; \delta) = \{(\langle x, y\rangle) \,:\, x, y \in E, x \text{ is an accumulation point for } \langle x, y\rangle \cap E, \text{ and } \langle x, y\rangle \subset (x - \delta(x), x + \delta(x)\}$. Clearly $\beta(E; \delta)$ is a covering relation (possibly empty).

Definition 4.2′. Let P be a real set. We denote by

- $\mathrm{Is}^+(P) = \{x \in P \,:\, x \text{ is a right isolated point of } P\}$;
- $\mathrm{Is}^-(P) = \{x \in P \,:\, x \text{ is a left isolated point of } P\}$;
- $\mathrm{Is}(P) = \mathrm{Is}^+(P) \cup \mathrm{Is}^-(P)$. This set is countable (see [15, p. 260]).

Definition 13.2. ([16]). A sequence $\{E_n\}$ of sets whose union is E is called an E-form with parts E_n. If, in addition, each part E_n is closed in E (i.e., $E_n = P_n \cap E$, where P_n is a closed set, so $P_n = \overline{E}_n$) then the E-form is said to be closed.

Definition 13.3. Let $\{E_i\}_i$ be a closed $[a, b]$-form, $\delta_i : E_i \to (0, +\infty)$, and A a set that contains $\cup_{i=1}^{\infty} \mathrm{Is}(E_i)$. For each $a \in A$ let σ_a be a set having the point a as a bilateral accumulation point. Let

$$\beta = \beta(\{E_i\}; \{\delta_i\}; (\sigma_a)_{a\in A}) = \cup_{i=1}^{\infty} \beta(E_i; \delta_i) \cup \Big(\cup_{a\in A} \{(\langle a, x\rangle, a) : x \in \sigma_a \setminus \{a\}\}\Big).$$

Clearly β is a covering relation. It contains the AD full cover of Lee and Soedijono (see [12, p. 265]), the cover $\mathcal{U}$ of Henstock (see [7, p. 56]), the covering relation called "composite path derivation" defined by Thomson (see [20, p. 104]), and Henstock's covering relation PC ([20, p. 115]).

Lemma 4.1′. *Let P be a perfect nowhere dense subset of $[a, b]$, $a, b \in P$, and let $\delta : P \to (0, +\infty)$. Then there exists a finite packing π contained in $\beta(P; \delta)$ such that $\sigma(\pi) \supset P$.*

Proof. Let $\{(a_i, b_i)\}$, $i = \overline{1, \infty}$ be the intervals contiguous to P, and let $\eta : [a, b] \to (0, +\infty)$,

$$\eta(x) = \begin{cases} \delta(x) & , \ if \quad x \in P \setminus \cup_{i=1}^{\infty} \{a_i, b_i\} \quad , \\ \min\{\frac{b_i - a_i}{3}, \delta(x)\} & , \ if \quad x \in \{a_i, b_i\} \quad , \quad i = \overline{1, \infty} \\ \min\{\frac{x - a_i}{2}, \frac{b_i - x}{2}\} & , \ if \quad x \in (a_i, b_i) \quad , \quad i = \overline{1, \infty}\,. \end{cases}$$

Let π be a $\beta([a, b], \eta)$ partition of $[a, b]$ (that such a partition exists follows for example by [1, p. 87]). Let $\pi = \{(\langle x_i, y_i\rangle, x_i)\}_{i=1}^{n}$ and $\mathcal{A} = \{i \in \{1, 2, \ldots, n\} \,:\, \mathrm{int}(\langle x_i, y_i\rangle) \cap P \neq \emptyset\}$ (here $\mathrm{int}(X)$ denotes the interior of the set X). For $x \in P \setminus \cup_{i=1}^{\infty} \{a_i, b_i\}$, it follows that

$x \in \langle x_i, y_i\rangle$ for some $i \in \{1, 2, \dots, n\}$. Clearly $\langle x_i, y_i\rangle \cap P$ is an infinite set, so $i \in \mathcal{A}$. Therefore

$$P \setminus \Big(\cup_{i=1}^{\infty}\{a_i, b_i\}\Big) \subset \cup_{i\in\mathcal{A}}\langle x_i, y_i\rangle\,.$$

It follows that

$$P = \overline{P \setminus (\cup_{i=1}^{\infty}\{a_i, b_i\})} \subset \cup_{i\in\mathcal{A}}\langle x_i, y_i\rangle\,. \tag{13}$$

If $i \in \mathcal{A}$ then $x_i \in P$ (because if $x_i \in (a_j, b_j)$ for some j then $y_i \in (x_i - \eta(x_i), x_i + \eta(x_i)) \subset (a_j, b_j)$, which implies that $i \notin \mathcal{A}$, a contradiction).

