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Vasile Ene,* Lohmühlenweg 34a, 63571 Gelnhausen, Germany.
e-mail: gabrielaene@hotmail.com†

ON THE T-INTEGRATION OF KARTÁK AND MAŘÌK

Abstract

General notions of integration have been introduced by Saks [25, p. 254], Karták [14, p. 482], Kubota [17, p. 389] and Sarkhel [28, p. 299]. Karták's T-integration was further studied by Karták and Mařì in [15], and by Kubota in [18].

In this paper, starting from Kartak and Mařì's definition, we introduce another general integration (see Definition 3.2), that allows a very general theorem of dominated convergence (see Theorem 3.1). Then we present a general definition for primitives, and this definition contains many of the known nonabsolutely convergent integrals: the Denjoy*-integral, the α-Ridder integral, the wide Denjoy integral, the β-Ridder integral, the Foran integral, the AF integral, the Gordon integral. Using this integration and Theorem 3.1, we obtain a generalization of a result on differential equations, of Bullen and Vyborny [5].

We further give a Banach-Steinhaus type theorem, a categoricity theorem, Riesz type theorems (as a particular case we obtain the Alexiewicz Theorem [1]), and study the weak convergence for the T-integration.

1 Essentially Bounded Variation and the Bounded Slope Variation

We denote by $m(A)$ Lebesgue measure of A, whenever $A \subseteq \mathbb{R}$ is Lebesgue measurable. For the definitions of VB, AC AC^*G and Lusin's condition (N), see [25]. Let χ_E denote the characteristic function of the set E.

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Definition 1.1 (Preiss)**.** ([24] or [8, p. 35]). Let $F : [a, b] \to \mathbb{R}$. F is said to be lower *internal**, if $F(x+) \geq F(x)$, whenever $x \in [a, b)$ and $F(x+)$ exists, and $F(x-) \leq F(x)$, whenever $x \in (a, b]$ and $F(x-)$ exists. F is said to be upper *internal** if $-F$ is lower *internal**. F is said to be *internal** if it is simultaneously upper and lower *internal**.

Definition 1.2. ([23]). Let $P \subset [a, b]$ be a set of positive measure, and let $f : P \to \overline{\mathbb{R}}$ be a measurable function, finite *a.e.* .

- f is said to be essentially upper bounded if there exists a real number M such that the set $\{x \in P : f(x) > M\}$ has measure zero.
- f is said to be essentially lower bounded if the function $-f$ is essentially upper bounded.
- f is said to be essentially bounded if it is simultaneously essentially upper bounded and essentially lower bounded; i.e., there exists $M > 0$ such that the set $\{x \in P : |f(x)| > M\}$ is of measure zero.
- Let $\sup_{ess}(f; P) = \inf\{M : M$ is given by the fact that f is essentially upper bounded$\}$ and $\sup_{ess}(f; P) = +\infty$ if f is not essentially upper bounded. Similarly we define $\inf_{ess}(f; P)$.
- Let $\mathcal{O}_{ess}(f; P) = \sup_{ess}(f; P) - \inf_{ess}(f; P)$.
- Let $\mathcal{O}_{ess}(f; X) = 0$, whenever X is a null subset of P.
- f is said to be of essentially bounded variation (short $f \in EVB$) on P, if there exists $M > 0$ such that $\sum_{i=1}^{n} \mathcal{O}_{ess}\big(f; [a_i, b_i] \cap P\big) < M$ whenever $[a_i, b_i], i = 1, 2, \ldots, n$ are nonoverlapping closed intervals with the endpoints in P.
- Let $EV(f; P) = \inf\{M : M$ is given by the fact that $f \in EVB$ on $P\}$, and let $EV(f; P) = +\infty$ if $f \notin EVB$ on P.
- Let $V(f; P) = \inf\{M : M$ is given by the fact that $f \in VB$ on $P\}$ and let $V(f; P) = +\infty$ if $f \notin VB$ on P.

Lemma 1.1. ([9]). *Let $f : [a, b] \to \overline{\mathbb{R}}$ be a measurable function. The following assertions are equivalent:*

(i) $f \in EVB$ *on* $[a, b]$,

(ii) There exists $\tilde{f} : [a, b] \to \mathbb{R}$, *such that* $\tilde{f} \in VB$ *and* $\tilde{f} = f$ *a.e. on* $[a, b]$. *Moreover* $EV\big(f; [a, b]\big) \leq V\big(\tilde{f}; [a, b]\big) \leq 2 \cdot EV\big(f; [a, b]\big)$.

Lemma 1.2. *Let P be a set of positive finite measure, and let $g : P \to \overline{\mathbb{R}}$ be a measurable function, which is finite a.e. on P. If g is not essentially upper (respectively lower) bounded on P then there exists a function $f : P \to \mathbb{R}$ such that:*

(i) f is summable on P;

(ii) $f \cdot g \geq 0$ on P;

(iii) $f \cdot g$ is not summable on P.

Proof. Suppose for example that g is not essentially upper bounded on P. For $\alpha, \beta \in \mathbb{R}$, we let $E_\alpha(g) = \{x \in P : g(x) \geq \alpha\}$, $E_\alpha^\beta(g) = \{x \in P : \alpha \leq g(x) < \beta\}$ and $E_\infty(g) = \{x \in P : |g(x)| = +\infty\}$. Clearly $|E_\infty(g)| = 0$. We show that there exists a strictly increasing sequence of positive integers $\{n_i\}_{i=1}^\infty$, such that

$$\left|E_{n_i}^{n_{i+1}}(g)\right| > 0, \quad i = 1, 2, \dots . \tag{1}$$

Let $n_1 = 1$. Then $E_{n_1}(g)$ has positive measure. Since

$$E_{n_1}(g) \setminus E_\infty(g) = \cup_{n=n_1+1}^\infty E_{n_1}^n(g)\,,$$

it follows that there exists $n_2 > n_1$ such that $E_{n_1}^{n_2}(g)$ has positive measure and $|E_{n_2}(g)| > 0$ (because g is not essentially upper bounded on P). Continuing in this way, we obtain (1). Let $\alpha_i = |E_{n_i}^{n_i+1}(g)|$ and let β_i be such that $\alpha_i \cdot n_i \cdot \beta_i = 1/i$, $i = 1, 2, \dots$. Let $f : P \to \mathbb{R}$,

$$f(x) = \begin{cases} \beta_i, & x \in E_{n_i}^{n_i+1}(g),\ i = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

(i) We have

$$(\mathcal{L}) \int_P f(t)\, dt = \sum_{i=1}^\infty \beta_i \cdot \alpha_i = \sum_{i=1}^\infty \frac{1}{n_i \cdot i} \leq \sum_{i=1}^\infty \frac{1}{i^2} < +\infty\,.$$

Hence f is summable on P.

(ii) We have $f(x) \cdot g(x) \geq \beta_i \cdot n_i$, if $x \in E_{n_i}^{n_i+1}$, $i = 1, 2, \dots$, and $f(x) \cdot g(x) = 0$ otherwise.

(iii) We have

$$(\mathcal{L}) \int_P f(t) \cdot g(t)\, dt \geq \sum_{i=1}^\infty \alpha_i \cdot n_i \cdot \beta_i = \sum_{i=1}^\infty \frac{1}{i} = +\infty\,.$$

Hence $f \cdot g$ is not summable on P. □

Lemma 1.3 (Sargent)**.** ([26]). *Let $g : [a,b] \to \overline{\mathbb{R}}$ be an essentially bounded, measurable function. If $g \notin EVB$ on $[a,b]$ then there exists $[\alpha,\beta] \subseteq [a,b]$ and a function $f : [\alpha,\beta] \to \mathbb{R}$ such that:*

- *f is Denjoy*-integrable (short $\mathcal{D}^*$-integrable) on $[\alpha,\beta]$;*
- *either $f \cdot g$ is summable on $[\alpha,x]$ whenever $x \in (\alpha,\beta)$, but*

$$\lim_{x\to\beta}(\mathcal{L})\int_\alpha^x f(t)g(t)\,dt = +\infty,$$

or $f \cdot g$ is summable on $[x,\beta]$ whenever $x \in (\alpha,\beta)$, but

$$\lim_{x\to\alpha}(\mathcal{L})\int_x^\beta f(t)g(t)\,dt = +\infty.$$

Proof. Let $J_o = [a,b]$. Since $g \notin EVB$ on J_o, it follows that $g \notin EVB$ on at least one of the intervals, $[a,(a+b)/2]$ or $[(a+b)/2,b]$. Denote this interval by $J_1 = [a_1,b_1]$. Continuing, we obtain a sequence of closed intervals $\{J_n\}_n$, $J_n = [a_n,b_n]$ such that $b_n - a_n = (b-a)/2^n$ and $g \in EVB$ on no J_n. Let $\{c\} = \cap_{n=1}^\infty J_n$, $J'_n = [a_n,c]$ and $J''_n = [c,b_n]$. Then there exist infinitely many subscripts n such that $g \notin EVB$ on J'_n for example. We may suppose without loss of generality that $g \in EVB$ on no J'_n, for no $n = 0,1,\dots$. Let $M = \sup_{ess}(g;[a,b])$ and $m = \inf_{ess}(g;[a,b])$. Because $g \notin EVB$ on $[a,b]$, $M - m > 0$. Since $g \notin EVB$ on J'_o, there exists a partition π_o of J'_o such that $\sum_{I\in\pi_o}\mathcal{O}_{ess}(g;I) > 3(M-m)$. Let $\pi'_o = \pi_o \setminus \{c\}$. Then

$$\sum_{I\in\pi'_o}\mathcal{O}_{ess}(g;I) > 2(M-m)\,.$$

Let I'_o be the last interval of the partition π_o. Then I'_o contains an interval J'_{n_1}; so $g \notin EVB$ on I'_o (because $g \notin EVB$ on J'_{n_1}). It follows that there exists a partition π_{n_1} of I'_o such that $\sum_{I\in\pi_{n_1}}\mathcal{O}_{ess}(g;I) > 3(M-m)$. Let I'_{n_1} be the last interval of the partition π_{n_1}. Let $\pi'_{n_1} = \pi_{n_1} \setminus \{c\}$. Then $\sum_{I\in\pi'_{n_1}}\mathcal{O}_{ess}(g;I) > 2(M-m)$. Continuing, we obtain a sequence of partitions $\{\pi'_{n_k}\}_k$ such that $\sum_{I\in\pi'_{n_k}}\mathcal{O}_{ess}(g;I) > 2(M-m)$, for each k. Let $x_1 < x_2 < x_3 < \dots < c$ be the endpoints of all intervals contained in $\cup_{k=0}^\infty \pi'_{n_k}$. We obtain that $\sum_{n=1}^\infty \mathcal{O}_{ess}\big(g;[x_n,x_{n+1}]\big) = +\infty$. Let $[\alpha,\beta] = [x_1,c]$. Let $M_n = \sup_{ess}\big(g;[x_n,x_{n+1}]\big)$ and $m_n = \inf_{ess}\big(g;[x_n,x_{n+1}]\big)$. Now the proof continues as in [20, p. 78], (see also [6, p. 46]).

