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BILIPSCHITZ MAPPINGS OF NETS†

Abstract

Let $0 < a < \sqrt{2}$. Suppose $\delta = \delta(d, \varepsilon)$ has the following property. If $\mathcal{N}$ is an a-net of the Euclidean ball in $\mathbb{R}^d$, $A \subset \mathcal{N}$, and $f : A \to \mathbb{R}^d$ is $(1+\varepsilon)$-bilipschitz, then f admits a $(1 + \delta)$-bilipschitz extension $f : \mathcal{N} \to \mathbb{R}^d$. We give some estimates of δ.

1 Introduction

Let A be a subset of a Hilbert space X and $f : A \to X$ a Lipschitz mapping. By the classical theorem of Kirszbraun and Valentine, f can be extended to X with the same Lipschitz constant. There are many simple examples of bilipschitz mappings (both the mapping and its inverse are Lipschitz) that cannot be extended; the paper [V] of Väisälä explains the subject. Nevertheless, if $\mathcal{N} \subset \mathbb{R}^d$ is finite, then clearly every bijection of $\mathcal{N}$ is bilipschitz. In this note we consider the following question. Fix some $0 < a < \sqrt{2}$. Let $\delta = \delta(d, \varepsilon)$ have the following property. Suppose $\mathcal{N}$ is an a-net of $B_{\mathbb{R}^d}$ and $A \subset \mathcal{N}$. If $f : A \to \mathbb{R}^d$ is $(1 + \varepsilon)$-bilipschitz, then f admits a $(1 + \delta)$-bilipschitz extension $f : \mathcal{N} \to \mathbb{R}^d$. How large does δ have to be? In particular, can we have $\delta = \delta(\varepsilon)$ not depending on the dimension, and at the same time $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$?

In Proposition 3.2 we show that independently of the dimension, $\delta \le c_a \sqrt{\varepsilon}$ if we wish to extend just to $\mathcal{N} \cap \operatorname{conv} A$. This makes it perhaps more natural to investigate extension properties of bilipschitz mappings defined on nets of S^{d-1} rather than on nets of $B_{\mathbb{R}^d}$. If we can extend to a net of the sphere, we can extend to some net of the ball as well.

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Suppose f is a $(1+\varepsilon)$-bilipschitz mapping of an a-net of S^{d-1} into $\mathbb{R}^d$. We adapt a proof from [K] to show in Theorem 4.4 that there exists an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ with $\|f(x) - T(x)\| \le c\frac{1}{2-a^2}\sqrt{\varepsilon}\ln d$ for all $x \in \mathcal{N}$, where c is an absolute constant.

Suppose A is a subset of an a-net $\mathcal{N}$ of S^{d-1}, and $f : A \to \mathbb{R}^d$ is $(1+\varepsilon)$-bilipschitz. By a result of [ATV], f can be extended to a $(1+c_d\sqrt{\varepsilon})$-bilipschitz mapping of $\mathcal{N}$. It is a corollary of Theorem 4.4, that c_d really does depend on d. In Proposition 4.5 we show that $c_d \ge c_a d^{\frac{1}{4}} \ln^{-2} d$. Notice though that this does not answer the question whether δ really does depend on d.

If a net $\mathcal{N}$ of S^{d-1} is symmetric and "thin" enough; that is, if $|\langle x, y\rangle|$ is not very far from ε for $x \ne \pm y \in \mathcal{N}$, extending independently of the dimension is possible. In Proposition 5.4 we give an example of such an extension. We show that if $0 < 1 - \frac{a^2}{2} \le \sqrt{\varepsilon}$ and $f : A \to \mathbb{R}^d$ is $(1+\varepsilon)$-bilipschitz, then f admits a $(1 + c(\varepsilon \ln 1/\varepsilon)^{\frac{1}{2}})$-bilipschitz extension $f : \mathcal{N} \to \mathbb{R}^d$. Here the point is that f goes into $\mathbb{R}^d$ again extending f so that $f : \mathcal{N} \to \ell_2$ is trivial in this case.

In Proposition 6.3 we show that for every $\varepsilon > 0$ and $d_0 > 0$ there exist $d > d_0$, $k \ge c\varepsilon d \ln d$ and a $(1+\varepsilon)$-bilipschitz antipodal mapping f of $B_{\mathbb{R}^k}$ onto itself such that $f(B_{\mathbb{R}^d})$ contains an orthonormal basis of $\mathbb{R}^k$ together with its negative. We leave open the question of how large $k = k(d, \varepsilon)$ can get in general.

2 Preliminaries

In this section we give the notation, terminology, and a few basic results we will use in the paper.

By c we denote absolute constants, which may have different values, even in the same formula; c_d is a function of d only. By S^{d-1}, $d \ge 2$, we denote the sphere of the Euclidean ball $B_{\mathbb{R}^d}$ in $\mathbb{R}^d$. P is the uniform measure on the sphere S^{d-1}. By $e_1, e_2, \ldots, e_d$ we denote the standard orthonormal basis of $\mathbb{R}^d$. By $\mathrm{D}(A)$ we denote the diameter of the set A.

Definition 2.1. Let M be a metric space and $a > 0$. An inclusion-maximal set $\mathcal{N} \subset M$ such that $\|x - y\| \ge a$ if $x, y \in \mathcal{N}$, $x \ne y$, is called an a-net of M.

We recall an example from [V]. The mapping $f : \{\pm 1, \pm\varepsilon\} \to \mathbb{R}$ defined by $f(\pm\varepsilon) = \pm\varepsilon$ and $f(\pm 1) = \mp 1$ is $(1 + 3\varepsilon)$-bilipschitz, if ε is small. If f is any continuous extension of the mapping to $\mathbb{R}$, then $\emptyset \ne [-\varepsilon, 1] \cap [-1, \varepsilon] \subset f([-1, -\varepsilon]) \cap f([\varepsilon, 1])$. Since $[-1, -\varepsilon] \cap [\varepsilon, 1] = \emptyset$, f is not bijective. In particular, f admits no bilipschitz extension to $\mathbb{R}$. This nice idea of [V] works also in $\mathbb{R}^d$. Let $A = \ker e_1 \cup \{e_1, -e_1, \varepsilon e_1, -\varepsilon e_1\}$ and $f : A \to \mathbb{R}^d$ be defined by $f(\pm e_1) = \mp e_1$ and as the identity on $\ker e_1 \cup \{\pm\varepsilon e_1\}$. Then f is $(1+3\varepsilon)$-bilipschitz, and f admits no continuous bijective extension to $\mathbb{R}^d$ with range in $\mathbb{R}^d$. Suppose now some $L > 1$ is given and choose $a > 0$ small and $R > 0$ large enough. Let $\mathcal{N}_0$ be an a-net of $A \cap B_{\mathbb{R}^d}(0, R)$. Extend $\mathcal{N}_0$ to an a-net $\mathcal{N}$ of $B_{\mathbb{R}^d}$. Let $g : \mathcal{N}_0 \to \mathbb{R}^d$ be the restriction of f to $\mathcal{N}_0$. It is quite easy to see that g admits no L-bilipschitz extension to $\mathcal{N}$ with range in $\mathbb{R}^d$. By scaling the whole picture down, we can get that $R = 1$ with $a > 0$ small enough. Similarly, by scaling it up we can get $a = 1$ with $R > 0$ large enough. Therefore in this paper, we only consider a-nets of the unit ball with some $a > 0$ fixed at the beginning, before the bilipschitz constant is given.