Fix some $i \in \mathcal{A}$. If $x_i \in P \setminus \big(\cup_{j=1}^{\infty}\{a_j, b_j\}\big)$ then x_i is an accumulation point for $\langle x_i, y_i\rangle \cap P$. Let $z_i \in P$ such that $\langle x_i, y_i\rangle \cap P = \langle x_i, z_i\rangle \cap P$. Then $\big(\langle x_i, z_i\rangle, x_i\big) \in \beta(P;\delta)$.
If $x_i = a_j$ for some j then $y_i < x_i$ (because if $x_i < y_i$ then $[x_i, y_i] \subset \big[a_j + \frac{b_j - a_j}{3}\big]$, so $i \notin \mathcal{A}$). It follows that $[y_i, x_i] \cap P$ has x_i as an accumulation point. Let $z_i = \inf[y_i, x_i] \cap P$, hence $[y_i, x_i] \cap P = [z_i, x_i] \cap P$. Consequently $([z_i, x_i], x_i) \in \beta(P;\delta)$.
If $x_i = b_j$ for some j, then $y_i > x_i$. Let $z_i = \sup(P \cap [x_i, y_i])$. Then $([x_i, z_i], x_i) \in \beta(P;\delta)$.

By (13), it follows easily that $\pi = \big\{(\langle x_i, z_i\rangle, x_i)\big\}_{i\in\mathcal{A}}$ satisfies our lemma. □

Lemma 4.2′ (Fundamental lemma)**.** *For each $\beta = \beta(\{E_i\};\{\delta_i\};(\sigma_a)_{a\in\mathcal{A}})$, the interval $[a, b]$ has a β-division.*

PROOF. We shall use the Romanovski Lemma (see for example [1, p. 10]). Let $\mathcal{A} = \{(p, q) \subseteq (a, b) : [p_1, q_1]$ has a β-division whenever $(p_1, q_1) \subseteq (p, q)\}$.

(i) If $(p, q) \in \mathcal{A}$ and $(q, r) \in \mathcal{A}$ then clearly $(p, r) \in \mathcal{A}$.

(ii) If $(p, q) \in \mathcal{A}$ and $(p_1, q_1) \subset (p, q)$ then $(p_1, q_1) \in \mathcal{A}$ (see the definition of $\mathcal{A}$).

(iii) Let $(c, d) \subseteq (a, b)$ such that $(p, q) \in \mathcal{A}$ whenever $[p, q] \subset (c, d)$. We show that $(c, d) \in \mathcal{A}$. Let $c \in E_n$. Let $c_1 \in (c, c + \delta_n(c)) \cap E_n \cap (c, (c + d)/2)$ if c is a right accumulation point for E_n, and let $c_1 \in \sigma_c \cap (c, (c+d)/2)$ if c is right isolated in $[a, b)\cap E_n$. Then $([c, c_1], c) \in \beta$. Similarly we find $d_1 \in ((c + d)/2, d)$ such that $([d_1, d], d) \in \beta$. But $(c_1, d_1) \in \mathcal{A}$ and $[c, d] = [c, c_1] \cup [c_1, d_1] \cup [d_1, d]$. Therefore $[c, d]$ admits a β-division. Analogously we obtain that $[c_2, d_2]$ has a β-division, whenever $(c_2, d_2) \subset (c, d)$. Hence $(c, d) \in \mathcal{A}$.

(iv) Let $E \subset [a, b]$ be a perfect set such that all intervals contiguous to E are contained in $\mathcal{A}$. We show that there exists $(p, q) \in \mathcal{A}$ such that $E \cap (p, q) \neq \emptyset$. Since $E = \cup_{n=1}^{\infty}(E \cap E_n)$, by the Baire Category Theorem (see for example [1, p. 10]) it follows that there exists a positive integer n and an interval (p, q) such that $\emptyset \neq (p, q)\cap E = (E\cap E_n)\cap(p, q)$. We may suppose without loss of generality that $p, q \in E$ and $[p, q] \cap E$ is perfect. Applying Lemma 4.1′ to $[p, q] \cap E$ and δ_n, there exists a finite packing π contained in $\beta(E;\delta_n)$ such that $\sigma(\pi) \supset E$. Clearly π is a finite packing contained in β. Since $[p, q] \setminus \sigma(\pi)$ consists of a finite number of intervals contiguous to E, it follows that $[p, q]$ admits a β-division. Similarly it follows that each $[p_1, q_1]$ admits such a division, whenever $(p_1, q_1) \subset (p, q)$. Therefore $(p, q) \in \mathcal{A}$.