Corresponding to each n, there exist distinct measurable subsets X_n and Y_n of $[x_n, x_{n+1}]$ such that $|X_n| = |Y_n| = \delta_n > 0$, $g(x) \geq (3/4)M_n + (1/4)m_n$ for $x \in X_n$, and $g(x) \leq (1/4)M_n + (3/4)m_n$ for $x \in Y_n$. Let

$$p_n = \frac{1}{\delta_n \cdot \sum_{i=1}^{n}(M_i - m_i)}$$

and

$$f(x) = \begin{cases} p_n & \text{for } x \in X_n,\ n = 1, 2, \dots \\ -p_n & \text{for } x \in Y_n,\ n = 1, 2, \dots \\ 0 & \text{elsewhere} \end{cases}$$

Clearly f is summable on each $[x_n, x_{n+1}]$ and $(\mathcal{L}) \int_{x_n}^{x_{n+1}} f(t)\, dt = 0$. For $u \in (x_n, x_{n+1}]$ we have

$$\left|(\mathcal{L}) \int_{x_n}^{u} f(t)\, dt\right| \leq (\mathcal{L}) \int_{x_n}^{x_{n+1}} \bigl|f(t)\bigr|\, dt \leq 2p_n\delta_n \to 0,\ n \to \infty.$$

Let

$$F(x) = \begin{cases} 0 & \text{for } x = \alpha \\ (\mathcal{L}) \int_{x_n}^{x} f(t)\, dt & \text{for } x \in [x_n, x_{n+1}],\ n = 1, 2, \dots \\ 0 & \text{for } x = \beta \end{cases}$$

Clearly f is $\mathcal{D}^*$-integrable on $[\alpha, \beta]$. Since f is summable on $[x_n, x_{n+1}]$ and g is essentially bounded, it follows that $f \cdot g$ is summable on $[x_n, x_{n+1}]$ and

$$\begin{aligned}(\mathcal{L}) \int_{x_n}^{x_{n+1}} f(t)g(t)\, dt &= (\mathcal{L}) \int_{X_n} f(t)g(t)\, dt + (\mathcal{L}) \int_{Y_n} f(t)g(t)\, dt \\ &= p_n \cdot (\mathcal{L}) \int_{X_n} g(t)\, dt - p_n (\mathcal{L}) \int_{Y_n} g(t)\, dt \\ &\geq \frac{p_n\delta_n}{2}(M_n - m_n) = \frac{1}{2} \cdot \frac{M_n - m_n}{\sum_{i=1}^{n}(M_i - m_i)} = r_n.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(M_n - m_n) = +\infty$, it follows that $\sum_{n=1}^{\infty} r_n = +\infty$ (see for example [20, p. 79]). We have

$$\begin{aligned}(\mathcal{L}) \int_{x_n}^{x_{n+1}} \bigl|f(t)g(t)\bigr|\, dt &\leq (M - m) \cdot (\mathcal{L}) \int_{x_n}^{x_{n+1}} \bigl|f(t)\bigr|\, dt \\ &\leq 2p_n\delta_n(M - m) \to 0.\end{aligned} \tag{2}$$

Let $\gamma_n = (\mathcal{L}) \int_{x_n}^{x_{n+1}} f(t)g(t)\, dt$. Then

$$\sum_{n=1}^{\infty} \gamma_n \geq \sum_{n=1}^{\infty} r_n = +\infty. \tag{3}$$

Let $G : [\alpha, \beta) \to \mathbb{R}$,

$$G(x) = (\mathcal{L}) \int_\alpha^x f(t)g(t)\, dt,\ x \in [x_n, x_{n+1}],\ n = 1, 2, \dots\ .$$

We observe that

$$G(x) = \sum_{i=1}^{n-1} \gamma_i + (\mathcal{L}) \int_{x_n}^x f(t)g(t)\, dt \quad \text{on} \quad [x_n, x_{n+1}],\ n \geq 2\,.$$

By (2) and (3) it follows now that $\lim_{x\to\beta} G(x) = +\infty$. □

Definition 1.3. ([9]). A function $F : [a, b] \to \mathbb{R}$ is said to be of bounded slope variation (short $F \in BSV$) on a subset P of $[a, b]$, if there exists $M > 0$ such that

$$\sum_{i=1}^{n} \left| \frac{F(b_{2i}) - F(a_{2i})}{b_{2i} - a_{2i}} - \frac{F(b_{2i-1}) - F(a_{2i-1})}{b_{2i-1} - a_{2i-1}} \right| < M,$$

whenever $a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_{2n} < b_{2n}$ are points in P. Let $SV(F; P) = \inf\{M : M$ is given by the fact that $F \in BSV$ on $P\}$. If $F \notin BSV$ on P. let $SV(F; P) = +\infty$.

Theorem 1.1. ([9]). *With the above notations we have the following results:*

(i) Let $f : [a, b] \to \mathbb{R}$, $f \in EVB$ and let $F(x) = (\mathcal{L}) \int_a^x f(t)\, dt$. Then $F \in BSV$ on $[a, b]$ and $SV\big(F; [a, b]\big) \leq EV\big(f; [a, b]\big)$.

(ii) Let $F : [a, b] \to \mathbb{R}$, $F \in BSV$ and let

$$F^*(x) = \begin{cases} F'(x) & \text{where } F \text{ is derivable} \\ 0 & \text{elsewhere} \end{cases}$$

Then F satisfies the Lipschitz condition, $F^* \in EVB$ on $[a, b]$, and $EV\big(F^*; [a, b]\big) \leq SV\big(F; [a, b]\big)$.

Remark 1.1.

(i) If f is essentially bounded on $[a, b]$, then $F(x) = (\mathcal{L}) \int_a^x f(t)\, dt$ is a Lipschitz function on $[a, b]$ and $F^{'} = f$ *a.e.* .

(ii) If $F : [a, b] \to \mathbb{R}$ is a Lipschitz function, then F^* is essentially bounded on $[a, b]$ and $F(x) = (\mathcal{L}) \int_a^x F^*(t)\, dt$ (for F^* see Theorem 1.1).

2 The T-integration of Karták and Mařì

Definition 2.1 (Karták and Mařì)**.** ([15], [14], [18], [19]). Let T be a functional by which there corresponds to each closed interval $J \subset I$ a linear space $\mathcal{K}(T, J)$ of real valued measurable functions defined on J, and to each function f of $\mathcal{K}(T, J)$ a real number $T(f, J)$. A functional T is called an integration (respectively a wide integration) on I if the following conditions are fulfilled:

(a) The functional $T(f, J)$ is linear on $\mathcal{K}(T, J)$.

(b) If $f \in \mathcal{K}(T, J)$, $J^{'} \subset J$, then $f \in \mathcal{K}(T, J^{'})$.

(c) If f is Lebesgue integrable (respectively f is Lebesgue integrable and essentially bounded) on J, then $f \in \mathcal{K}(T, J)$ and $T(f, J) = (\mathcal{L}) \int_J f$.

(d) If J_1 and J_2 are abutting intervals and if $f \in \mathcal{K}(T, J_1) \cap \mathcal{K}(T, J_2)$, then $f \in \mathcal{K}(T, J_1 \cup J_2)$ and $T(f, J_1 \cup J_2) = T(f, J_1) + T(f, J_2)$.

(e) If $f \in \mathcal{K}(T, J)$, $f \geq 0$, then f is Lebesgue integrable on J.

(f) If $f \in \mathcal{K}(T, J)$, $J = [\alpha, \beta]$, then $F(x) = T\big(f, [\alpha, x]\big)$ is continuous on J, where $F(\alpha) = 0$.

Let T be an integration (respectively a wide integration) on I. A function f in $\mathcal{K}(T, J)$ is said to be T-integrable (respectively wide T-integrable) on J. Given two integrals (respectively wide integrals) T_1 and T_2 on I, T_2 includes T_1, written $T_1 \subset T_2$, if $f \in \mathcal{K}(T_2, J)$ and $T_1(f, J) = T_2(f, J)$, whenever $f \in \mathcal{K}(T_1, J)$ and $J \subset I$.

Lemma 2.1 (Karták and Mařì)**.** ([15, p. 746]). *There exist an integration T, a function $f \in \mathcal{K}(T, I)$ and $g \in AC$ on the closed interval I such that $f \cdot g \notin \mathcal{K}(T, I)$.*

Lemma 2.1 leads us to the following definition.

Definition 2.2. Let T be a wide integration on $I = [a, b]$, satisfying the following conditions:

(i) $f \cdot g \in \mathcal{K}(T, I)$, whenever $f \in \mathcal{K}(T, I)$ and $g \in VB$,

(ii) $T(f \cdot g, I) = F(b)g(b) - (\mathcal{RS}) \int_a^b F(x)\, dg(x)$, where $F(x) = T(f, [a, x])$, $x \in [a, b]$, $F(a) = 0$, whenever $f \in \mathcal{K}(T, I)$ and $g \in VB$ (here $(\mathcal{RS})$ denotes the Riemann Stieltjes integral). Let $\langle f|g\rangle = T(f \cdot g, I)$.

We shall not make distinction between f and g belonging to $\mathcal{K}(T, I)$ if $f = g$ *a.e.* . We define the following real normed spaces:

- $\big(\mathcal{K}(T,I),\|\cdot\|\big)$, where $\|f\|=\|F\|_\infty=\sup\big\{|F(x)|:x\in[a,b]\big\}$;
- $\big(VB,\|\cdot\|_{VB}\big)$, where $\|g\|_{VB}=\big|g(b)\big|+V\big(g,[a,b]\big)$. (This is in fact a Banach space).

Example 2.1. Some particular wide integrals which satisfy Definition 2.2 are:

1. the $S\mathcal{F}$-integral (see [8, pp. 210-211]);
2. the Foran integral (see [10] or [8, p. 208]),
3. the Denjoy and Denjoy* integrals (see [6, pp. 31-34]),
4. the Lebesgue integral (because the product of a VB function and a Lebesgue integrable function is still a Lebesgue integrable function),
5. the Lebesgue integral restricted to essentially bounded functions (because the product of a VB function and an essentially bounded function is still an essentially bounded function).

3 A General Notion of Integration

Definition 3.1 (Sarkhel)**.** ([28]) By $f\dot{:}A\to\overline{\mathbb{R}}$ we mean a function with values in $\overline{\mathbb{R}}$, whose domain contains almost all points of the set A such that f is finite almost everywhere on A.

Let $\mathcal{L}_{comp}=\{f:\mathbb{R}\to\mathbb{R}:\mathrm{supp}(f)$ is compact and f is Lebesgue integrable$\}$.

Starting from Definition 2.1, we introduce the following general integration.