We can equivalently describe a net of the sphere by estimating the angles between its points. If the net is symmetric, we also get an upper estimate for the distances between points in it.

Lemma 2.2. *Let $\mathcal{N} \subset S^{d-1}$, $0 < a < \sqrt{2}$, and $b = 1 - \frac{1}{2}a^2$.*

(i) *Then $\mathcal{N}$ is an a-net if and only if $\mathcal{N}$ is an inclusion-maximal set such that $\langle x, y\rangle \le b$ if $x, y \in \mathcal{N}$, $x \ne y$.*

(ii) *$\mathcal{N}$ is a symmetric a-net if and only if $\mathcal{N}$ is a symmetric inclusion-maximal set such that $|\langle x, y\rangle| \le b$ if $x, y \in \mathcal{N}$, $x \ne \pm y$.*

(iii) *If $\mathcal{N}$ is a symmetric a-net, then $\|x - y\| \le \sqrt{4 - a^2}$ for any $x, y \in \mathcal{N}$, $x \ne -y$.*

Proof. For $x, y \in \mathcal{N}$ we have

$$\|x \pm y\|^2 = 2 \pm 2\langle x, y\rangle. \tag{1}$$

If $\mathcal{N}$ is an a-net and $x \ne y$, then (1) implies $a^2 \le 2 - 2\langle x, y\rangle$, and $a^2 \le 2 \pm 2\langle x, y\rangle$ in the symmetric case. Conversely, if $\langle x, y\rangle \le 1 - \frac{1}{2}a^2$, then (1) implies that $\|x - y\|^2 \ge 2 - 2(1 - \frac{1}{2}a^2) = a^2$. Finally, if $\mathcal{N}$ is symmetric, $\|x - y\|^2 = 2 - 2\langle x, y\rangle \le 2 + 2(1 - \frac{1}{2}a^2) = 4 - a^2$. □

The thicker the net is, the larger a ball contained in its convex hull is.

Lemma 2.3. *Let $0 < a < \sqrt{2}$, and $\mathcal{N}$ be an a-net, or a symmetric a-net of S^{d-1}. Then $B_{\mathbb{R}^d}(0, b) \subset \operatorname{conv} \mathcal{N}$, where $b = 1 - \frac{1}{2}a^2$.*

Proof. Suppose not. By the Hahn-Banach theorem, there is $v \in S^{d-1}$ so that $\langle x, v\rangle < 1 - \frac{1}{2}a^2 = b$ for all $x \in \mathcal{N}$. Then $\operatorname{dist}(v, \mathcal{N}) > a$. Hence $\mathcal{N}$ is not maximal, which is a contradiction.

If $\mathcal{N}$ is symmetric, we get $|\langle x, v\rangle| < b$ from the Hahn-Banach theorem, and we may, for a contradiction, enlarge $\mathcal{N}$ by both v and $-v$. □

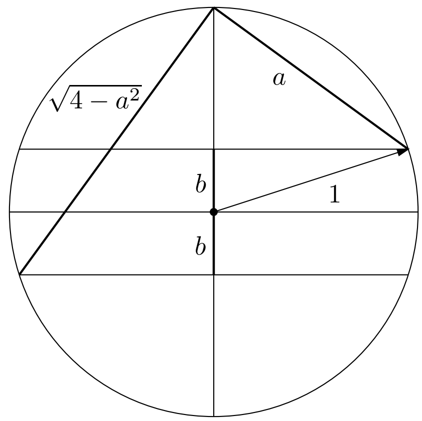

Figure 1: Illustration to the statement of Lemma 2.2.

Definition 2.4. Let $\varepsilon > 0$. A mapping f from a subset A of a Banach space X into a Banach space Y is called *ε-rigid* if it is $(1+\varepsilon)$-bilipschitz; that is $(1+\varepsilon)^{-1}\|x-y\| \leq \|f(x)-f(y)\| \leq (1+\varepsilon)\|x-y\|$ for all $x, y \in A$.

We will mostly deal with ε-rigid mappings of nets. It will be convenient to have another description for them.

Definition 2.5. Let $\varepsilon > 0$. A mapping f from a subset A of a Banach space X into a Banach space Y is called an *ε-nearisometry* if $\|x-y\| - \varepsilon \leq \|f(x)-f(y)\| \leq \|x-y\| + \varepsilon$ for all $x, y \in A$.

Suppose that $\mathrm{D}(A) < \infty$ and that A is a discrete set such that $\|x-y\| \geq a$ for $x, y \in A$, $x \neq y$. If $f : A \to Y$ is ε-rigid, then f is an $\varepsilon\mathrm{D}(A)$-nearisometry. Conversely, for each $0 < \varepsilon \leq a/2$ every ε-nearisometry of A is a $\frac{2}{a}\varepsilon$-rigid mapping. In other words, for nets these two notions basically coincide.

The following estimate of linearity of Lipschitz mappings appears in [Za] (also see [BL, p. 81]).

Lemma 2.6. *Let X be a Hilbert space, $\alpha > 0$ and let $f : X \to X$ be an α-Lipschitz mapping. Let $x_1, \dots, x_n \in X$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then*

$$\|f(\textstyle\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i f(x_i)\|^2 \leq \alpha D \cdot \max(\alpha\|x_i - x_j\| - \|f(x_i) - f(x_j)\|),$$

where $D = \mathrm{D}(\{x_1, \dots, x_n\})$.

Notice that if f is ε-rigid on some set $A \subset X$ with $D = \mathrm{D}(A) < \infty$, then

$$0 \leq (1+\varepsilon)\|x-y\| - \|f(x)-f(y)\| \leq \varepsilon(\varepsilon+2)\|x-y\| \tag{2}$$

for $x, y \in A$. If $0 < \varepsilon < 1$, then by Lemma 2.6

$$\|f(\textstyle\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \leq 3D\sqrt{\varepsilon} \tag{3}$$

for $x_1, \dots, x_n \in A$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. We will also use the following iteration of Lemma 2.6.