By (i)-(iv) and the Romanovski Lemma, it follows that $(a, b) \in \mathcal{A}$. □

6′ The Lusin Type $[\mathcal{SD}]$ Integral

Definition 6.1′. Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$, and let E a bounded nonempty set, with $a = \inf E$, $b = \sup E$. f is said to be $[\mathcal{SD}]$-integrable on E if there is a real number I and a function $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ I & \text{if } x \ge b \end{cases}$$

such that F is $\mathcal{SC}$ on $[a,b]$, $F \in [ACG]$ on $[a,b]$, and $F'_{ap}(x) = \chi_E(x) \cdot f(x)$ *a.e.* on $[a,b]$, where χ_E is the characteristic function of E. We write $[\mathcal{SD}] \int_E f(t)\,dt = I$. F is said to be the (unique) indefinite integral of f on E.

Lemma 6.2′. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be $[\mathcal{SD}]$-integrable on $[a,b]$ and let $c \in (a,b)$. Then f is $[\mathcal{SD}]$-integrable on both $[a,c]$ and $[a,d]$, and we have*

$$[\mathcal{SD}] \int_a^b f(t)\,dt = [\mathcal{SD}] \int_a^c f(t)\,dt + [\mathcal{SD}] \int_c^b f(t)\,dt\,.$$

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be the indefinite $[\mathcal{SD}]$-integral of f on $[a,b]$. Let $F_1, F_2 : \mathbb{R} \to \mathbb{R}$,

$$F_1(x) = \begin{cases} F(x) & \text{if } x \in (-\infty, c] \\ F(c) & \text{if } x \in [c, +\infty) \end{cases}, \quad F_2(x) = \begin{cases} 0 & \text{if } x \in (-\infty, c] \\ F(x) - F(c) & \text{if } x \in [c, +\infty)\,. \end{cases}$$

Then F_1 (respectively F_2) is the indefinite $[\mathcal{SD}]$-integral of f on $[a,c]$ (respectively $[c,b]$) and we have the relation from above. □

Lemma 6.3′. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be $[\mathcal{SD}]$-integrable on $[a,c]$ and on $[c,b]$, where $c \in (a,b)$. Then f is $[\mathcal{SD}]$-integrable on $[a,b]$ and we have:*

$$[\mathcal{SD}] \int_a^b f(t)\,dt = [\mathcal{SD}] \int_a^c f(t)\,dt + [\mathcal{SD}] \int_c^b f(t)\,dt\,.$$

Proof. Let F_1 (respectively F_2) be the indefinite $[\mathcal{SD}]$-integral of f on $[a,c]$ (respectively $[c,b]$). Then $F = F_1 + F_2$ is the indefinite $[\mathcal{SD}]$-integral of f on $[a,b]$ and we have the relation from above. □

7′ The Riemann Type $[\mathcal{SR}]$ Integral

Definition 7.1′. Let $\mathcal{S} = \{\mathcal{S}(x)\})x \in \mathbb{R}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be $[\mathcal{SR}]$ integrable on $[a,b]$ to $I \in \mathbb{R}$ if for $\epsilon > 0$ there exist a closed $[a,b]$-form $\{E_i\}$, $\delta_i : E_i \to (0,+\infty)$, for $A \supseteq \cup_{i=1}^{\infty} \mathrm{Is}(E_i)$ countable there is $\Delta_{A,\epsilon} : A \to \mathcal{P}(\mathbb{R})$, $\Delta_{A,\epsilon}(x) \in \mathcal{S}(x)$, and for $B \supset A$, B countable there is a $\Delta_{B,\epsilon}$ such that

$$|s(f;\alpha) - I| < \epsilon\,,$$

whenever $\alpha \subset \beta\big(\{E_i^\epsilon\}, \{\delta_i^\epsilon\}, \Delta_{A,\epsilon} \vee \Delta_{B\epsilon}\big)$ is a division of $[a,b]$.

Theorem 7.1′. *The number I in Definition 7.1′ is unique, and it will be denoted by $[\mathcal{SR}] \int_a^b f(t)\,dt$.*

Proof. Suppose that there exist two numbers I_1 and I_2 as in Definition 7.1′. Let $\{E_i^k\}_{i=1}^{\infty}$, $k = 1,2$, be a closed $[a,b]$-form given for I_k and ϵ. Let $E_{ij} = E_i^1 \cap E_j^2$, $\delta_{ij}^k = E_{ij} \to (0,+\infty)$ and $\Delta^k : \cup_{i,j} \mathrm{Is}(E_{ij}) \to \mathcal{P}(\mathbb{R})$, $\Delta_k(x) \in \mathcal{S}(x)$, be such that $|s(f;\pi_k) - I - k| < \epsilon$, whenever $\pi_k \subset \beta_k = \beta(\{E_{ij}\}; \delta_{ij}^k; \Delta_k)$ is a division of $[a,b]$. Let $\delta_{ij} : E_{ij} \to (0,+\infty)$, $\delta_{ij}(x) = \min\{\delta_{ij}^1(x), \delta_{ij}^2(x)\}$, and let $\Delta : \cup_{i,j} \mathrm{Is}(E_{ij}) \to \mathcal{P}(\mathbb{R})$, $\Delta(x) = \Delta_1(x) \cap \Delta_2(x) \in \mathcal{S}_{\infty,\infty}(x)$. By Lemma 4.2′ there exists $\pi \subset \beta = \beta(\{E_{ij}\}; \{\delta_{ij}\}; \Delta)$. Clearly $\pi \subset \beta_1 \cap \beta_2$. Hence $|s(f;\pi) - I_k| < \epsilon$, $k = 1,2$. It follows that $|I_1 - I_2| < 2\epsilon$. Since ϵ is arbitrary we obtain that $I_1 = I_2$. □