Definition 3.2. Let $\mathcal{A}=\{(f,I):I$ is a compact interval, $f\dot{:}I\to\overline{\mathbb{R}},\ f$ is measurable on $I\}$. A mapping $\mathcal{J}:\mathcal{A}_o\to\mathbb{R}$, $\mathcal{A}_o\subset\mathcal{A}$ is said to be an integral if the following conditions are fulfilled:

(a) If $(f,I)\in\mathcal{A}$, f is Lebesgue integrable on I, $(g,I)\in\mathcal{A}_o$ and $\alpha,\beta\in\mathbb{R}$, then $(\alpha f+\beta g,I)\in\mathcal{A}_o$ and $\mathcal{J}(\alpha f+\beta g,I)=\alpha\cdot(\mathcal{L})\int_I f(t)\,dt+\beta\cdot\mathcal{J}(g,I)$.

(b) $(f,J)\in\mathcal{A}_o$ whenever $(f,I)\in\mathcal{A}_o$ and $J\subseteq I$.

(c) If (f,I) and (g,I) belong to $\mathcal{A}_o$ and $f\geq g$ *a.e.* on I then $f-g$ is Lebesgue integrable on $[a,b]$.

(d) If $(f,[a,b])$ and $(f,[b,c])$ belong to $\mathcal{A}_o$, then $(f,[a,c])\in\mathcal{A}_o$ and $\mathcal{J}(f,[a,b])+\mathcal{J}(f,[b,c])=\mathcal{J}(f,[a,c])$.

Let $\mathcal{J}$ be an integral. Then f is said to be $\mathcal{J}$-integrable on $[a,b]$ if $(f,[a,b]) \in \mathcal{A}_o$. In this case the function $F : [a,b] \to \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x = a \\ \mathcal{J}(f,[a,x]) & \text{if } x = (a,b] \end{cases}$$

is called the indefinite $\mathcal{J}$-integral of f on $[a,b]$. Clearly F is well defined and $\mathcal{J}(f,[c,d]) = F(d) - F(c)$ whenever $[c,d] \subseteq [a,b]$ (see (b) and (d)). A function $G : [a,b] \to \mathbb{R}$ of the form $G(x) = F(x) + \alpha$, $\alpha \in \mathbb{R}$ is called a $\mathcal{J}$-primitive of f on $[a,b]$. Let $(\mathcal{J}) \int_a^x f(t)\,dt := \mathcal{J}(f;[a,x])$.

Lemma 3.1. *Let $\mathcal{J} : \mathcal{A}_o \to \mathbb{R}$ be an integral as above. If $f : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$, then $(f,[a,b]) \in \mathcal{A}_o$ and*

$$(\mathcal{J}) \int_a^b f(t)\,dt = (\mathcal{L}) \int_a^b f(t)\,dt\,.$$

Proof. Let $(g,[a,b]) \in \mathcal{A}_o$. By Definition 3.2, (a),

$$(\mathcal{J}) \int_a^b (1f + 0g)(t)\,dt = 1(\mathcal{L}) \int_a^b f(t)\,dt + 0(\mathcal{J}) \int_a^b g(t)\,dt.$$

Hence $(\mathcal{J}) \int_a^b f(t)\,dt = (\mathcal{L}) \int_a^b f(t)\,dt$. □

Definition 3.3. ([22, p. 151]). Let $\mathcal{M} = \{f\}$ be a family of Lebesgue integrable functions defined on a set P. If for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|(\mathcal{L}) \int_A f| < \epsilon$ for all $f \in \mathcal{M}$, whenever $A \subset P$, $m(A) < \delta$ then the functions of $\mathcal{M}$ are said to have equi-absolutely continuous integrals.

Lemma 3.2. *Let $\{f_n\}_n$ be a sequence of nonnegative Lebesgue integrable functions, converging in measure to a function f defined on a measurable set P. The following assertions are equivalent:*

- *(i) f is Lebesgue integrable and $\lim_{n\to\infty} (\mathcal{L}) \int_P f_n = (\mathcal{L}) \int_P f$,*
- *(ii) The functions of the sequence $\{f_n\}_n$ have equi-absolutely continuous integrals.*

Proof. $(i) \Rightarrow (ii)$ By Theorem 5 of [22, p. 157] we have $\lim_{n\to\infty} (\mathcal{L}) \int_A f_n = (\mathcal{L}) \int_A f$ whenever A is a measurable subset of P. Now (ii) follows by [22] (Corollary 1, p. 156 of Theorem 3, p. 153).

$(ii) \Rightarrow (i)$ See Vitali's Theorem 2 of [22, p. 152]. □

Theorem 3.1. *Let $\mathcal{J}$ be an integral as in Definition 3.2.*

(i) *If f is measurable and $\big(|f|, I\big) \in \mathcal{A}_o$, then $f \in \mathcal{L}_{comp}$.*

(ii) *If $(f, I) \in \mathcal{A}_o$ and $g = f$ a.e. then $(g, I) \in \mathcal{A}_o$ and $(\mathcal{J})\int_I f = (\mathcal{J})\int_I g$.*

(iii) *If $(f, I), (g, I) \in \mathcal{A}_o$ and $f \le g$ a.e., then $(\mathcal{J})\int_I f \le (\mathcal{J})\int_I g$.*

(iv) *If $(g, I), (h, I) \in \mathcal{A}_o$, $\{f_n\}_n$ is a sequence of measurable functions on R, $g \le f_n \le h$, a.e. and $f_n \to f$ (f_n converges in measure to f), then $(f_n, I), (f, I) \in \mathcal{A}_o$ and $(\mathcal{J})\int_I f = \lim_{n\to\infty}(\mathcal{J})\int_I f_n$.*

(v) *If $(f, I) \in \mathcal{A}_o$, $g \in \mathcal{L}_{comp}$ and $f \ge g$ a.e., then $f \in \mathcal{L}_{comp}$ and $(\mathcal{J})\int_I f = (\mathcal{L})\int_I f$.*

(vi) *Let $\{f_n\}_n$ be a sequence of functions on I having the following properties:*

(1) *$(f_n, I) \in \mathcal{A}_o$ for each n,*

(2) *there exists g, with $(g, I) \in \mathcal{A}_o$, such that $f_n \ge g$ a.e. for each n,*

(3) *$\{f_n\}$ converges in measure to f.*

Then

(a) *each $f_n - g \in \mathcal{L}_{comp}$ and $(\mathcal{L})\int_I (f_n - g) + (\mathcal{J})\int_I g = (\mathcal{J})\int_I f_n$;*

(b) *$(f, I) \in \mathcal{A}_o$ if and only if $f - g \in \mathcal{L}_{comp}$;*

(c) *$(f, I) \in \mathcal{A}_o$ and $\lim_{n\to\infty}(\mathcal{J})\int_I f_n = (\mathcal{J})\int_I f$ if and only if the functions of the sequence $\{f_n - g\}_n$ have equi-absolutely continuous integrals.*

(vii) *Let $\{(f_n, I)\}_n \subset \mathcal{A}_o$, $f_1 \le f_2 \le \ldots \le f_n \le \ldots$ a.e. , and $f_n \to f$ a.e. . Then $(f, I) \in \mathcal{A}_o$ if and only if $\lim_{n\to\infty}(\mathcal{J})\int_I f_n \ne +\infty$. In this case we have $\lim_{n\to\infty}(\mathcal{J})\int_I f_n = (\mathcal{J})\int_I f$.*

(viii) *Let g be a measurable function on I. If $(fg, I) \in \mathcal{A}_o$ whenever f is a Lebesgue integrable function on I,, then g is essentially bounded on I.*

(ix) *Suppose that $F(x) = (\mathcal{J})\int_\alpha^x f(t)\,dt$ is internal* on J, whenever $(f, J) \in \mathcal{A}_o$ and $J = [\alpha, \beta]$. Let g be a measurable function on I. If the $\mathcal{J}$-integral contains the $\mathcal{D}^*$-integral and $(f \cdot g, I) \in \mathcal{A}_o$ whenever $f \in \mathcal{D}^*$, then g equals a VB function a.e. on I.*

Proof. (i) By Definition 3.2, (a), $(0, I) \in \mathcal{A}_o$. Since $|f| \ge 0$ *a.e.* on I, by Definition 3.2, (c), $|f|$ is Lebesgue integrable on I. Therefore so is f.

(ii) Since $g - f = 0$ *a.e.*, it follows that $g - f$ is Lebesgue integrable on I. Because $g = (g - f) + f$ and by Definition 3.2, (a) we obtain that $(g, I) \in \mathcal{A}_o$ and $(\mathcal{L})\int_I (g - f) + (\mathcal{J})\int_I f = (\mathcal{J})\int_I g$. Therefore $(\mathcal{J})\int_I f = (\mathcal{J})\int_I g$.

(iii) By Definition 3.2, (c), we have that $g-f$ is Lebesgue integrable on I and by Definition 3.2, (a), $(\mathcal{L})\int_I(g-f)+(\mathcal{J})\int_I f=(\mathcal{J})\int_I g$. But $(\mathcal{L})\int_I(g-f)\geq 0$. Hence $(\mathcal{J})\int_I f\leq(\mathcal{J})\int_I g$.

(iv) By Definition 3.2, (c) we have that $h-g$ is Lebesgue integrable on I. But $0\leq f_n-g\leq h-g$ *a.e.* and each f_n is measurable. It follows that each f_n-g is Lebesgue integrable and $f_n-g\to f-g$ (convergence in measure). By the Lebesgue Dominated Convergence Theorem, $f-g$ is Lebesgue integrable and $\lim_{n\to\infty}(\mathcal{L})\int_I(f_n-g)=(\mathcal{L})\int_I(f-g)$. Because $f=(f-g)+g$ and by Definition 3.2, (a) we obtain that $(f,I)\in\mathcal{A}_o$,

$$(\mathcal{L})\int_I(f_n-g)+(\mathcal{J})\int_I g=(\mathcal{J})\int_I f_n$$

and

$$(\mathcal{L})\int_I(f-g)+(\mathcal{J})\int_I g=(\mathcal{J})\int_i f,I\,.$$

Therefore $\lim_{n\to\infty}(\mathcal{J})\int_I f_n=(\mathcal{J})\int_I f$.

(v) By Definition 3.2, (a), $(g,I)\in\mathcal{A}_o$, and by Definition 3.2, (c), $f-g$ is Lebesgue integrable on I. It follows that $f=(f-g)+g$ is Lebesgue integrable on I and by Lemma 3.1, $(\mathcal{J})\int_I f=(\mathcal{L})\int_I f$.

(vi) (a) This follows by Definition 3.2, (c), (a).

b) Since $f_n\geq g$ *a.e.* it follows that $f\geq g$ *a.e.* . The assertion follows by Definition 3.2, (a), (c).

c) By (vi), (b) and (a) it follows that the statement $(f,I)\in\mathcal{A}_o$ and $\lim_{n\to\infty}(\mathcal{J})\int_I f_n=(\mathcal{J})\int_I f$ is equivalent to $f-g$ is Lebesgue integrable and $\lim_{n\to\infty}(\mathcal{L})\int_I(f_n-g)=(\mathcal{L})\int_I(f-g)$. Now Lemma 3.2 completes the proof.