Lemma 2.7. *Let X be a Hilbert space, $A \subset X$, $\mathrm{D}(A) < \infty$. Let $f : X \to X$ be α-Lipschitz and such that $\alpha\|x - y\| - \|f(x) - f(y)\| \leq \delta$ for $x, y \in A$. Suppose $X_1, \dots, X_n \in \operatorname{conv} A$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then*

$$\|f(\textstyle\sum_{i=1}^n \lambda_i X_i) - \sum_{i=1}^n \lambda_i f(X_i)\|^2 \leq 4\alpha\delta\mathrm{D}(A).$$

Proof. Let $X_i = \sum_{j=1}^N a_j^i x_j^i$, where $x_j^i \in A$, $a_j^i \geq 0$, and $\sum_{j=1}^N a_j^i = 1$. Lemma 2.6 implies that

$$\begin{aligned}\|f(\textstyle\sum_{i=1}^n \lambda_i X_i) - \sum_{i=1}^n \lambda_i f(X_i)\| &\leq \|f(\textstyle\sum_{i=1}^n \lambda_i X_i) - \sum_{i=1}^n \sum_{j=1}^N \lambda_i a_j^i f(x_j^i)\| \\ &+ \lambda_1 \|f(X_1) - \textstyle\sum_{j=1}^N a_j^1 f(x_j^1)\| + \dots + \lambda_n \|f(X_n) - \sum_{j=1}^N a_j^n f(x_j^n)\| \\ \leq &(\alpha\delta\mathrm{D}(A))^{\frac{1}{2}}(1 + (\lambda_1 + \dots + \lambda_n)) \leq 2(\alpha\delta\mathrm{D}(A))^{\frac{1}{2}}.\end{aligned}$$

□

3 Extensions to Convex Hulls

In this paper, we investigate extension properties of mappings defined on a subset of a net of S^{d-1} rather than on a subset of a net of $B_{\mathbb{R}^d}$. The reason for this is, that, as we will show in Proposition 3.2, an ε-rigid mapping of a bounded subset A of a Hilbert space X can be extended to a net of the convex hull of A without altering the bilipschitz constant too much. This is a simple corollary of Zarantonello's Lemma 2.6. Here is the idea. By the Kirszbraun-Valentine extension theorem for Lipschitz mappings [BL, p. 18], the mapping $f : A \to X$ can be extended to a $(1+\varepsilon)$-Lipschitz mapping $f : X \to X$. Similarly, the mapping $f^{-1} : f(A) \to X$ can be extended to a $(1+\varepsilon)$-Lipschitz mapping $\tilde{f} : X \to X$. Lemma 2.6 then implies that $\tilde{f}$ well approximates the inverse of $f : \operatorname{conv} A \to X$. Consequently, f is bilipschitz on a net of $\operatorname{conv} A$.

Lemma 3.1. *Let X be a Hilbert space, $A \subset X$ such that $\mathrm{D}(A) < \infty$, $0 < \varepsilon < 1$, and let $f : A \to X$ be $(1+\varepsilon)$-bilipschitz. Let f be a $(1+\varepsilon)$-Lipschitz extension of f to X, and let $\tilde{f} : X \to X$ be a $(1+\varepsilon)$-Lipschitz extension of $f^{-1} : f(A) \to A$. (Both extensions exist by the theorem of Kirszbraun and Valentine.) Then $\|\tilde{f}(f(x)) - x\| \leq 12\mathrm{D}(A)\sqrt{\varepsilon}$ for each $x \in \operatorname{conv} A$.*

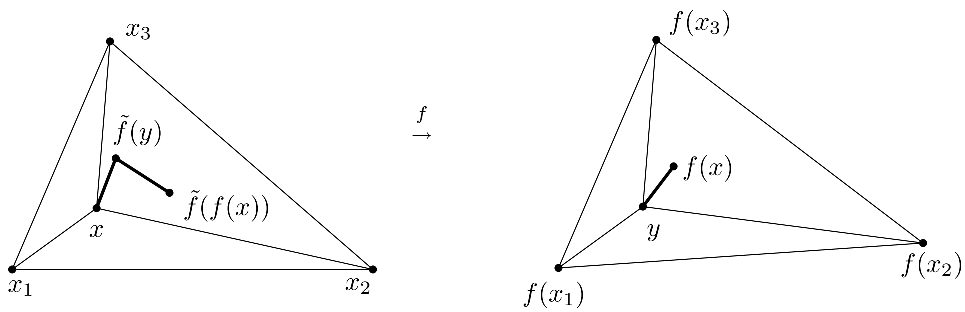

Figure 2: Lemma 3.1 when $|A| = 3$. Here $x = \sum a_i x_i$, $y = \sum a_i f(x_i)$ and the thick segments have length at most $cD\sqrt{\varepsilon}$.

PROOF. Put $D = \mathrm{D}(A)$. Let $x \in \operatorname{conv} A$, where $x = \sum a_i x_i$, $x_i \in A$, $a_i \geq 0$, and $\sum a_i = 1$. Fig. 2 illustrates the situation when $|A| = 3$. Lemma 2.6, and (3) in particular, imply that $\|f(x) - \sum a_i f(x_i)\| \leq 3D\sqrt{\varepsilon}$. Since $\tilde{f}$ is $(1+\varepsilon)$-Lipschitz,

$$\|\tilde{f}(f(x)) - \tilde{f}(\textstyle\sum a_i f(x_i))\| \leq 3(1+\varepsilon)D\sqrt{\varepsilon} \leq 6D\sqrt{\varepsilon}.$$

Since $\tilde{f}$ is $(1+\varepsilon)$-bilipschitz on $f(A)$, and $\mathrm{D}(f(A)) \leq (1+\varepsilon)D$, by Lemma 2.6 applied to $\tilde{f}$, we have that

$$\|\tilde{f}(\textstyle\sum a_i f(x_i)) - x\| = \|\tilde{f}(\sum a_i f(x_i)) - \sum a_i \tilde{f}(f(x_i))\| \leq 6D\sqrt{\varepsilon},$$

and the statement of the lemma follows from the triangle inequality. □

Proposition 3.2. *Let X be a Hilbert space, and $A \subset X$ be such that $\mathrm{D}(A) < \infty$; let $0 < \varepsilon < 1$. Let $f : A \to X$ be $(1+\varepsilon)$-bilipschitz. Again denote by f its $(1+\varepsilon)$-Lipschitz extension to X (It exists by the theorem of Kirszbraun and Valentine.) and put $D = \max\{1, \mathrm{D}(A)\}$. Then*

(i) *f is a $(50D\sqrt{\varepsilon})$-nearisometry on $\operatorname{conv} A$;*

(ii) *if, moreover, $\mathcal{N} \subset X$ is such that $\|x - y\| \geq a > 0$ if $x, y \in \mathcal{N}$ and $x \neq y$, and $0 < \varepsilon < 10^{-4}(a/D)^2$, then f is $(1 + 100\frac{1}{a}D\sqrt{\varepsilon})$-bilipschitz on $\operatorname{conv} A \cap \mathcal{N}$.*