Remark 13.1. If $\alpha \in \mathbb{R}$ and f is $[\mathcal{SR}]$-integrable to I on $[a,b]$ then αf is $[\mathcal{SR}]$-integrable to αI on $[a,b]$.

Lemma 7.1′. *Let $\mathcal{S}$ be a filtering local system on $[a,b]$. Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be $[\mathcal{SR}]$-integrable to I_1 respectively I_2 on $[a,b]$. Then $f_1 + f_2$ is $[\mathcal{SR}]$-integrable on $[a,b]$ to $I_1 + I_2$.*

PROOF. For $\frac{\epsilon}{2} > 0$ and I_k, $k = 1, 2$, let $\{E_i^k\}$ be a closed $[a,b]$-form given by Definition 7.1′. Let $E_{ij} = E_i^1 \cap E_j^2$. Then $\{E_{ij}\}_{i,j}$ is a closed $[a,b]$-form. Let $\{X_i\}$ be a closed $[a,b]$-form finer than $\{E_{ij}\}$. Clearly $\{X_i\}$ is finer than $\{E_i\}^k$. By Definition 7.1′, there exist $\delta_i^k : X_i \to (0, +\infty)$ and $\Delta^k : \cup_i \mathrm{Is}(X_i) \to \mathcal{P}(\mathbb{R})$, $\Delta_k(x) \in \mathcal{S}(x)$ such that

$$|s(f_k;\pi_k) - I_k| < \frac{\epsilon}{2}\,,$$

whenever $\pi_k \subset \beta_k = \beta(\{X_i\};\{\delta_i^k\};\Delta_k)$ is a division of $[a,b]$. Let $\delta_i : X_i \to (0,+\infty)$, $\delta_i(x) = \min\{\delta_i^1(x), \delta_i^2(x)\}$, and let $\Delta : \cup_i \mathrm{Is}(X_i) \to \mathcal{P}(\mathbb{R})$, $\Delta(x) = \Delta_1(x) \cap \Delta_(x) \in \mathcal{S}(x)$ (because $\mathcal{S}$ is filtering). Let $\pi \subset \beta = \beta(\{X_i\};\{\delta_i\};\Delta)$ be a division of $[a,b]$ (this is possible by Lemma 4.2′). Clearly π is a β_k division. Hence

$$\big|s(f_1+f_2;\pi) - (I_1+I_2)\big| < \big|s(f_1;\pi) - I_1\big| + \big|s(f_2;\pi) - I_2\big| < \epsilon\,.$$

Thus $f_1 + f_2$ is $[\mathcal{SR}]$-integrable to $I_1 + I_2$ on $[a,b]$. □

Lemma 7.2′ (A Cauchy criterion)**.** *Let $\mathcal{S} = \{\mathcal{S}(x)\})x \in \mathbb{R}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering on $[a,b]$, and let $f : \mathbb{R} \to \mathbb{R}$. The following assertions are equivalent:*

(i) $f \in [\mathcal{SR}]$ integrable on $[a,b]$;

(ii) for $\epsilon > 0$ there exist a closed $[a,b]$-form $\{E_i^\epsilon\}$, $\delta_i^\epsilon : E_i^\epsilon \to (0,+\infty)$, for $A \supseteq \cup_{i=1}^\infty Is(E_i^\epsilon)$ there is a $\Delta_{A,\epsilon}$ and for $B \supset A$, B countable, there is a $\Delta_{B,\epsilon}$ such that

$$\big|s(f,\pi_1) - s(f;\pi_2)\big| < \epsilon\,,$$

whenever $\pi_1, \pi_2 \subset \beta\big(\{E_i^\epsilon\}, \{\delta_i^\epsilon\}, \Delta_{A,\epsilon} \vee \Delta_{B,\epsilon}\big)$.

PROOF. (i) ⇒ (ii) This is obvious.