(vii) By (iii), $(\mathcal{J})\int_I f_n\leq(\mathcal{J})\int_I f_{n+1}$ for each n. Then $\lim_{n\to\infty}(\mathcal{J})\int_I f_n$ exists (finite or infinite). By Definition 3.2, (c), (a), each f_n-f_1 is Lebesgue integrable on I and

$$(\mathcal{J})\int_I f_n=(\mathcal{L})\int_I(f_n-f_1)+(\mathcal{J})\int_I f_1\,.$$

By the Beppo-Levi Theorem it follows that

$$\lim_{n\to\infty}(\mathcal{J})\int_I f_n=(\mathcal{L})\int_I(f-f_1)+(\mathcal{J})\int_I f_1\,.$$

Therefore $f-f_1$ is Lebesgue integrable if and only if $\lim_{n\to\infty}(\mathcal{L})\int_I(f_n-f_1)$ is finite, and since $f=(f-f_1)+f_1$, it follows that $(f,I)\in\mathcal{A}_o$ and

$$\lim_{n\to\infty}(\mathcal{J})\int_I f_n=(\mathcal{J})\int_I f\,.$$

If $\lim_{n\to\infty}(\mathcal{J})\int_I f_n = +\infty$ then $f - f_1$ is not Lebesgue integrable on I. But $f - f_1 \geq 0$ *a.e.*; so by Definition 3.2, (a), (c), $(f, I) \notin \mathcal{A}_o$.

(viii) Suppose on the contrary that g is not essentially bounded on I. By Lemma 1.2 there exists a function $f : I \to \mathbb{R}$ such that f is Lebesgue integrable, $fg \geq 0$ and fg is not Lebesgue integrable on I. Since $fg \geq 0$, by (v), it follows that fg is Lebesgue integrable, a contradiction.

(ix) By (viii), g is essentially bounded. Suppose on the contrary that $g \notin EVB$ on $[a, b]$ (see Lemma 1.1). Then, by Lemma 1.3, there exist $[\alpha, \beta] \subseteq [a, b]$ and a function $f : [\alpha, \beta] \to \mathbb{R}$ such that f is $\mathcal{D}^*$-integrable on $[\alpha, \beta]$, fg is Lebesgue integrable on $[\alpha, x]$ for example, whenever $x \in (\alpha, \beta)$, and

$$\lim_{x\nearrow\beta}(\mathcal{L})\int_\alpha^x fg = +\infty\,.$$

By Definition 3.2, (c), we obtain that $\lim_{x\nearrow\beta}\mathcal{J}\big(fg, [\alpha, x]\big) = +\infty$ (see Lemma 3.1). This contradicts the hypothesis. □

Remark 3.1. Theorem 3.1, (viii) extends Theorem 12.8 of [20].

4 A Riesz Type Representation Theorem for T-integration

Lemma 4.1. *In the conditions of Definition 2.2, let $g \in VB$ be fixed. Let $L : \mathcal{K}(T, I) \to \mathbb{R}$, $L(f) = \langle f|g\rangle$. Then:*

(i) $\langle\cdot|g\rangle$ is linear.

(ii) $\big|\langle f|g\rangle\big| \leq \|f\| \cdot \|g\|_{VB}$.

(iii) L is a continuous linear functional and $\|L\| \leq \|g\|_{VB}$.

Proof. (i) This follows by Definition 2.1, (a) and Definition 2.2, (i).

(ii) We have

$$\begin{aligned}\big|\langle f|g\rangle\big| &= \big|T(f\cdot g, [a, b])\big| = \Big|F(b)g(b) - (\mathcal{RS})\int_a^b F(x)\,dg(x)\Big|\\ &\leq \big|F(b)\big|\cdot\big|g(b)\big| + \|F\|_\infty \cdot V\big(g, [a, b]\big)\\ &\leq \|F\|_\infty \cdot \big(\big|g(b)\big| + V(g, [a, b]\big) = \|f\|\cdot\|g\|_{VB}\,.\end{aligned}$$

(iii) This follows by (i) and (ii). □

Lemma 4.2. *Let $\big(X, \|\cdot\|_1\big)$ and $\big(Y, \|\cdot\|_2\big)$ be normed real spaces and let $\langle\cdot|\cdot\rangle : X \times Y \to \mathbb{R}$ be such that:*

(1) $\langle \cdot | y \rangle$ *is linear in the first variable, for each* $y \in Y$,

(2) $\big|\langle x|y\rangle\big| \le \|x\|_1 \cdot \|y\|_2$, *whenever* $x \in X$, $y \in Y$.

If $f : X \to \mathbb{R}$ *is a continuous linear functional and if there exist* $y_o \in Y$ *and a dense subset* X_o *of* X *such that* $f(x) = \langle x|y_o\rangle$ *for each* $x \in X_o$, *then* $f(x) = \langle x|y_o\rangle$ *on* X *and* $\|f\| \le \|y_o\|_2$.

PROOF. Since $\overline{X}_o = X$, for $x \in X$ there exists a sequence $\{x_n\}_n \subset X_o$ such that $\|x_n - x\|_1 \to 0$, for $n \to \infty$. But $\big|\langle x_n|y_o\rangle - \langle x|y_o\rangle\big| = \big|\langle x_n - x|y_o\rangle\big| \le \|x_n - x\|_1 \cdot \|y_o\|_2$. Since f is continuous, $f(x_o) = \lim_{n\to\infty}\langle x_n|y_o\rangle = \langle x|y_o\rangle$. Hence $f(x) = \langle x|y_o\rangle$, for each $x \in X$ and $\|f\| \le \|y_o\|_2$. □

Theorem 4.1. *In the conditions of Definition 2.2, let* $L : \mathcal{K}(T, I) \to \mathbb{R}$ *be a continuous linear functional. Then there exists* $g \in VB$ *such that*

$$L(f) = \langle f|g\rangle = T(fg, I) \text{ and} \tag{4}$$

$$EV(g; I) \le \|L\| \le \|g\|_{VB}. \tag{5}$$

PROOF. Let

$$\begin{aligned}\mathcal{S}(I) = \big\{ s : [a,b] \to \mathbb{R} : & \; s \text{ is a step function of the form} \\ & s(t) = \sum_{i=1}^{n-1} \alpha_i \chi_{[t_{i-1}, t_i)} + \alpha_n \chi_{[t_{n-1}, t_n]} \text{ for some positive integer } n, \\ & \text{where each } \alpha_i \in \mathbb{R},\ a = t_0 < t_1 < \ldots < t_n = b] \big\}.\end{aligned}$$

We show that $\overline{\mathcal{S}(I)} = \mathcal{K}(T, I)$. Let $f \in \mathcal{K}(T, I)$. Then $F(x) = T\big(f, [a, x]\big)$ is continuous on $[a, b]$. Let $a = x_0 < x_1 < \ldots < x_n = b$, $x_i - x_{i-1} = (b-a)/n$ for each $i = 1, 2, \ldots, n$. Let $F_n(x_i) = F(x_i)$, $i = 0, 1, \ldots, n$ and let F_n be linear on each closed interval $[x_{i-1}, x_i]$. Then $F_n \to F$ [unif] on $[a, b]$. Let

$$s_n(x) = \begin{cases} \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} & \text{for } x \in [x_{i-1}, x_i),\ i = 1, 2, \ldots, n-1 \\ \frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}} & \text{for } \in [x_{n-1}, x_n] \end{cases}$$

Then $s_n \in \mathcal{S}(I)$ and $\|s_n - f\| = \|F_n - F\|_\infty \to 0$ (because $F_n \to F$ [unif]). Let $G(t) = L(\chi_{[a,t]})$ and let $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_{2n} < b_{2n} \le b$. Since L is linear and continuous, we have

$$\begin{aligned}&\sum_{i=1}^{n} \left| \frac{G(b_{2i}) - G(a_{2i})}{b_{2i} - a_{2i}} - \frac{G(b_{2i-1}) - G(a_{2i-1})}{b_{2i-1} - a_{2i-1}} \right| \\ &= \sum_{i=1}^{n} \big|L(\varphi_i)\big| = \sum_{i=1}^{n} \epsilon_i L(\varphi_i) = L\Big(\sum_{i=1}^{n} \epsilon \varphi_i\Big) \le \|L\| \cdot \Big\|\sum_{i=1}^{n} \epsilon_i \varphi_i\Big\|_1 \le \|L\|\end{aligned}$$

where $\epsilon_i = \text{sign} L(\varphi_i)$ and

$$\varphi_i = \frac{1}{b_{2i} - a_{2i}} \cdot \chi_{(a_{2i}, b_{2i}]} - \frac{1}{b_{2i-1} - a_{2i-1}} \cdot \chi_{(a_{2i-1}, b_{2i-1}]}.$$

It follows that $G \in BSV$ and

$$SV\big(G; [a,b]\big) \leq \|L\|. \tag{6}$$

By Theorem 1.1, (ii) there exists $g = G^* \in EVB$ and

$$EV\big(g, [a,b]\big) \leq SV\big(G; [a,b]\big). \tag{7}$$

Clearly

$$G(t) = (\mathcal{L}) \int_a^t g(x)\, dx = (\mathcal{L}) \int_a^b \chi_{[a,t]}(x) g(x)\, dx = L(\chi_{[a,t]})\,.$$

Since L is linear it follows that $L(s) = \langle s|g\rangle$ whenever $s \in \mathcal{S}(I)$. Then $L(f) = \langle f|g\rangle$ for every $f \in \mathcal{K}(T, I)$ and $\|L\| \leq \|g\|_{VB}$ (see Lemma 4.2). By (7) and (6), $EV\big(g; [a,b]\big) \leq \|L\|$, hence $EV\big(g; [a,b]\big) \leq \|L\| \leq \|g\|_{VB}$. □

Remark 4.1. Particularly, if in Theorem 4.1, T stands for the $\mathcal{D}^*$-integral, then we obtain the Alexiewicz Theorem (see [20, Theorem 12.7]; see also [1]).

5 Banach-Steinhaus Type Theorems for T-integration

Definition 5.1. ([20, p. 67]).

- A sequence $\{X_n\}_n$ of sets in a normed real linear space X is said to be an α-sequence if $0 \in X_1$ and if for every n, $x + y$ and $x - y$ belong to X_{n+1}, whenever $x, y \in X_n$.
- X is called an α-space if $X = \cup_{n=1}^{\infty} X_n$. where $\{X_n\}_n$ is an α-sequence of closed sets each of which being nowhere dense in X.
- A normed real space is said to be a Sargent space or a β-space if it is not an α-space.