PROOF. Let $x, y \in \operatorname{conv} A$, $x \neq y$. Since f is $(1+\varepsilon)$-Lipschitz,

$$\|f(x) - f(y)\| \leq (1+\varepsilon)\|x - y\| \leq \|x - y\| + \varepsilon D.$$

Let $\tilde{f} : X \to X$ be a $(1+\varepsilon)$-Lipschitz extension of $f^{-1} : f(A) \to A$. By Lemma 3.1

$$\begin{aligned}\|f(x) - f(y)\| \geq & (1+\varepsilon)^{-1}\|\tilde{f}(f(x)) - \tilde{f}(f(y))\| \\ \geq & (1-\varepsilon)(\|x-y\| - 24D\sqrt{\varepsilon}) \geq \|x-y\| - 50D\sqrt{\varepsilon}.\end{aligned}$$

This means that f is a $(50D\sqrt{\varepsilon})$-nearisometry on $\operatorname{conv} A$. If, moreover, $0 < \varepsilon < 10^{-4}(a/D)^2$, then f is $(1 + 100\frac{1}{a}\mathrm{D}(A)\sqrt{\varepsilon})$-bilipschitz on $\operatorname{conv} A \cap \mathcal{N}$ by the remark after Definition 2.5. □

4 Approximation by Linear Mappings

Let $f : B_{\mathbb{R}^d} \to \mathbb{R}^d$ be a $(1+\varepsilon)$-bilipschitz mapping. By a result of Kalton [K], there exists an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ so that $\|f(x) - T(x)\| \leq c\varepsilon \ln d$, where c is an absolute constant. By [M], this estimate is sharp. In this section we modify Kalton's proof to get an approximation of a bilipschitz mapping of a net of S^{d-1} by an isometry; see Theorem 4.4.

For more general mappings $f : A \to \ell_2$, where $A \subset \mathbb{R}^d$ is some compact set (not necessarily a sphere or a net of a sphere), there is the following approximation result of Alestalo, Trotsenko, and Väisälä.

Theorem 4.1. *[ATV] Let $A \subset \mathbb{R}^d$ be bounded, $0 < \varepsilon < 1$, and let $f : A \to \ell_2$ be an $\varepsilon\mathrm{D}(A)$-nearisometry. There is a surjective isometry $S : \ell_2 \to \ell_2$ so that $\|f(x) - S(x)\| \leq c_d\mathrm{D}(A)\sqrt{\varepsilon}$, where c_d depends only on d. If $f(A) \subset \mathbb{R}^d$, we can choose S so that $S(\mathbb{R}^d) \subset \mathbb{R}^d$. In particular, if we extend f by setting $f(x) = S(x)$ for $x \in \ell_2 \setminus A$, then $f : \ell_2 \to \ell_2$ is a δ-nearisometry with $\delta = c_d\mathrm{D}(A)\sqrt{\varepsilon}$.*

Nearisometries and rigid mappings basically coincide for finite discrete sets, as we already mentioned in Section 2. Theorem 4.1 immediately implies the following.

Corollary 4.2. *There exists $c_d > 0$ depending only on d with the following property. Let $0 < a < \sqrt{2}$, let $\mathcal{N}$ be an a-net of $B_{\mathbb{R}_d}$ and let $0 < \varepsilon < 1$. Let $A \subset \mathcal{N}$ and $f : A \to \ell_2$ be $(1+\varepsilon)$-bilipschitz. There exists an isometry $S : \ell_2 \to \ell_2$ with $\|f(x) - S(x)\| \leq c_d\sqrt{\varepsilon}$ for $x \in A$. If $f(A) \subset \mathbb{R}^d$, we can choose S so that $S(\mathbb{R}^d) \subset \mathbb{R}^d$. Moreover, f admits a $(1+\delta)$-bilipschitz extension to $\mathcal{N}$ with $\delta \leq c_d\frac{1}{a}\sqrt{\varepsilon}$. If $f(A) \subset \mathbb{R}^d$, then we also have $f(\mathcal{N}) \subset \mathbb{R}^d$.*

Proof. The mapping f is an $\varepsilon\mathrm{D}(A)$-nearisometry, as it is $(1+\varepsilon)$-bilipschitz. By Theorem 4.1, it can be approximated by an isometry with an error of no more than $c_d\mathrm{D}(A)\sqrt{\varepsilon}$, and extended to a δ-nearisometry with $\delta \leq c_d\mathrm{D}(A)\sqrt{\varepsilon}$.

Suppose $\varepsilon \leq a^2/(2c_d D(A))^2$. Then $\delta \leq a/2$ and the δ-nearisometry is $(1+\delta')$-bilipschitz on $\mathcal{N}$, where $\delta' = \frac{2}{a}\delta \leq \frac{2}{a}c_d \mathrm{D}(\mathcal{N})\sqrt{\varepsilon} \leq c_d \frac{4}{a}\sqrt{\varepsilon}$, since $\mathrm{D}(\mathcal{N}) \leq 2$. Next we will observe that for any $0 < \varepsilon < 1$, the mapping $f : A \to \ell_2$ can be extended to a $10/a$-bilipschitz mapping of $\mathcal{N}$ (while preserving $f(\mathcal{N}) \subset \mathbb{R}^d$ if $f(A) \subset \mathbb{R}^d$). This will finish the proof, by enlarging c_d to $\max\{4c_d, 10\}$.

Indeed, let M and N be two discrete sets of diameters at most $\alpha > 0$ and so that the distances between the points in M and the distances between the points in N are at least a. Then any bijection from M to N is α/a-bilipschitz. Place a copy $\tilde{\mathcal{N}}$ of $\mathcal{N}$ at a distance a from $f(A)$ and extend f as a bijection of $\mathcal{N}$ into $f(A) \cup \tilde{\mathcal{N}}$. As $D(f(A) \cup \tilde{\mathcal{N}}) \leq 10$, this extension is $10/a$-bilipschitz. □

In Proposition 4.5 we will show that if we wish to extend the mapping f from Corollary 4.2 to a $(1 + c_{a,d}\sqrt{\varepsilon})$-bilipschitz mapping with range in $\mathbb{R}^d$, then $c_{a,d}$ really does depend on the dimension, independently of the method we use to extend f. In other words, the function $\sqrt{\varepsilon}$ does not go to zero slowly enough to extend ε-rigid mappings to $c_a\sqrt{\varepsilon}$-rigid mappings on a-nets (with c_a independent of the dimension).

We will approximate ε-rigid mappings on nets of S^{d-1} by isometries, by reducing the situation to the following theorem, as was done by Kalton in [K].