(ii) ⇒ (i) For $\frac{1}{k}$ let $\{E_i^{\frac{1}{k}}\}$ be a closed $[a,b]$-form, $\delta^{\frac{1}{k}} : E_i^{\frac{1}{k}} \to (0,+\infty)$ be given by (ii). Let A_k be countable, $A_k \supseteq \cup_{i=1}^\infty \mathrm{Is}(E_i^{\frac{1}{k}})$ and $B_k \supset A_k$, B_k countable. Let $A_o = \cup_{i=1}^\infty A_k$ and $B_o = \cup_{i=1}^\infty B_k$.Again by (ii), for A_o there is a $\Delta_{A_o,\frac{1}{k}}$ and for B_o there is a $\Delta_{B_o,\frac{1}{k}}$ such that

$$\big|s(f;\pi_k') - s(f;\pi_k'')\big| < \frac{1}{k}\,,$$

whenever $\pi_k', \pi_k'' \subset \beta_k = \beta\big(\{E_i^{\frac{1}{k}}\}, \{\delta_i^{\frac{1}{k}}\}, \Delta_{A_o,\frac{1}{k}} \vee \Delta_{B_o,\frac{1}{k}}\big)$ are divisions of $[a,b]$. Let $\pi_k \subset \beta_k^* = \beta\big(\{E_i^{\frac{1}{k}}\}, \{\delta_i^{\frac{1}{k}}\}, \Delta_{A_k,\frac{1}{k}}\big) \subset \beta_k$ be a fixed division of $[a,b]$, where $\Delta_{A_k,\frac{1}{k}} = \Delta_{A_o,\frac{1}{k}}/A_k$. Let $\epsilon > 0$ and let k_ϵ be a positive integer such that$1/k_\epsilon < \epsilon/2$. Let $k_\epsilon \le m < n$. Let

$$\pi_{mn} \subset \beta_{mn} = \beta\Big(\{E_i^{\frac{1}{m}} \cap E_j^{\frac{1}{n}}\}, \{\delta_{mnij}\}, \Delta_{A_o,\frac{1}{m}} \cap \Delta_{A_o,\frac{1}{n}}\Big)$$

be a division od $[a,b]$ (see Lemma 4.2′). Then $\pi_{mn} \subset \beta_m$ and $\pi_{mn} \subset \beta_n$. But $\pi_m \subset \beta_m$ and $\pi_n \subset \beta_n$, so

$$\big|s(f;\pi_m) - s(f;\pi_{mn})\big| < \frac{1}{m} \quad \text{and} \quad \big|s(f;\pi_n) - s(f;\pi_{mn})\big| < \frac{1}{n}$$

imply that

$$\big|s(f;\pi_m) - s(f;\pi_n)\big| < \frac{1}{m} + \frac{1}{n} < \epsilon\,,.$$

hence $\{s(f;\pi_k)\}_k$ is a Cauchy sequence. Let $I=\lim_{k\to\infty} s(f;\pi_k)$. Fix $m\geq k_\epsilon$. Then

$$\big|s(f;\pi_m)-I\big|<\frac{1}{m}\,.$$

Now we show that I satisfies the conditions in Definition 7.1′. Let $\Delta_{B_o,\frac{1}{k}}=\Delta_{B_o,\frac{1}{k}}/B_k$ and let $\alpha\subset\beta^{**}_m=\beta\Big(\{E_i^{\frac{1}{k}}\},\{\delta_i^{\frac{1}{k}}\},\Delta_{A_k,\frac{1}{k}}\vee\Delta_{B_k,\frac{1}{k}}\Big)\subset\beta_m$ be a division of $[a,b]$. Then

$$\big|s(f;\alpha)-I\big|\leq\big|s(f;\alpha)-s(f;\pi_m)\big|+\big|s(f;\pi_m)-I\big|<\frac{1}{m}+\frac{1}{m}<\epsilon\,.$$

□

Lemma 7.3′. *Let $\mathcal{S}=\{\mathcal{S}(x)\})x\in\mathbb{R}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering. Let $f:\mathbb{R}\to\mathbb{R}$.*

(i) If $a<c<b$, f is $[\mathcal{SR}]$ integrable on $[a,c]=[a_1,b_1]$ to I_1, and f is $[\mathcal{SR}]$ integrable on $[c,b]=[a_2,b_2]$ to I_2, then f is $[\mathcal{SR}]$ integrable on $[a,b]$ to I_1+I_2.

(ii) If $a\leq c<d\leq b$ and f is $[\mathcal{SR}]$ integrable on $[a,b]$ then f is $]\mathcal{SR}]$ integrable on $[c,d]$.