Lemma 5.1. ([20, p. 70]). *A normed real linear space X is a Sargent space if and only if for every representation of the form $X = \cup_{n=1}^{\infty} X_n$, where $\{X_n\}_n$ is an α-sequence, there is an X_n for some n which is dense in a ball B of X.*

Lemma 5.2. *Let T be a wide integration on $I = [a, b]$ as in Definition 2.2 satisfying the Cauchy property*

$$\begin{aligned}&If f \in \mathcal{K}\big(T, [\alpha, \beta]\big) for every interval [\alpha, \beta] with a \le c < \alpha < \beta < d \le b\\ &and \lim_{\substack{\alpha\to c\\ \beta\to d}} T\big(f, [\alpha, \beta]\big) = A, then f \in \mathcal{K}\big(T, [c, d]\big) and T\big(f, [c, d]\big) = A.\end{aligned} \tag{C}$$

Then $\big(\mathcal{K}(T, I), \|\cdot\|\big)$ is a Sargent space.

Proof. The proof is similar to that of Example 11.3 of [20, pp. 68-69]. Condition (C) is necessary to show the convergence of the sequence $\{X_n\}_n$ in the proof of Example 11.3. $\square$

Theorem 5.1. (A Banach-Steinhaus type theorem for a Sargent space, [20, Theorem 11.6]). *Let T_n be a sequence of continuous linear operators from a Sargent space X into a normed linear space Y. If $\sup_{n=1}^{\infty} \|T_n(x)\| < +\infty$ for every $x \in X$, then $\sup_{n=1}^{\infty} \|T_n\| < +\infty$.*

Theorem 5.2. (A Banach-Steinhaus type theorem for the T-integral). *Let T be an integration as in Lemma 5.2, containing the $\mathcal{D}^*$-integral. The following assertions are equivalent:*

(i) For every $f \in \mathcal{K}\big(T, [a, b]\big)$ there exists a constant $M(f)$ such that for all n we have $\big|T(fg_n, [a, b])\big| \le M(f)$;.

(ii) There exists $c > 0$ such that $\sup_{ess} |g_n| < c$ and $EV\big(g_n, [a, b]\big) < c$ for all n.

Proof. (i) $\Rightarrow$ (ii) Each function g_n equals a VB function *a.e.* (see Theorem 3.1, (ix)). and is therefore essentially bounded. Let $L_n(f) = T\big(fg_n, [a, b]\big)$ for $f \in \mathcal{K}\big(T, [a, b]\big)$. If f is Lebesgue integrable, then fg_n is also Lebesgue integrable. Hence $L_n(f) = (\mathcal{L}) \int_a^b fg_n$ (see Definition 2.1, (c)). By the Banach-Steinhaus Theorem (see [6, p. 45]) it follows that for some $M_1 > 0$ we have $\sup_{ess} |g_n| < M_1$, for all $n = 1, 2, \dots$. By Theorem 5.1 and Lemma 5.2, there exists $M_2 > 0$ such that $\|L_n\| \le M_2$ for all $n = 1, 2, \dots$ and by Theorem 4.1, $EV\big(g_n, [a, b]\big) \le \|L_n\|$. Therefore $EV\big(g_n; [a, b]\big) < M_2$, for all $n = 1, 2, \dots$. Let $c = \max\{M_1, M_2\}$.

(ii) $\Rightarrow$ (i) For this implication, condition (C) is not needed. By Lemma 1.1, there exists $G_n : [a, b] \to \mathbb{R}$, $G_n \in VB$ such that $G_n = g_n$ *a.e.* and

$$V\big(G_n, [a, b]\big) \le 6V(G_n, A) \le 12EV\big(G_n, [a, b]\big) < 12c$$

(where A is defined in the proof of Lemma 1.1). Since $\sup_{ess} |g_n| < c$, it follows that $\sup |G_n| < 13c$. By Theorem 3.1, (ii) we have that

$$T\big(fg_n, [a, b]\big) = T\big(fG_n, [a, b]\big)\,.$$

Now the proof follows applying Definition 2.2. □

Remark 5.1. Theorem 5.2 is an extension of Theorem 12.10 of [20] or of a lemma of [6, p. 47].

6 The Categoricity of $\mathcal{K}\big(\mathbf{T};[\mathbf{a},\mathbf{b}]\big)$ for Wide T-integration

Theorem 6.1. ([14, p. 511]). *There exist an integration T (as in Definition 2.1) and a function $f \in \mathcal{K}\big(T,[a,b]\big)$ such that the identity $F^{'} = f$ a.e. does not hold, where $F(x) = T\big(f,[a,x]\big)$.*

Lemma 6.1. ([12, p. 49]). *Let (X,τ) be a topological space and let X_o be a dense subset of X. Let $\tau_o = \tau_{/X_o}$. If X_o is of the second category in (X,τ_o), then X_o is of the second category in (X,τ).*

Lemma 6.2 (Jarnik)**.** ([4, p. 213]). *Let $\big(C([a,b]),\|\|_\infty\big)$ and let $\mathcal{A} = \big\{f : [a,b] \to \mathbb{R} : f$ is continuous and f has every extended real number as a derived number at every point$\big\}$. Then $C\big([a,b]\big)\setminus\mathcal{A}$ is of the first category in $C\big([a,b]\big)$.*

Remark 6.1. For a wide T-integration let $\tilde{\mathcal{K}}\big(T,[a,b]\big) = \big\{F : [a,b] \to \mathbb{R} :$ there exists $f \in \mathcal{K}\big(T,[a,b]\big)$ such that $F(x) = T(f,[a,x]),\ \forall\ x \in [a,b]\big\}$ endowed with the norm $\|\cdot\|_\infty$. Then $\mathcal{K}\big(T,[a,b]\big)$ with the norm $\|\cdot\|$ given by Definition 2.2 is isomorphic to $\big(\tilde{\mathcal{K}}(T,[a,b]),\|\cdot\|_\infty\big)$.

Let $C([a,b]) = \{f : [a,b] \to \mathbb{R} : f$ is continuous on $[a,b]\}$. Clearly $\big(\tilde{\mathcal{K}}(T,[a,b]),\|\cdot\|_\infty\big) \subset \big(C([a,b]),\|\cdot\|_\infty\big)$ (see Definition 2.1, (f)). Since each polynomial on $[a,b]$ is a Lipschitz function, and because by the Weierstrass theorem, each function $f \in C\big([a,b]\big)$ is the uniform limit of a sequence of polynomials, it follows that $\tilde{\mathcal{K}}\big(T,[a,b]\big)$ is dense in $\big(C([a,b]),\|\cdot\|_\infty\big)$. Therefore the completion of $\big(\mathcal{K}(T,[a,b]),\|\cdot\|\big)$ is the Banach space $\big(C([a,b]),\|\cdot\|_\infty\big)$.

Theorem 6.2. *Let T be a wide integration on $[a,b]$ which satisfies the hypotheses of Lemma 5.2. If for each $f \in \mathcal{K}\big(T,[a,b]\big)$ the equality $F^{'}(x) = f(x)$ holds on a set of positive measure, where $F(x) = T\big(f;[a,x]\big)$, $x \in [a,b]$), then $\big(\mathcal{K}(T,[a,b]),\|\cdot\|\big)$ is of the first category on itself.*

Proof. Suppose on the contrary that $\big(\mathcal{K}(T,[a,b]),\|\cdot\|\big)$ is of the second category on itself. Since $\overline{\mathcal{K}\big(T,[a,b]\big)} = \overline{\tilde{\mathcal{K}}\big(T,[a,b]\big)} = C\big([a,b]\big)$, by Lemma 6.1 it follows that $\big(\tilde{\mathcal{K}}(T,[a,b]),\|\cdot\|\big)$ is of the second category in $\big(C([a,b]),\|\cdot\|_\infty\big)$. By Lemma 6.2, $\tilde{\mathcal{K}}\big(T,[a,b]\big)$ is of the first category. This contradicts the fact that $\big(C([a,b]),\|\cdot\|_\infty\big)$ is a Banach space. □

Theorem 6.3. *For a wide T integration on $[a,b]$ let $L : \big(\tilde{\mathcal{K}}(T,[a,b]), \|\cdot\|_\infty\big) \to \mathbb{R}$ be a continuous linear functional. Then there exists $g \in VB$ on $[a,b]$ such that $L(F) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)$, whenever $F \in \tilde{\mathcal{K}}\big(T,[a,b]\big)$.*

Proof. For $F \in \tilde{\mathcal{K}}\big(T,[a,b]\big)$ there exists $f \in \mathcal{K}\big(T,[a,b]\big)$ such that $F(x) = T\big(f;[a,x]\big)$. Let $L^*(f) = L(F)$. Since $\|f\| = \|F\|_\infty$ and L is a continuous linear functional, by Theorem 4.1, there exists $G \in VB$ such that

$$L^*(f) = F(b) \cdot G(b) - (\mathcal{RS}) \int_a^b F(t)\, dG(t) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)\,,$$

where $g(x) = -G(x)$, $x \in [a,b)$ and $g(b) = 0$. So $L(F) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)$. $\square$

Corollary 6.1. (The Riesz representation theorem, [20, Theorem 12.12]). *Let $L : \big(C([a,b]), \|\cdot\|_\infty\big) \to \mathbb{R}$ be a continuous linear functional. Then there exists $g \in VB$ on $[a,b]$ such that $L(F) = (\mathcal{RS}) \int_a^b F(t)\, dg(t)$ whenever $F \in C\big([a,b]$*

Proof. Since $\tilde{\mathcal{K}}\big(T,[a,b]\big)$ is dense in $C\big([a,b]\big)$, it follows that for each F in $C\big([a,b]\big)$ there exists a sequence $\{F_n\}_n \subset \tilde{\mathcal{K}}\big(T,[a,b]\big)$ such that $F_n \to F$ [unif] on $[a,b]$. Applying the uniform convergence theorem for the $(\mathcal{RS})$-integral we obtain

$$L(F) = \lim_{n\to\infty} L(F_n) = \lim_{n\to\infty} (\mathcal{RS}) \int_a^b F_n(t)\, dg(t) = (\mathcal{RS}) \int_a^b F(t)\, dg(t). \qquad \square$$

7 Weak Convergence in $\mathcal{K}\big(\mathbf{T},[\mathbf{a},\mathbf{b}]\big)$ for Wide T-integration

Theorem 7.1. ([16, p. 259]). *Let $f, f_n : [a,b] \to \mathbb{R}$, $n = 1,2,\dots$ be such that f, f_n are continuous and $\big|f_n(x)\big| < M$ for some M, for every $x \in [a,b]$ and each $n = 1,2,\dots$. Let $g : [a,b] \to \mathbb{R}$, $g \in VB$. If $f_n \to f$ on $[a,b]$, then*

$$(\mathcal{RS}) \int_a^b f(t)\, dg(t) = \lim_{n\to\infty} (\mathcal{RS}) \int_a^b f_n(t)\, dg(t)\,.$$

Theorem 7.2. *Let T be a wide integration on $[a,b]$ as in Definition 2.2. Let $f, f_n \in \mathcal{K}\big(T,[a,b]\big)$, $n = 1,2,\dots$. The following assertions are equivalent:*

(i) $f_n \to f$ weakly on $\mathcal{K}\big(T,[a,b]\big)$,

(ii) Let $F_n(x) = T\big(f_n,[a,x]\big)$ and $F(x) = T\big(f,[a,x]\big)$, $x \in [a,b]$.