Theorem 4.3. *[K] There is an absolute constant c with the following property. Let $\varepsilon > 0$ and $\Omega : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, be a continuous mapping such that*

(i) $\Omega(\lambda x) = \lambda\Omega(x)$, *if* $\lambda \in \mathbb{R}$ *and* $x \in \mathbb{R}^d$*;*

(ii) $\|\Omega(x + y) - \Omega(x) - \Omega(y)\| \leq \varepsilon(\|x\| + \|y\|)$ *for* $x, y \in \mathbb{R}^d$*.*

Then there is a linear mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ with $\|\Omega(x) - T(x)\| \leq c\varepsilon\|x\| \ln d$ for all $x \in \mathbb{R}^d$.

Theorem 4.4. *Let $0 < a < \sqrt{2}$, $d \geq 2$, let $\mathcal{N}$ be an a-net of S^{d-1} and let $0 < \varepsilon < 1$. Let $f : \mathcal{N} \to \mathbb{R}^d$ be $(1 + \varepsilon)$-bilipschitz. Then there exists an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ with $\|f(x) - T(x)\| \leq c\frac{1}{2-a^2}\sqrt{\varepsilon}\ln d$ for $x \in \mathcal{N}$, where c is an absolute constant.*

PROOF. Denote also by f a $(1 + \varepsilon)$-Lipschitz extension of f to $\mathbb{R}^d$. We can assume that $f(0) = 0$ (otherwise we add $-f(0)$ to f and at the end $f(0)$ to the approximating isometry we will find). By Lemma 2.3, $B_{\mathbb{R}^d}(0, b) \subset \operatorname{conv}\mathcal{N}$, where $b = 1 - \frac{a^2}{2}$.

Claim 1. Let $x, y \in \operatorname{conv}\mathcal{N} \supset B_{\mathbb{R}^d}(0, b)$ and let $t \in [0, 1]$. Then

$$\|f(tx + (1 - t)y) - tf(x) - (1 - t)f(y)\| \leq 15\sqrt{\varepsilon}.$$

In particular, $\|f(tx) - tf(x)\| \le 15\sqrt{\varepsilon}$, and for $x \in B_{\mathbb{R}^d}(0,b)$ we have $\|f(x) + f(-x)\| \le 30\sqrt{\varepsilon}$. Indeed, since f is $(1+\varepsilon)$-bilipschitz on $\mathcal{N}$, we have by (2) for every $x, y \in \mathcal{N}$ that

$$0 \le (1+\varepsilon)\|x-y\| - \|f(x) - f(y)\| \le 3\varepsilon\|x-y\|.$$

Hence by Lemma 2.7, if $x, y \in \operatorname{conv}\mathcal{N}$, then

$$\|f(tx + (1-t)y) - tf(x) - (1-t)f(y)\| \le 2\sqrt{6}\sqrt{\varepsilon}\mathrm{D}(\mathcal{N}) \le 15\sqrt{\varepsilon}.$$

Since $f(0) = 0$ we get immediately that $\|f(tx) - tf(x)\| \le 15\sqrt{\varepsilon}$. If $x \in B_{\mathbb{R}^d}(0,b)$, then $-x \in \operatorname{conv}\mathcal{N}$. Hence

$$\tfrac{1}{2}\|f(x) + f(-x)\| = \|f(\tfrac{x-x}{2}) - \tfrac{1}{2}(f(x) + f(-x))\| \le 15\sqrt{\varepsilon}.$$

Define $\Omega(0) = 0$ and for $x \ne 0$

$$\Omega(x) = \tfrac{1}{2b}\|x\|(f(b\tfrac{x}{\|x\|}) - f(-b\tfrac{x}{\|x\|})).$$

Claim 2. If $x \in B_{\mathbb{R}^d}(0,b)$, then $\|\Omega(x) - f(x)\| \le 30\sqrt{\varepsilon}$. If $x \in \operatorname{conv}\mathcal{N} \setminus B_{\mathbb{R}^d}(0,b)$, then $\|\Omega(x) - f(x)\| \le 30\frac{1}{b}\sqrt{\varepsilon}$. Indeed, by the definition of Ω,

$$\|\Omega(x) - f(x)\| = \|\tfrac{\|x\|}{2b}(f(b\tfrac{x}{\|x\|}) - f(-b\tfrac{x}{\|x\|})) - f(x)\|.$$

If $x \in B_{\mathbb{R}^d}(0,b)$, we have by Claim 1 that

$$\begin{aligned}\|\Omega(x) - f(x)\| \le &\tfrac{1}{2}\|\tfrac{\|x\|}{b}f(b\tfrac{x}{\|x\|}) - f(x)\| + \tfrac{1}{2}\|\tfrac{\|x\|}{b}f(b\tfrac{-x}{\|x\|}) - f(-x)\| \\ &+ \tfrac{1}{2}\|f(x) + f(-x)\| \\ \le &\tfrac{1}{2}(15\sqrt{\varepsilon} + 15\sqrt{\varepsilon} + 30\sqrt{\varepsilon}) = 30\sqrt{\varepsilon}.\end{aligned}$$

If $x \in \operatorname{conv}\mathcal{N} \setminus B_{\mathbb{R}^d}(0,b)$, then by Claim 1

$$\begin{aligned}\|\Omega(x) - f(x)\| \le &\tfrac{\|x\|}{b}\|f(b\tfrac{x}{\|x\|}) - \tfrac{b}{\|x\|}f(x)\| + \tfrac{\|x\|}{2b}\|f(b\tfrac{x}{\|x\|}) + f(-b\tfrac{x}{\|x\|})\| \\ \le &\tfrac{1}{b}15\sqrt{\varepsilon} + \tfrac{1}{2b}30\sqrt{\varepsilon} = 30\tfrac{1}{b}\sqrt{\varepsilon}.\end{aligned}$$

Let $x, y \in \mathbb{R}^d$ be such that $\beta = (\|x\| + \|y\|)/b \ne 0$. Then $(x+y)/\beta$, x/β, $y/\beta \in B_{\mathbb{R}^d}(0,b)$, and by Claim 2 and Claim 1 we get that

$$\begin{aligned}\|\Omega(x+y) - \Omega(x) - \Omega(y)\| = &\beta\|2\Omega(\tfrac{x+y}{2\beta}) - \Omega(\tfrac{x}{\beta}) - \Omega(\tfrac{y}{\beta})\| \\ \le &\beta(2\|f(\tfrac{x+y}{2\beta}) - \tfrac{1}{2}(f(\tfrac{x}{\beta}) + f(\tfrac{y}{\beta}))\| + 4\cdot 30\sqrt{\varepsilon}) \\ \le &\beta(2\cdot 15\sqrt{\varepsilon} + 4\cdot 30\sqrt{\varepsilon}) = 150\sqrt{\varepsilon}(\|x\| + \|y\|)/b.\end{aligned}$$