Proof. (i) Let $\epsilon>0$. For $\epsilon/2$ let $\{E_i^{k,\frac{\epsilon}{2}}\}$ be a closed $[a_k,b_k]$-form, $\delta_i^{k,\frac{\epsilon}{2}}:E_i^{k,\frac{\epsilon}{2}}\to(0,+\infty)$ be given by Definition 7.1′, $k=1,2$. Clearly $\{E_i^{k,\frac{\epsilon}{2}}\}_{i,k}$ is a closed $[a,b]$-form. Let $A\supset\cup_{k=1}^2\cup_{i=1}^\infty \mathrm{Is}\big(E_i^{k,\frac{\epsilon}{2}}\big)$ be a countable set and $B\supset A$ another countable set. Let $A_k=A\cap[a_k,b_k]$, $B_k=B\cap[a_k,b_k]$, $k=1,2$. Clearly $B_k\supset A_k\supset\cup_{i=1}^\infty\mathrm{Is}(E_i^{k,\frac{\epsilon}{2}})$. Again by Definition 7.1′, there exist $\Delta_{A_k,\frac{\epsilon}{2}}$ and $\Delta_{B_k,\frac{\epsilon}{2}}$ such that

$$\big|s(f;\pi'_k)-s(f;\pi''_k)\big|<\frac{\epsilon}{2}\,,\quad k=1,2 \tag{14}$$

whenever $\pi'_k,\pi''_k\subset\beta_k=\beta\big(\{E_i^{k,\frac{\epsilon}{2}}\},\{\delta_i^{k,\frac{\epsilon}{2}}\},\Delta_{A_k,\frac{\epsilon}{2}}\vee\Delta_{B_k,\frac{\epsilon}{2}}\big)$ are divisions of $[a,b]$. Let

$$\Delta_{A,\epsilon}(x)=\begin{cases}\Delta_{A_1,\frac{\epsilon}{2}}(a)\cap(-\infty,c) & \text{if } x=a\\ \Delta_{A_1,\frac{\epsilon}{2}}(x)\cap(a,c) & \text{if } x\in A_1\cap(a,c)\\ \Delta_{A_2,\frac{\epsilon}{2}}(x)\cap(c,b) & \text{if } x\in A_2\cap(c,b)\\ \big(\Delta_{A_1,\frac{\epsilon}{2}}(c)\cap(a,c]\big)\cup\big(\Delta_{A_2,\frac{\epsilon}{2}}(c)\cap[c,b)\big) & \text{if } x=c\\ \Delta_{A_2,\frac{\epsilon}{2}}(b)\cap(c,+\infty) & \text{if } x=b\,.\end{cases}$$

Let

$$\delta_i^{1,\epsilon}(x)=\begin{cases}\min\big\{\delta_i^{1,\frac{\epsilon}{2}}(x),c-x\big\} & \text{if } x\in E_i^{1,\frac{\epsilon}{2}}\cap[a,c)\\ \delta_i^{1,\frac{\epsilon}{2}}(c) & \text{if } x=c\in E_i^{1,\frac{\epsilon}{2}}\end{cases}$$

and

$$\delta_i^{2,\epsilon}(x)=\begin{cases}\min\big\{\delta_i^{2,\frac{\epsilon}{2}}(x),x-c\big\} & \text{if } x\in E_i^{2,\frac{\epsilon}{2}}\cap(c,b]\\ \delta_i^{2,\frac{\epsilon}{2}}(c) & \text{if } x=c\in E_i^{2,\frac{\epsilon}{2}}\,.\end{cases}$$

For B, $\Delta_{B,\epsilon}(x)$ is defined similarly with $\Delta_{A,\epsilon}(x)$.

Let $E_i^{k,\epsilon}=E_i^{k,\frac{\epsilon}{2}}$. Let $\pi\subset\beta=\beta\big(\{E_i^{k,\epsilon}\}_{i,k},\{\delta_i^{k,\epsilon}\}_{i,k},\Delta_{A,\epsilon}\vee\Delta_{B,\epsilon}\big)$ be a division of $[a,b]$, and let $\big(\langle x,y\rangle,x\big)\in\pi$. Then we have:

1) $x < c \;\Rightarrow\; y < c \;\Rightarrow\; \langle x, y\rangle \subset [a, c) \;\Rightarrow\; \big(\langle x, y\rangle, x\big) \in \beta_1$;

2) $x > c \;\Rightarrow\; y > c \;\Rightarrow\; \langle x, y\rangle \subset (c, b] \;\Rightarrow\; \big(\langle x, y\rangle, x\big) \in \beta_2$;

3) $x = c$. Let $c_1 = \sup_{\substack{(\langle x,y\rangle,x)\in\pi\\ x<c}}\{x, y\}$ and $c_2 = \inf_{\substack{(\langle x,y\rangle,x)\in\pi\\ x>c}}\{x, y\}$. Then $\big([c_1, c], c\big)$ and $\big([c, c_2], c\big)$ belong to π. Note that $\big([c_1, c], c\big) \in \beta_1$ and $\big([c, c_2], c\big) \in \beta_2$.