(1) $\left|F_n(x)\right| \le M$ *for some* M*, for every* $x \in [a,b]$ *and each* $n = 1, 2, \dots,$

(2) $F_n(x) \to F(x)$ *for every* $x \in [a,b]$.

Proof. Our proof follows the proof of Theorem 3, # 3, Chapter VIII of [13]. Let $L : \mathcal{K}\big(T, [a,b]\big) \to \mathbb{R}$ be a continuous linear functional. By Theorem 4.1 there exists $g_L \in VB$ on $[a,b]$ such that $L(f) = T\big(fg_L, [a,b]\big)$, for every $f \in \mathcal{K}\big(T, [a,b]\big)$.

(i) $\Rightarrow$ (ii) We shall use the following classical result (see [7] or [13], Theorem 2, # 1 of Chapter VIII): $x_n \to x$ weakly in a normed space if and only if $\sup_n \|x_n\| < +\infty$ and $\big\{f : f(x_n) \to f(x),\ x \in [a,b]\big\}$ is a dense set of functionals in X^*. Since $f_n \to f$ weakly, we have $\|f_n\| = \|F_n\|_\infty \le M$ for some positive number M. So we have (ii), (1). For $x \in [a,b]$ let $L_x : \mathcal{K}\big(T, [a,b]\big) \to \mathbb{R}$ be a continuous linear functional defined by $L_x(f) = T\big(f\chi_{[a,x]}, [a,b]\big) = T\big(f, [a,x]\big) = F(x)$. Since $f_n \to f$ weakly, we obtain (ii), (2).

(ii) $\Rightarrow$ (i) It is sufficient to show that $L(f_n) \to L(f)$. By Theorem 7.1,

$$\begin{aligned}\left|L(f_n) - L(f)\right| &= \left|T\big((f_n - f)g_L, [a,b]\big)\right| \\ &= \left|(F_n - F)(b)g_L(b) - (\mathcal{RS})\int_a^b (F_n - F)(t)\, dg_L(t)\right| \to 0.\end{aligned}$$

□

8 General Classes of Primitives

Let $a \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Let's denote by

- $T_a f : \mathbb{R} \to \mathbb{R}\,,\ T_a f(x) := f(x - a)$, whenever $f : \mathbb{R} \to \mathbb{R}$,
- $f_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$,

  $$f_{\alpha,\beta}(x) = \begin{cases} f(\alpha) & \text{if } x < \alpha \\ f(x) & \text{if } x \in [\alpha, \beta] \\ f(\beta) & \text{if } x > \beta \end{cases}$$

  whenever $f : \mathbb{R} \to \mathbb{R}$,
- $f_Q : \mathbb{R} \to \mathbb{R}$,

  $$f_Q(x) = \begin{cases} f(x) & \text{if } x \in Q \\ ,0 & \text{if } x \notin Q \end{cases}$$

  whenever $f : E \to \mathbb{R}$ and $Q \subset E \subset \mathbb{R}$.

Definition 8.1. A family $\mathcal{S} \subset \{f : \mathbb{R} \to \mathbb{R} : \text{supp}(f) \text{ is compact}\}$ is said to be a space of integrable functions if it satisfies the following conditions:

1) $\mathcal{L}_{comp} + \mathcal{S} = \mathcal{S}$ and $\mathbb{R} \cdot \mathcal{S} = \mathcal{S}$ i.e., if $f \in \mathcal{L}_{comp}$, $g \in \mathcal{S}$, $\alpha \in \mathbb{R}$, then $f + g \in \mathcal{S}$ and $\alpha g \in \mathcal{S}$,

2) $\mathcal{S}$ is invariant to translations: i.e., $T_a f \in \mathcal{S}$ whenever $f \in \mathcal{S}$ and $a \in \mathbb{R}$,

3) $\mathcal{S} \cdot \chi_{[a,b]} \subset \mathcal{S}$ for any $[a, b] \subset \mathbb{R}$; i.e., if $f \in \mathcal{S}$ then $f \cdot \chi_{[a,b]} \in \mathcal{S}$,

4) If $f, g \in \mathcal{S}$ and $f - g \geq 0$ *a.e.* on some closed interval $[a, b]$ then $(f - g) \cdot \chi_{[a,b]} \in \mathcal{L}_{comp}$,

5) If $f, g \in \mathcal{S}$, $\mathrm{supp}(f) \subseteq [a, b]$ and $\mathrm{supp}(g) \subseteq [b, c]$, then $f + g \in \mathcal{S}$.

Definition 8.2. Let $\mathcal{S}$ be a space of integrable functions. A functional $\mathcal{I} : \mathcal{S} \to \mathbb{R}$ is said to be an integral if:

1) $\mathcal{I}(\alpha f + \beta g) = \alpha(\mathcal{L}) \int_{\mathbb{R}} f(t)\, dt + \beta \mathcal{I}(g)$, whenever $f \in \mathcal{L}_{comp}$, $g \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{R}$,

2) $\mathcal{I}(T_a f) = \mathcal{I}(f)$ whenever $a \in \mathbb{R}$ and $f \in \mathcal{S}$,

3) $\mathcal{I}(f + g) = \mathcal{I}(f) + \mathcal{I}(g)$ whenever $f, g \in \mathcal{S}$, $\mathrm{supp}(f) \subseteq [a, b]$, $\mathrm{supp}(g) \subseteq [b, c]$.

Let $f : E \to \mathbb{R}$, $Q \subset E \subset \mathbb{R}$, Q bounded. f is said to be $\mathcal{I}$-integrable on Q if $f_Q \in \mathcal{S}$. We denote by $(\mathcal{I}) \int_Q f(t)\, dt = \mathcal{I}(f_Q)$.

Definition 8.3. Let $f : [a, b] \to \mathbb{R}$ be $\mathcal{I}$–integrable on $[a, b]$, and let $\alpha \in \mathbb{R}$. The function $G : [a, b] \to \mathbb{R}$ defined by $G(x) = \alpha + (\mathcal{I}) \int_{[a,x]} f(t)\, dt$ is called an $\mathcal{I}$-primitive of f on $[a, b]$.

A function $G : [a, b] \to \mathbb{R}$ is called an $\mathcal{I}$-primitive if there exists $g : [a, b] \to \mathbb{R}$, such that g is $\mathcal{I}$-integrable on $[a, b]$ and there exists $\alpha \in \mathbb{R}$ so that

$$G(x) = \alpha + (\mathcal{I}) \int_{[a,x]} g(t)\, dt\,.$$

Definition 8.4. Let $AC_{\mathbb{R}} = \{F : \mathbb{R} \to \mathbb{R} : F \in AC \text{ on each compact interval}\}$. A class $\mathcal{G} \subset \{F : \mathbb{R} \to \mathbb{R} : F \text{ is a measurable function approximately derivable } a.e.\}$ is said to be a general class of primitives if it has the following properties:

1) $AC_{\mathbb{R}} + \mathcal{G} = \mathcal{G}$ and $\mathbb{R} \cdot \mathcal{G} = \mathcal{G}$,

2) $\mathcal{G}$ is invariant to translations; i.e., $T_a F \in \mathcal{G}$ whenever $F \in \mathcal{G}$ and $a \in \mathbb{R}$,

3) If $F \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, then $F_{\alpha,\beta} \in \mathcal{G}$,

4) If $F \in \mathcal{G}$ and $F'_{ap} \geq 0$ *a.e.* on some interval $[a, b]$, then F is increasing on $[a, b]$,

5) Let $F, G \in \mathcal{G}$. If $F = F_{a,b}$ for some $[a,b] \subset \mathbb{R}$ and $G = G_{b,c}$ for some $[b,c]$, then $F + G \in \mathcal{G}$.

Let $g : \mathbb{R} \to \mathbb{R}$. A function $F : \mathbb{R} \to \mathbb{R}$, $F \in \mathcal{G}$ with $F'_{ap} = g$ *a.e.* is said to be a $(\mathcal{G})$- primitive of g on $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ with compact support is said to be $\mathcal{G}$-integrable if it admits $\mathcal{G}$-primitives. The definite $\mathcal{G}$-integral of f will be denoted by

$$(\mathcal{G}) \int_{\mathbb{R}} f(t)\, dt = F(b) - F(a)$$

where F is a $\mathcal{G}$-primitive of f such that $\mathrm{supp}(f) \subseteq [a,b]$.

In what follows we show that the $\mathcal{G}$-integral is well defined.

Lemma 8.1. *Let $g : \mathbb{R} \to \mathbb{R}$ which admits $\mathcal{G}$-primitives. Suppose that $F, G : \mathbb{R} \to \mathbb{R}$ are two $\mathcal{G}$-primitives of g. Then $F - G$ is a constant on $\mathbb{R}$.*

Proof. By Definition 8.4, 4), it follows that $F - G$ is a constant on each $[a,b] \subset \mathbb{R}$. Since $\mathbb{R} = \cup_{n=1}^{\infty} [-n, n]$, we get that $F - G$ is a constant on $\mathbb{R}$. □

Lemma 8.2. *The $\mathcal{G}$-integral is well-defined.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{G}$-integrable function and F, G two $\mathcal{G}$-primitives of f. By Lemma 8.1, $F - G$ is a constant on $\mathbb{R}$. Let $c = \inf \mathrm{supp}(f)$, $d = \sup \mathrm{supp}(f)$ and $[a,b] \supset [c,d]$. By Definition 8.4, 3), $F_{c,d}, G_{c,d}$ belong to $\mathcal{G}$ and they obviously are $\mathcal{G}$-primitives of f. Hence, by Lemma 8.1 again, $F = F_{c,d}$ and $G = G_{c,d}$. It follows that $F(b) - F(a) = G(b) - G(a)$. □

Definition 8.5. A function $f : E \to \mathbb{R}$ is said to be $\mathcal{G}$-integrable on a bounded set $Q \subset E$, if the function f_Q is $\mathcal{G}$-integrable. Then we write

$$(\mathcal{G}) \int_{Q} f(t)\, dt = (\mathcal{G}) \int_{\mathbb{R}} f_Q(t)\, dt\,.$$

Theorem 8.1. *Let $\mathcal{S}_{\mathcal{G}} = \{f : \mathbb{R} \to \mathbb{R} : \mathrm{supp}(f)$ is compact and f is $\mathcal{G}$-integrable$\}$. Then $\mathcal{S}_{\mathcal{G}}$ is a space of integrable functions.*

Proof. We verify conditions 1)–5) of Definition 8.1.