Therefore, by Theorem 4.3, there exists a linear $T:\mathbb{R}^d\to\mathbb{R}^d$ so that

$$\|\Omega(x)-Tx\|\le c\sqrt{\varepsilon}\tfrac{1}{b}\|x\|\ln d \text{ for } x\in\mathbb{R}^d.$$

It remains to replace T by an isometry. Let $\|x\|=b$. By Claim 2,

$$\|Tx-f(x)\|\le\|\Omega(x)-Tx\|+\|\Omega(x)-f(x)\|\le c\sqrt{\varepsilon}\ln d+30\sqrt{\varepsilon}\le c\sqrt{\varepsilon}\ln d.$$

By Proposition 3.2, if $\|x\|=b$, then $1-c\frac{1}{b}\sqrt{\varepsilon}\le\frac{1}{b}\|f(x)\|\le 1+\varepsilon$. Consequently,

$$1-c\tfrac{1}{b}\sqrt{\varepsilon}\ln d\le\tfrac{\|Tx\|}{\|x\|}\le 1+c\tfrac{1}{b}\sqrt{\varepsilon}\ln d$$

and there exists an isometry U so that $\|U-T\|\le c\frac{1}{b}\sqrt{\varepsilon}\ln d$. From Claim 2 it then follows that for $x\in\operatorname{conv}\mathcal{N}$,

$$\|Ux-f(x)\|\le\|f(x)-\Omega(x)\|+\|\Omega(x)-Tx\|+\|Tx-Ux\|\le c\tfrac{1}{b}\sqrt{\varepsilon}\ln d. \qquad \square$$

Recall that if $d=2^k$ for some $k\in\mathbb{N}$, then $\mathbb{R}^d$ contains an orthonormal basis comprising vectors of the form $\frac{1}{\sqrt{d}}\sum_{i=1}^d \pm e_i$. This is the so-called Walsh basis.

Proposition 4.5. *Let $0<a<\sqrt{2}$. Suppose $\gamma:\mathbb{N}\to\mathbb{R}^+$, $\gamma=\gamma_a$, has the following property. If $0<\varepsilon<1$, $d\ge 2$ and $\mathcal{N}\subset S^{d-1}$ is an a-net, $A\subset\mathcal{N}$ and $f:A\to\mathbb{R}^d$ is $(1+\varepsilon)$-bilipschitz, then f admits a $(1+\gamma(d)\sqrt{\varepsilon})$-bilipschitz extension $f:\mathcal{N}\to\mathbb{R}^d$. Then $\gamma(d)\ge c_a d^{\frac14}\ln^{-2}d$, where $c_a>0$ depends only on a.*

Proof. Let d be so that $\sqrt{2}(1-\frac{1}{\sqrt{d}})\ge a$. For the finitely many smaller d we just adjust c_a. Choose $k\in\mathbb{N}$ so that $2^k\le d<2^{k+1}$, and put $d_0=2^k$. Let $v_1,\dots,v_{d_0/2}$ be the Walsh basis in $\mathbb{R}^{d_0/2}$. Let

$$A=\{\pm e_1,\dots,\pm e_{d_0/2},\pm v_1,\dots,\pm v_{d_0/2}\},$$

and let $\mathcal{N}$ be an a-net of S^{d-1} containing A. Define $f:A\to\mathbb{R}^d$ by $f(\pm e_i)=\pm e_i$ and $f(\pm v_i)=\pm e_{i+d_0/2}$ for $i=1,\dots,d_0/2$. An elementary computation shows that f is $(1+cd^{-\frac12})$-bilipschitz for some absolute constant $c>0$. Suppose f admits a $(1+\gamma(d)d^{-\frac14})$-bilipschitz extension to $\mathcal{N}$. By Theorem 4.4, there exists an isometry T so that $\|f(x)-Tx\|\le c\frac{1}{2-a^2}\gamma^{\frac12}(d)d^{-\frac18}\ln d=\delta(d)$. Let $Z=T(\mathbb{R}^{d_0/2})$. Then Z is a $d_0/2$-dimensional affine subspace of $\mathbb{R}^d$. Put $Q=\{\pm e_1,\dots,\pm e_{d_0}\}\subset f(\mathcal{N})$. By [T, p. 237] there exists $q\in Q$ so that $\operatorname{dist}(Z,q)\ge 1/\sqrt{2}$. Therefore $\delta(d)\ge 2^{-\frac12}$ and, consequently, $\gamma(d)\ge c_a d^{\frac14}/\ln^2 d$. $\square$

5 Extension to Sparse Nets of the Sphere Is Possible

Let $\varepsilon > 0$ be fixed and let $\mathcal{N} \subset B_{\mathbb{R}^d}$ be an a-net with $a > 0$ small enough (how small depends only on ε). In Section 2 we mentioned a simple example of a $(1+\varepsilon)$-bilipschitz mapping of a subset A of $\mathcal{N}$ which cannot be extended to a 2-bilipschitz mapping of $\mathcal{N}$ into $\mathbb{R}^d$.

In this section we will deal with the other extreme case when a is very close to $\sqrt{2}$. Let $\varepsilon > 0$ be fixed and let $\mathcal{N}$ be a symmetric a-net of S^{d-1} with $0 < a < \sqrt{2}$ close enough to $\sqrt{2}$ (how close depends only on ε, but not on the dimension d). Suppose $A \subset \mathcal{N}$, and that $f : \{0\} \cup A \to \mathcal{N}$ is $(1+\varepsilon)$-bilipschitz. Then f can be extended to $\mathcal{N}$ with a bilipschitz constant not much larger. We prove this assertion in Proposition 5.4 as a simple corollary of an estimate of the size of $\mathcal{N}$ in Lemma 5.3. For an easy reference we include a simple proof of Lemma 5.3 and the results needed for it.

Lemma 5.1. *Let $0 < \varepsilon < 1$ and $d \in \mathbb{N}$ be given. If $A_0 \subset S^{d-1}$, then there exists $A \subset S^{d-1}$ so that $|A_0 \cup A| \geq \frac{1}{4} e^{\varepsilon^2 d/2}$ and $|\langle x, y\rangle| < \varepsilon$ for each $x \in A_0 \cup A$ and $y \in A$, $x \neq y$.*

Proof. If $u \in S^{d-1}$, then by the concentration of measure on the sphere $P[|\langle u, x\rangle| \geq \varepsilon] \leq 4e^{-\varepsilon^2 d/2}$. If $A' \subset S^{d-1}$ consists of vectors as required above, but $|A'| < \frac{1}{4} e^{\varepsilon^2 d/2}$, then $P[|\langle u, x\rangle| < \varepsilon$ for all $u \in A'] > 1 - 4e^{-\varepsilon^2 d/2}|A'| > 0$ and the set A' can be enlarged. $\square$

Theorem 5.2. *[Al] Let $B = (b_{i,j})$ be an $n \times n$ matrix with $b_{i,i} = 1$ for all i and $|b_{i,j}| \leq \varepsilon$ for all $i \neq j$. If the rank of B is d and $\frac{1}{\sqrt{n}} \leq \varepsilon \leq \frac{1}{3}$, then*

$$d \geq c \frac{\ln n}{\varepsilon^2 \ln \frac{1}{\varepsilon}},$$

where $c > 0$ is an absolute constant.