Let $\pi_k = \pi \cap [a_k, b_k]$, $k = 1, 2$. Since $\pi = \pi_1 \cup \pi_2$, by (14) we have

$$\big|s(f;\pi) - (I_1 + I_2)\big| \le \big|s(f;\pi_1) - I_1\big| + \big|s(f;\pi_2) - I_2\big| < \epsilon\,.$$

It follows that f is $[\mathcal{SR}]$ integrable on $[a, b]$ to $I_1 + I_2$.

(ii) Consider $a < c < d < b$, $[a, c] = [a_1, b_1]$, $[c, d] = [a_2, b_2]$, $[d, b] = [a_3, b_3]$. Suppose that f is $[\mathcal{SR}]$ integrable to I on $[a, b]$. For $\epsilon > 0$ there exist a closed $[a, b]$-form $\{E_i\}$, $\delta_i : E_i \to (0, +\infty)$, for $A \supseteq \cup_{i=1}^{\infty}\mathrm{Is}(E_i)$, A countable, there is a $\Delta_{A,\epsilon}$, and for $B \supset A$, B countable, there is a $\Delta_{B,\epsilon}$ such that $\big|s(f;\pi) - I\big| < \epsilon$, whenever $\pi \subset \beta = \beta\big(\{E_i\}, \{\delta_i\}, \Delta_{A,\epsilon} \vee \Delta_{B,\epsilon}\big)$ is a division of $[a, b]$. Let $E_{ik} = [a_k, b_k] \cap E_i$, $k = 1, 2, 3$. Then $\{E_{ik}\}$ is a closed $[a_k, b_k]$-form, $k = 1, 2, 3$. Let $\delta_{ik} : E_{ik} \to (0, +\infty)$, $\delta_{ik}(x) = \delta_{i|E_{ik}}$. Let $[a_k, b_k] \supseteq A_k \supset \cup_{i=1}^{\infty}\mathrm{Is}(E_{ik})$, $k = 1, 2, 3$, A_k countable, and let $[a_k, b_k] \supseteq B_2 \supset A_2$ be another countable set. Let $A = A_1 \cup A_2 \cup A_3$ and $B = A_1 \cup B_2 \cup A_3$. Clearly $A \supset \cup_{i=1}^{\infty}\mathrm{Is}(E_i)$. Let $\Delta_{A_k,\epsilon} = \Delta_{A,\epsilon}/A_k$, $k = 1, 2, 3$ and $\Delta_{B_2,\epsilon} = \Delta_{B,\epsilon}/B_2$. Let

$$\pi_2', \pi_2'' \subset \beta_2 = \beta\big(\{E_{i2}\}, \{\delta_{i2}\}, \Delta_{A_2,\epsilon} \vee \Delta_{B_2,\epsilon}\big) \subset \beta$$

be a division of $[a_2, b_2]$, and let $\pi_k \subset \beta_k = \beta\big(\{E_{ik}\}, \{\delta_{ik}\}, \Delta_{A_k,\epsilon}\big) \subset \beta$ be a division of $[a_k, b_k]$, $k = 1, 3$. Then $\pi_1 \cup \pi_2' \cup \pi_3$ and $\pi_1 \cup \pi_2'' \cup \pi_3$ are divisions of $[a, b]$. It follows that

$$\big|s(f;\pi_2') - s(f;\pi_2'')\big| = \big|s(f;\pi_1 \cup \pi_2' \cup \pi_3) - s(f;\pi_1 \cup \pi_2'' \cup \pi_3)\big| <$$
$$< \big|s(f;\pi_1 \cup \pi_2' \cup \pi_3) - I\big| + \big|s(f;\pi_1 \cup \pi_2'' \cup \pi_3) - I\big| < 2\epsilon\,.$$

By Lemma 7.2$'$, f is $[\mathcal{SR}]$ integrable on $[a_2, b_2] = [c, d]$. □

Lemma 7.4$'$ (A quasi Saks-Henstock Lemma)**.** *Let $\mathcal{S} = \{\mathcal{S}(x)\})x \in \mathbb{R}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering on $[a, b]$. Let $f : \mathbb{R} \to \mathbb{R}$ be $[\mathcal{SR}]$ integrable to I on $[a, b]$. Let $F : \mathbb{R} \to \mathbb{R}$,*

$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ [\mathcal{SR}]\int_a^x f(t)\,dt & \text{if } x \in (a, b) \\ I & \text{if } x \ge b \end{cases}$$