1) Let $f \in \mathcal{L}_{comp}$, $g \in \mathcal{S}_{\mathcal{G}}$ and $\alpha \in \mathbb{R}$. Clearly $\alpha g \in \mathcal{S}_{\mathcal{G}}$. Let $a_1 = \inf(\mathrm{supp}(f))$, $b_1 = \sup(\mathrm{supp}(f))$ and $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \le a_1 \\ (\mathcal{L}) \int_a^x f(t)\, dt & \text{if } x \in [a_1, b_1] \\ F(b_1) & \text{if } x \ge b_1 \end{cases}$$

Then $F \in AC_{\mathbb{R}}$ and $F' = f$ *a.e.* on $\mathbb{R}$. For $g \in \mathcal{S}_{\mathcal{G}}$, there exists $G \in \mathcal{G}$ such that $G'_{ap} = g$ *a.e.* on $\mathbb{R}$. By Definition 8.4, 1), it follows that $F + G \in \mathcal{G}$ and $(F+G)'_{ap} = f + g$ *a.e.* on $\mathbb{R}$.

2) Suppose that $f \in \mathcal{S}_{\mathcal{G}}$. Then there exists $F \in \mathcal{G}$ such that $F'_{ap} = f$ *a.e.* on $\mathbb{R}$. Let $a \in \mathbb{R}$. Then

$$(T_aF)'_{ap}(x) = (F(x-a))'_{ap} = F'_{ap}(x-a) = f(x-a) = T_af(x) \text{ } a.e. \text{ on } \mathbb{R}.$$

By Definition 8.4, 2), it follows that $T_af \in \mathcal{S}_{\mathcal{G}}$.

3) Suppose that $f \in \mathcal{S}_{\mathcal{G}}$ and $[a,b] \subset \mathbb{R}$. Then there exists $F \in \mathcal{G}$ such that $F'_{ap} = f$ *a.e.* on $\mathbb{R}$. By Definition 8.4, 3), it follows that $F_{a,b} \in \mathcal{G}$ and

$$(F_{a,b})'_{ap} = f\chi_{[a,b]} \text{ } a.e. \text{on } \mathbb{R},$$

so $f\chi_{[a,b]} \in \mathcal{S}_{\mathcal{G}}$.

4) Suppose that $f, g \in \mathcal{S}_{\mathcal{G}}$ and $f - g \geq 0$ *a.e.* on some $[a,b] \subset \mathbb{R}$. Then there exists $F, G \in \mathcal{G}$ such that $F'_{ap} = f$ and $G'_{ap} = g$ *a.e.* on $\mathbb{R}$. But $(F-G)'_{ap} = f - g \geq 0$ *a.e.* on $[a,b]$. By Definition 8.4, 4), $F - G$ is increasing on $[a,b]$; so $f - g$ is Lebesgue integrable on $[a,b]$. It follows that $(f-g) \cdot \chi_{[a,b]} \in \mathcal{L}_{comp}$.

5) Suppose that $f, g \in \mathcal{S}_{\mathcal{G}}$ such that $\text{supp}(f) \subset [a,b]$ and $\text{supp}(g) \subset [b,c]$. Then there exist $F, G \in \mathcal{S}_{\mathcal{G}}$ such that $F'_{ap} = f$ and $G'_{ap} = g$ *a.e.* on $\mathbb{R}$. By Definition 8.4, 3), $F_{a,b}, G_{b,c} \in \mathcal{S}_{\mathcal{G}}$. Clearly

$$(F_{a,b})'_{ap} = f \quad \text{and} \quad (G_{b,c})'_{ap} = g \text{ } a.e. \text{ on } \mathbb{R}\,.$$

By Lemma 8.1, $F = F_{a,b}$ and $G = G_{b,c}$, Hence by Definition 8.4, 5), $F+G \in \mathcal{G}$ and $(F+G)'_{ap} = f + g$ *a.e.* on $\mathbb{R}$. Therefore $f + g \in \mathcal{S}_{\mathcal{G}}$. □

Example 8.1 (Examples of general classes of primitives)**.** Let

- $\mathcal{C} = \{F : \mathbb{R} \to \mathbb{R} : F \text{ is continuous on } \mathbb{R}\}$,
- $\mathcal{C}_{ap} = \{F : \mathbb{R} \to \mathbb{R} : F \text{ is approximately continuous on } \mathbb{R}\}$,
- $\mathcal{C}_{pro} = \{F : \mathbb{R} \to \mathbb{R} : F \text{ is proximally continuous on } \mathbb{R}\}$.

The definition of the proximal continuity is somewhat technical, and it was introduced by Sarkhel and De in [29]. We don't give this definition here, but we mention that $\mathcal{C}_{pro}$ is a real linear space contained in the class of Darboux Baire one functions and $\mathcal{C} \cdot \mathcal{C}_{pro} = \mathcal{C}_{pro}$. That $\mathcal{C}_{ap}$ is contained in the class Darboux Baire one is well known, and of course $\mathcal{C} \cdot \mathcal{C}_{ap} = \mathcal{C}_{ap}$.

Let

- $AC^*G_{\mathbb{R}} = \{F : \mathbb{R} \to \mathbb{R} : F$ is AC^*G on each compact interval $[a, b]\}$,
- $ACG_{\mathbb{R}} = \{F : \mathbb{R} \to \mathbb{R} : F$ is ACG on each compact interval $[a, b]\}$,
- $\mathcal{F}_{\mathbb{R}} = \{F : \mathbb{R} \to \mathbb{R} : F$ satisfies Foran's condition $\mathcal{F}$ on each compact interval $[a, b]\}$.

We have the following examples of $\mathcal{G}$-integrals:

- $\mathcal{G} = \mathcal{C} \cap AC^*G_{\mathbb{R}}$ is the Denjoy*-integral,
- $\mathcal{G} = \mathcal{C}_{ap} \cap AC^*G_{\mathbb{R}}$ is the α-Ridder integral,
- $\mathcal{G} = \mathcal{C}_{pro} \cap AC^*G_{\mathbb{R}}$ seems to be new,
- $\mathcal{G} = \mathcal{C} \cap ACG_{\mathbb{R}}$ is the wide Denjoy-integral,
- $\mathcal{G} = \mathcal{C}_{ap} \cap ACG_{\mathbb{R}}$ is the β-Ridder integral,
- $\mathcal{G} = \mathcal{C}_{pro} \cap ACG_{\mathbb{R}}$ seems to be new,
- $\mathcal{G} = \mathcal{C} \cap \mathcal{F}_{\mathbb{R}}$ is the Foran integral,
- $\mathcal{G} = \mathcal{C}_{ap} \cap \mathcal{F}_{\mathbb{R}}$ is called the AF-integral (see [11]),
- $\mathcal{G} = \mathcal{C}_{ap} \cap VBG \cap (N)$ is the Gordon integral,
- $\mathcal{G} = \mathcal{C}_{pro} \cap VBG \cap (N)$ seems to be new.

9 A Generalization of a Result on Differential Equations of Bullen and Vyborny

Definition 9.1. Let $I_o = [t_o - \alpha_o, t_o + \alpha_o]$ and $J_o = [x_o - \beta_o, x_o + \beta_o]$, where $t_o, x_o \in \mathbb{R}$ and $\alpha_o, \beta_o > 0$. Given $f : I_o \times J_o \to \overline{\mathbb{R}}$, I a compact interval, $I \subset I_o$ and $g : I \to J_o$, we define $f_g : I \to \mathbb{R}$ by $f_g(t) = f(t, g(t))$.

Lemma 9.1 (Helly)**.** ([22, p. 221]). *Let $F = \{f(x)\}$ be an infinite family of increasing functions, defined on $[a, b]$. If all functions of the family are bounded by one and the same number, $\big|f(x)\big| \le K$, $f \in F$, $a \le x \le b$, then there is a sequence of functions $\big\{f_n(x)\big\}$ in F which converges to an increasing function $\varphi(x)$ at every point of $[a, b]$.*

Theorem 9.1. *Let $\mathcal{I} : \mathcal{S} \to \mathbb{R}$ be an integral as in Definition 8.2, and let $f : I_o \times J_o \to \overline{\mathbb{R}}$ satisfy the following properties:*

(i) $f(t, \cdot)$ is continuous on J_o for almost all $t \in I_o$,

(ii) There exists a subinterval $I=[t_o-\alpha, t_o+\alpha]$ of I_o, and two $\mathcal{I}$-integrable functions $m, M : I \to \overline{\mathbb{R}}$ such that

- $\left|(\mathcal{I})\int_{t_o}^{t} m(s)\,ds\right| < \beta_o$
- $\left|(\mathcal{I})\int_{t_o}^{t} M(s)\,ds\right| < \beta_o$
- *if $g : I \to J_o$ is an $\mathcal{I}$-primitive with $g(t_o)=x_o$, then f_g is measurable on I and $m(t) \le f_g(t) \le M(t)$ a.e. on I.*

Then there exists an $\mathcal{I}$-primitive $\varphi : I \to J_o$ such that $\varphi(t) = x_o + (\mathcal{I})\int_{t_o}^{t} f_\varphi(s)\,ds$.

Proof. We prove for example the case $t \ge t_o$. On the interval $[t_o, t_o+\alpha]$ we define the approximations φ_k, $k=1,2,\dots$ by

$$\varphi_k(t) = \begin{cases} x_o & \text{if } t \in [t_o, t_o+\frac{\alpha}{k}] \\ x_o + (\mathcal{I})\int_{t_o}^{t-\frac{\alpha}{k}} f_{\varphi_k}(s)\,ds & \text{if } t \in [t_o+\frac{\alpha}{k}, t_o+\alpha]. \end{cases}$$

Since the integral $\mathcal{I}$ is invariant to translations, it follows that

$$\varphi_k(t) = \begin{cases} x_o & \text{if } t \in [t_o, t_o+\frac{\alpha}{k}] \\ x_o + (\mathcal{I})\int_{t_o+\frac{\alpha}{k}}^{t} f_{\varphi_k}(s-\frac{\alpha}{k})\,ds & \text{if } t \in [t_o+\frac{\alpha}{k}, t_o+\alpha]. \end{cases}$$

Let $\varphi_{k,1} : [t_o, t_o+\alpha] \to J_o$, $\varphi_{k,1}(t) = x_o$. Clearly $\varphi_{k,1}$ is an $\mathcal{I}$-primitive on $[t_o, t_o+\alpha]$. By hypotheses we have

$$\begin{aligned} -\beta_o &< (\mathcal{I})\int_{t_o}^{t-\frac{\alpha}{k}} m(s)\,ds \le (\mathcal{I})\int_{t_o}^{t-\frac{\alpha}{k}} f_{\varphi_{k,1}}(s)\,ds \\ &\le (\mathcal{I})\int_{t_o}^{t-\frac{\alpha}{k}} M(s)\,ds < \beta_o. \end{aligned} \tag{8}$$