Using these two results we can easily estimate the size of a symmetric a-net when a is close to $\sqrt{2}$.

Lemma 5.3. *Let $\frac{4}{3} \leq a < \sqrt{2}$, let $\mathcal{N} \subset S^{d-1}$ be a symmetric a-net of S^{d-1}, and $b = 1 - \frac{a^2}{2}$. If $\frac{1}{\sqrt{d}} \leq b$, then $e^{cdb^2} \leq |\mathcal{N}| \leq e^{Cdb^2 \ln \frac{1}{b}}$ where $c, C > 0$ are absolute constants.*

Proof. First recall that by Lemma 2.2, $\mathcal{N}$ is a symmetric inclusion-maximal subset of S^{d-1} such that $|\langle x, y\rangle| \leq b$ for all $x, y \in \mathcal{N}$, $x \neq y$. Notice that for the lower estimate we do not have to assume that $\mathcal{N}$ is symmetric. Suppose $|\mathcal{N}| < \frac{1}{4} e^{b^2 d/2}$. Then by Lemma 5.1 applied to $A_0 = \mathcal{N}$ and $\varepsilon = b$ the set $\mathcal{N}$ can be enlarged, and this contradicts its maximality.

To get the upper estimate, we let $\mathcal{N}_0$ be "one half" of the set $\mathcal{N}$ (that is, $\mathcal{N} = -\mathcal{N}_0 \cup \mathcal{N}_0$ and $-\mathcal{N}_0 \cap \mathcal{N}_0 = \emptyset$) and define the matrix $B = (\langle x, y\rangle)_{x,y\in\mathcal{N}_0}$. Since $\mathcal{N}_0 \subset \mathbb{R}^d$, the rank of B is at most d. By Lemma 2.2, if $x, y \in \mathcal{N}_0$, then $|\langle x, y\rangle| \le b$. Theorem 5.2 then implies that $d \ge c \ln |\mathcal{N}_0|/b^2 \ln \frac{1}{b}$, and the estimate in the lemma follows. □

Suppose a net $\mathcal{N}$ is symmetric and "thin" enough; that is, $|\langle x, y\rangle|$ is not very far from $\sqrt{\varepsilon}$ for $x \ne \pm y \in \mathcal{N}$. If $A \subset \mathcal{N}$ and $f : \{0\} \cup A \to \mathbb{R}^d$ is $(1 + \varepsilon)$-bilipschitz, then f can be extended to $\mathcal{N}$ without enlarging the bilipschitz constant too much. This is trivial if the range-space is ℓ_2. We first extend f symmetrically to $-A \cup A$, then choose a large enough orthonormal set Q orthogonal to $f(A)$ and finally map $\mathcal{N} \setminus (-A\cup A)$ symmetrically and bijectively into $-Q \cup Q$. If the range-space is only $\mathbb{R}^d$, then the same idea works; we just have to make sure (using Lemma 5.3) that S^{d-1} accommodates an "almost orthogonal" set of cardinality at least $\mathcal{N}$. Here is an example of such an extension.

Proposition 5.4. *Let $0 < \varepsilon < \varepsilon_1$, where $\varepsilon_1 > 0$ is an absolute constant, and let $a \le \sqrt{2}$ be such that $b = 1 - \frac{a^2}{2} \le \sqrt{\varepsilon}$. Suppose $\mathcal{N} \subset S^{d-1}$ is a symmetric a-net, $A \subset \mathcal{N}$ and $f : A \cup \{0\} \to \mathbb{R}^d$ is $(1 + \varepsilon)$-bilipschitz. Then f admits a $(1 + c(\varepsilon \ln \frac{1}{\varepsilon})^{\frac{1}{2}})$-bilipschitz extension $f : \mathcal{N} \to \mathbb{R}^d$.*

Proof. Let $\varepsilon_1 > 0$ be small; just how small can in principle be determined by an inspection of the estimates below. We can assume that $f(0) = 0$ which implies that $1/(1 + \varepsilon) \le \|f(x)\| \le 1 + \varepsilon$ for $x \in A$. Let $A_0 \subset A$ be such that $-A_0 \cap A_0 = \emptyset$ and $A \subset -A_0 \cup A_0$. Similarly, let $\mathcal{N}_0 \subset \mathcal{N}$ be such that $-\mathcal{N}_0 \cap \mathcal{N}_0 = \emptyset$ and $\mathcal{N} = -\mathcal{N}_0 \cup \mathcal{N}_0$. Suppose $x \ne y \in A$. Then

$$\begin{aligned}\Big\|\tfrac{f(x)}{\|f(x)\|} - \tfrac{f(y)}{\|f(y)\|}\Big\| \ge& \|f(x) - f(y)\| - \Big\|f(x) - \tfrac{f(x)}{\|f(x)\|}\Big\| - \Big\|f(y) - \tfrac{f(y)}{\|f(y)\|}\Big\| \\ \ge& a/(1 + \varepsilon) - 2\varepsilon = a_1.\end{aligned}$$

Let $\beta = c' b \ln^{\frac{1}{2}} \frac{1}{b}$, where $c' = \sqrt{C/c}$ and c, C are the absolute constants from Lemma 5.3. Put $a_2 = \sqrt{2 - 2\beta}$, $\alpha = \min\{a_1, a_2\}$ and extend the set $\tilde{A} = \{f(x)/\|f(x)\| : x \in A_0\}$ to a symmetric α-net $\mathcal{M}$ of S^{d-1}. Choose $\mathcal{M}_0 \supset \tilde{A}$ so that $\mathcal{M} = -\mathcal{M}_0 \cup \mathcal{M}_0$ and $-\mathcal{M}_0 \cap \mathcal{M}_0 = \emptyset$. By Lemma 5.3,

$$|\mathcal{N}| \le e^{Cdb^2 \ln \frac{1}{b}} \le e^{cd\beta^2} \le |\mathcal{M}|.$$

Therefore it is possible to extend f as a bijection of $\mathcal{N}_0 \setminus A_0$ into $\mathcal{M}_0 \setminus \tilde{A}$, and for $x \in -\mathcal{N}_0 \setminus A$ put $f(x) = -f(-x)$. If both $x \in A$ and $-x \in A$ for some $x \in \mathcal{N}$ we have by (3) that $\|f(x) + f(-x)\| \le c\sqrt{\varepsilon}$, and, also $\|f(x)/\|f(x)\| + f(-x)\| \le$