Let $\{E_i\}$ be a closed $[a, b]$-form, and let $\delta_i : E_i \to (0, +\infty)$ be given for $\epsilon > 0$ and I. Let A_o be a fixed countable subset of $[a, b]$, $A_o \supseteq \cup_{i=1}^{\infty} Is(E_i)$, and let B be another countable subset of $[a, b]$ containing A_o. Let $\Delta_{A_o,\epsilon}$ and $\Delta_{B,\epsilon}$ be such that

$$\big|s(f;\pi) - I\big| < \epsilon\,,$$

whenever $\pi \subset \beta = \beta\big(\{\{E_i\}, \{\delta_i\}, \Delta_{A_o,\epsilon} \vee \Delta_{B,\epsilon}\big)$ is a division of $[a, b]$. Then we have:

(i) $\big|s(f;\alpha) - S(F;\pi)\big| < \frac{3\epsilon}{2}$ *whenever $\alpha \subset \beta^* = \beta\big(\{E_i\}, \{\delta_i\}, \Delta_{A_o,\epsilon}\big)$ is a finite packing;*

(ii) $\sum_{(\langle x,y\rangle,x)\in\alpha}\big|f(x)(y - x) - (F(y) - F(x))\big| < 3\epsilon$ *whenever $\alpha \subset \beta^*$ is a finite packing.*

PROOF. (i) Let $\alpha \subset \beta^*$ be a finite packing. If α is a division of $[a, b]$ then we have nothing to prove. Suppose that $[a, b] - \sigma(\alpha) \ne \emptyset$. Let (c_k, d_k), $k = 1, 2, \dots, n$ be the components of the open set $(a, b) \setminus \sigma(\alpha)$. By Lemma 7.3$'$ we have

$$[\mathcal{SR}]\int_{c_k}^{d_k} f(t)\,dt = F(d_k) - F(c_k) = I_k\,.$$

For $\frac{\epsilon}{2n}$ and I_k, let $\{P_i^k\}$ be a closed $[c_k,d_k]$-form, $k=1,2,\dots,n$, and $\eta_i^k : P_i^k \to (0,+\infty)$ be given by Definition 7.1$'$. Let $E_i^k = [c_k,d_k]\cap E_i$, $k=1,2,\dots,n$, $\delta_i^k : E_i^k \to (0,+\infty)$, $\delta_i^k = \delta_{i|E_i^k}$. Let B_k be a countable subset of $[c_k,d_k]$ containing $\cup_{i,j}\mathrm{Is}(P_i^k \cap E_j^k)$. Again by Definition 7.1$'$, for B_k and $\frac{\epsilon}{2n}$ there is a $\Delta_{B_k,\frac{\epsilon}{2n}}$ such that

$$\left|s(f;\pi_k) - I_k\right| < \frac{\epsilon}{2n}\,,$$

whenever $\pi_k \subset \beta\big(\{P_i^k\},\{\eta_i^k\},\Delta_{B_k,\frac{\epsilon}{2n}}\big)$ is a division of $[c_k,d_k]$. Let $B = A_o \cup \big(\cup_{k=1}^n B_k\big)$ and let $\Delta_{B,\epsilon}$ be such that

$$\left|s(f;\pi) - I\right| < \epsilon\,,$$

whenever $\pi \subset \beta = \beta\big(\{E_i\},\{\delta_i\},\Delta_{A,\epsilon}\vee\Delta_{B,\epsilon}\big)$ is a division of $[a,b]$. Let

$$\pi_k^* \subset \beta\Big(\{P_i^k \cap E_j^k\},\{\delta_{ij}^k\},(\Delta_{B,\epsilon})_{|B_k}\cap\Delta_{B_k,\frac{\epsilon}{2n}}\Big)$$

be a division of $[c_k,d_k]$ (see Lemma 7.2$'$). Clearly $\alpha\cup\big(\cup_{k=1}^n \pi_k^*\big)\subset\beta$ is a division of $[a,b]$. Hence

$$\left|s(f;\alpha\cup(\cup_{k=1}^n\pi_k^*)) - I\right| < \epsilon \quad\text{and}\quad \left|s(f;\pi_k^*) - I_k\right| < \frac{\epsilon}{2n}\,.$$

Since $I = S(F;\alpha) + \sum_{k=1}^n S(F;\pi_k^*) = S(F;\alpha) + \sum_{k=1}^n I_k$, it follows that

$$\left|s(f;\alpha) - S(F;\alpha)\right| = \left|s\big(f;\alpha\cup(\cup_{k=1}^n\pi_k^*)\big) - I - \sum_{k=1}^n\big(s(f;\pi_k^*) - I_k\big)\right| \le$$

$$\le \left|s\big(f;\alpha\cup(\cup_{k=1}^n\pi_k^*)\big) - I\right| + \sum_{k=1}^n\left|s(f;\pi_k^*) - I_k\right| < \epsilon + n\cdot\frac{\epsilon}{2n} = \frac{3\epsilon}{2}\,.$$

(ii) This follows by definitions using (i). □

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