Let $\varphi_{k,2} : [t_o, t_o+\alpha] \to J_o$,

$$\varphi_{k,2}(t) = \begin{cases} \varphi_{k,1}(t) & \text{if } t \in [t_o, t_o+\frac{\alpha}{k}] \\ x_o + (\mathcal{I})\int_{t_o}^{t-\frac{\alpha}{k}} f_{\varphi_{k,1}}(s)\,ds & \text{if } t \in [t_o+\frac{\alpha}{k}, t_o+\frac{2\alpha}{k}] \\ \varphi_{k,2}(t_o+\frac{2\alpha}{k}) & \text{if } t \in [t_o+\frac{2\alpha}{k}, t_o+\alpha]. \end{cases}$$

By (8), it follows that $\varphi_{k,2}$ takes indeed values in J_o. Since the integral $\mathcal{I}$ is invariant to translations, it follows that

$$x_o + (\mathcal{I})\int_{t_o}^{t-\frac{\alpha}{k}} f_{\varphi_{k,1}}(s)\,ds = x_o + (\mathcal{I})\int_{t_o+\frac{\alpha}{k}}^{t} f_{\varphi_{k,1}}\big(s-\frac{\alpha}{k}\big)\,ds\,,$$

for $t \in [t_o + \frac{\alpha}{k}, t_o + \frac{2\alpha}{k}]$. Therefore $\varphi_{k,2}$ is well defined and a $\mathcal{I}$-primitive on $[t_o, t_o + \alpha]$, with $\varphi_{k,2}(t_o) = x_o$. Suppose that $\varphi_{k,j-1} : [t_o, t_o + \alpha] \to J_o$, $j \geq 2$ are already defined and let $\varphi_{k,j} : [t_o, t_o + \alpha] \to J_o$ be defined by

$$\varphi_{k,j}(t) = \begin{cases} \varphi_{k,j-1}(t) & t \in [t_o, t_o + \frac{(j-1)\alpha}{k}] \\ \varphi_{k,j-1}(\frac{(j-1)\alpha}{k}) + (\mathcal{I}) \int_{t_o+\frac{(j-2)\alpha}{k}}^{t-\frac{\alpha}{k}} f_{\varphi_{k,j-1}}(s)\, ds & t \in [t_o + \frac{(j-1)\alpha}{k}, t_o + \frac{j\alpha}{k}] \\ \varphi_{k,j-1}(t_o + \frac{j\alpha}{k}) & t \in [t_o + \frac{j\alpha}{k}, t_o + \alpha] \end{cases}$$

But

$$(\mathcal{I}) \int_{t_o+\frac{(j-2)\alpha}{k}}^{t-\frac{\alpha}{k}} f_{\varphi_{k,j-1}}(s)\, ds = (\mathcal{I}) \int_{t_o+\frac{(j-1)\alpha}{k}}^{t} f_{\varphi_{k,j-1}}\big(s - \frac{\alpha}{k}\big)\, ds\,,$$

for $t \in [t_o + \frac{(j-1)\alpha}{k}, t_o + \frac{j\alpha}{k}]$. Clearly $\varphi_{k,j}$ is a $\mathcal{I}$-primitive on $[t_o, t_o + \alpha]$, with $\varphi_{k,j}(t_o) = x_o$. We show that $\varphi_{k,j}$ takes values only in J_o. We first show inductively that $\varphi_k = \varphi_{k,j}$ on $\big[t_o, t_o + \frac{j\alpha}{k}\big]$. Suppose that $\varphi_k = \varphi_{k,j-1}$ on $[t_o, t_o + \frac{j\alpha}{k}]$. Then clearly $\varphi_{k,j} = \varphi_{k,j-1} = \varphi_k$ on $[t_o, t_o + \frac{(j-1)\alpha}{k}]$. Let $t \in [t_o + \frac{(j-1)\alpha}{k}, t_o + \frac{j\alpha}{k}]$. It follows that

$$\begin{aligned} \varphi_{k,j}(t) &= \varphi_k\big(t_o + \frac{(j-1)\alpha}{k}\big) + (\mathcal{I}) \int_{t_o+\frac{(j-1)\alpha}{k}}^{t} \varphi_k\big(s - \frac{\alpha}{k}\big)\, ds \\ &= x_o + (\mathcal{I}) \int_{t_o+\frac{\alpha}{k}}^{t_o+\frac{(j-1)\alpha}{k}} f_{\varphi_k}\big(s - \frac{\alpha}{k}\big)\, ds + (\mathcal{I}) \int_{t_o+\frac{(j-1)\alpha}{k}}^{t} \varphi_k\big(s - \frac{\alpha}{k}\big)\, ds \\ &= x_o + (\mathcal{I}) \int_{t_o+\frac{\alpha}{k}}^{t} f_{\varphi_k}\big(s - \frac{\alpha}{k}\big)\, ds = \varphi_k(t). \end{aligned}$$

Suppose that $\varphi_{k,j-1} \in J_o$. We prove that $\varphi_{k,j} \in J_o$. For $t \in [t_o \frac{(j-1)\alpha}{k}, t_o + \frac{j\alpha}{k}]$ we have

$$-\beta_o < (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} m(s) ds \leq (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} f_{\varphi_k}(s) ds \leq (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} M(s)\, ds < \beta_o.$$

Hence $\varphi_{k,j} \in J_o$ in this case. Since $\varphi_{k,j} = \varphi_{k,j-1} = \varphi_k$ on $\big[t_o, t_o + \frac{(j-1)\alpha}{k}\big]$ we have $\varphi_{k,j}(t) \in J_o$ for all $t \in \big[t_o, t_o + \frac{j\alpha}{k}\big]$. Clearly $\varphi_{k,k} = \varphi_k$ on $[t_o, t_o + \alpha]$, hence φ_k is well defined and is a $\mathcal{I}$-primitive on $[t_o, t_o + \alpha]$.

Let $h, H : [t_o, t_o + \alpha] \to \mathbb{R}$ be defined as follows:

$$h(t) = \begin{cases} x_o & \text{if } t \in [t_o, t_o + \frac{\alpha}{k}] \\ x_o + (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} m(s)\, ds & \text{if } t \in [t_o + \frac{\alpha}{k}, t_o + \alpha] \end{cases}$$

$$H(t) = \begin{cases} x_o & \text{if } t \in [t_o, t_o + \frac{\alpha}{k}] \\ x_o + (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} M(s)\, ds & \text{if } t \in [t_o + \frac{\alpha}{k}, t_o + \alpha]. \end{cases}$$

Let $h_k : [t_o, t_o + \alpha] \to \mathbb{R}$, $h_k(t) = \varphi_k(t) - h(t)$. Then, for $t \in [t_o, t_o + \frac{\alpha}{k}]$ we have $h_k(t) = 0$, and for $t \in [t_o + \frac{\alpha}{k}, t_o + \alpha]$,

$$\begin{aligned} h_k(t) &= (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} (f_{\varphi_k}(s) - m(s))\, ds = (\mathcal{L}) \int_{t_o}^{t-\frac{\alpha}{k}} (f_{\varphi_k} - m)(s)\, ds \\ &\leq (\mathcal{L}) \int_{t_o}^{t-\frac{\alpha}{k}} (M - m)(s)\, ds \\ &= (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} M(s)\, ds - (\mathcal{I}) \int_{t_o}^{t-\frac{\alpha}{k}} m(s)\, ds < 2\beta_o. \end{aligned}$$

Therefore $\{h_k\}_k$ is an increasing sequence of functions on $[t_o, t_o + \alpha]$ and

$$0 \leq h_k(t_o) \leq h_k(t_o + \alpha) \leq 2\beta_o\,.$$

By Lemma 9.1, there exists a subsequence of $\{h_k\}_k$ which converges punctually to an increasing function G on $[t_o, t_o + \alpha]$. We may suppose without loss of generality that $\{h_k\}_k$ converges punctually to G on $[t_o, t_o + \alpha]$, hence $\{\varphi_k\}_k$ converges punctually to $\varphi := h + G$ on $[t_o, t_o + \alpha]$. By (i), it follows that $f_{\varphi_k} \to f_\varphi$ *a.e.* on $[t_o, t_o+\alpha]$. By Theorem 3.1, it follows that f_{φ_k} and f_φ belong to $\mathcal{S}$ on $[t_o, t]$ and $\lim_{k\to\infty} \mathcal{I}(f_{\varphi_k}) = \mathcal{I}(f_\varphi)$ on $[t_o, t]$. From the definition of φ_k, we obtain that $\varphi(t) = x_o + (\mathcal{I}) \int_{t_o}^{t} f_\varphi(s)\, ds$. $\square$

Corollary 9.1 (Bullen and Vyborny)**.** ([5]). *Let $f : I_o \times J_o \to \mathbb{R}$ be such that*

(i) $f(t, \cdot)$ is continuous on J_o for almost all $t \in I_o$.

(ii) there exists $\alpha > 0$ and two continuous functions $h, H : [t_o - \alpha, t_o + \alpha] \to [-\beta_o, \beta_o]$ satisfying the following properties:

- *$h(t_o) = H(t_o) = 0$.*
- *if $g : [t_o-\alpha, t_o+\alpha] \to J_o$, $g \in AC^*G$, g is continuous and $g(t_o) = x_o$, then f_g is measurable and $\overline{D}h \leq f_g \leq \underline{D}H$.*

Then there exists a continuous function $\varphi : [t_o - \alpha, t_o + \alpha] \to J_o$, such that $\varphi(t) = x_o + (\mathcal{D}^*) \int_{t_o}^{t} f_\varphi(s)\, ds$.

Proof. Let $g_o : [t_o - \alpha, t_o + \alpha] \to J_o$, $g_o(t) = x_o$. By hypothesis $\overline{D}h \leq f_{g_o} \leq \underline{D}H$ and f_{g_o} is measurable. From Marcinkiewicz' theorem of [25, p. 253], it follows that f_{g_o} is $\mathcal{D}^*$-integrable. Since $\overline{D}h \leq \underline{D}H$ on $[t_o-\alpha, t_o+\alpha]$, and $\overline{D}h$, $\underline{D}H$

are Borel measurable (hence Lebesgue measurable), by Marcinkiewicz' theorem again, we obtain that $\overline{D}h$, $\underline{D}H$ are $\mathcal{D}^*$-integrable on $[t_o-\alpha, t_o+\alpha]$. Let $m(x)=\overline{D}h(t)$ and $M(t)=\underline{D}H(t)$ for $t\in[t_o-\alpha, t_o+\alpha]$. Then

$$\left|(\mathcal{D}^*)\int_{t_o}^{t} m(s)\,ds\right| < \beta_o \quad \text{and} \quad \left|(\mathcal{D}^*)\int_{t_o}^{t} M(s)\,ds\right| < \beta_o\,,$$

because

$$-\beta_o \le h(t) \le (\mathcal{D}^*)\int_{t_o}^{t} m(s)\,ds \le (\mathcal{D}^*)\int_{t_o}^{t} M(s)\,ds \le H(t) \le \beta_o,$$

for all $t\in[t_o-\alpha, t_o+\alpha]$ (see for example [8]) Now the proof follows applying Theorem 9.1. □

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