$c\sqrt{\varepsilon}$. This and Lemma 2.2 imply the following estimates for the bilipschitz constants of f.

$$
\begin{array}{lll}
\frac{1}{1+\varepsilon} \le \frac{\|f(x)-f(y)\|}{\|x-y\|} \le 1+\varepsilon & & \text{if } x=-y \text{ or } x,y\in A \\
\frac{\alpha}{\sqrt{4-a^2}} \le \frac{\|f(x)-f(y)\|}{\|x-y\|} \le \frac{\sqrt{4-\alpha^2}}{a} & & \text{if } x\ne -y \text{ and } x,y\in \mathcal{N}\setminus A \\
\frac{\alpha-c\sqrt{\varepsilon}}{\sqrt{4-a^2}} \le \frac{\|f(x)-f(y)\|}{\|x-y\|} \le \frac{\sqrt{4-\alpha^2}+c\sqrt{\varepsilon}}{a} & & \text{if } x\ne -y \text{ and } x\in A,\ y\in \mathcal{N}\setminus A.
\end{array}
$$

Since $\alpha^2 \ge 2 - c\sqrt{\varepsilon}\ln^{\frac12}\frac{1}{\varepsilon}$ and $a^2 \ge 2-2\sqrt{\varepsilon}$, an elementary computation shows that the bilipschitz constant of f is at most $1 + c\sqrt{\varepsilon}\ln^{\frac12}\frac{1}{\varepsilon}$. □

Note that we did not assume in Proposition 5.4 that the subset A was symmetric. We did assume, though, that f was bilipschitz on $A\cup\{0\}$. The bilipschitz constant of the extension does not increase too much if instead we assume that for at least one $x\in A$ we also have $-x\in A$.

6 How Large Can an Almost-Isometric Image of $B_{\mathbb{R}^n}$ Get?

It is a simple and a well known application of the concentration of measure on the sphere that S^{d-1} contains a large "almost orthogonal" set. Let $A\subset S^{d-1}$ be such a set. Namely, let $|\langle x,y\rangle| < \varepsilon$ for all $x,y\in A$, $x\ne y$, and let $N=|A|\ge \frac14 e^{\varepsilon^2 d/2}$ which is possible by Lemma 5.1. Let f be a bijection of A and of the set $\{e_1,e_2,\dots,e_N\}$. Define f also on $-A$ by $f(-x)=-f(x)$. The mapping $f: -A\cup A\to\mathbb{R}^N$ is $(1+\varepsilon)$-bilipschitz, if $0<\varepsilon<1/2$. A natural question arises, if f can be extended to S^{d-1}, or at least to some a-net of S^{d-1} containing $-A\cup A$ without altering the bilipschitz constant of f too much.

For $0<\delta<1$, $0<a<\sqrt{2}$, and $d\in\mathbb{N}$, let $k(d,a,\delta)$ be the largest dimension k such that there exists an a-net $\mathcal{N}$ of S^{d-1} and a $(1+\delta)$-bilipschitz mapping $f:\mathcal{N}\to\ell_2$ so that $f(\mathcal{N})$ contains an orthonormal basis of $\mathbb{R}^k$ together with its negative. Does there exist $\delta>0$ so that $\lim_{d\to\infty} k(d,a,\delta)e^{-\varepsilon d}=0$ for every $\varepsilon>0$? Suppose that the answer is *yes.* (We do not know if it is.) This would mean, that for an arbitrarily small $\varepsilon>0$ we could find a large enough dimension d and a $(1+\varepsilon)$-bilipschitz mapping f (the one defined above) which admits no $(1+\delta)$-bilipschitz extension to an a-net containing $-A\cup A$. In other words, the answer to the question in the introduction, whether one can have $\delta=\delta(\varepsilon)$ not depending on the dimension, and at the same time, $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$ would be negative.

In this section we give a lower estimate for k. To do so, we first modify an example of F. John (see [J], or [BL, p. 352]).

Lemma 6.1. *Let $0 < \varepsilon < 1$. If $0 < r \le e^{-\pi/2\varepsilon}$ and $n \in \mathbb{N}$, then there exists a norm-preserving $(1+\varepsilon)$-bilipschitz mapping H of $\mathbb{R}^{n+1}$ onto itself so that $H(x) = -H(-x)$ for $x \in \mathbb{R}^{n+1}$, $H(\pm e_0) = \pm e_0$, and $H(\pm r^{4k-3}e_0) = \mp r^{4k-3}e_k$, for $k = 1, \dots, n$.*

Proof. We prove the assertion of the lemma for $r = e^{-\pi/2\varepsilon}$. If $0 < s < r$, we simply construct an ε'-quasi isometry as below with $\varepsilon' \ln s = -\pi/2$. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be defined in polar coordinates by $h(r,\varphi) = (r, \varphi + \varepsilon \ln r)$ if $r \le 1$ and $h(r,\varphi) = (r,\varphi)$ if $r \ge 1$. This is a norm-preserving $(1+\varepsilon)$-bilipschitz mapping of $\mathbb{R}^2$ onto itself (see [J], or [BL, p. 352]). Let h_k, $k = 1, \dots, n$, be the mapping h written in Cartesian coordinates and considered as a mapping $h_k : \operatorname{span}\{e_0, e_k\} \to \operatorname{span}\{e_0, e_k\}$. Let

$$A_0 = \{u : \|u\| \le r^{4n} \text{ or } 1 \le \|u\|\}$$
$$A_k = \{u : r^{4k} \le \|u\| \le r^{4(k-1)}\},$$

and define $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$H(x) = \begin{cases} x & \text{if } x \in A_0 \\ h_k(x_0, x_k) + \sum_{i=1, i\ne k}^n x_i e_i & \text{if } x = \sum_{i=0}^n x_i e_i \in A_k. \end{cases}$$

Fig. 3 illustrates that H rotates the points $re_0, r^5 e_0, \dots$ by $-\frac{\pi}{2}$ into orthogonal directions $-re_1, -r^5 e_2, \dots$. Notice, that H is well defined as $H(x) = x$ if $\|x\| = r^{4k}$. By the Pythagorean theorem, H is a $(1+\varepsilon)$-bilipschitz mapping of each of the sets $A_0, A_1, \dots, A_n$ onto itself. By Lemma 2 of [IP], H is a $(1+\varepsilon)$-bilipschitz mapping of $\mathbb{R}^{n+1}$ onto itself. If $\|x\| = r^{4k-3}$, H acts on x as a rotation by $\varepsilon \ln r^{4k-3} = -\frac{\pi}{2} - 2(k-1)\pi$ in $\operatorname{span}\{x_0, x_k\}$. Hence $H(r^{4k-3}e_0) = -r^{4k-3}e_k$. ☐

Lemma 6.2. *Let $m, n \in \mathbb{N}$ and let $d = 2^{m(4n+1)}$. There exist orthonormal bases $O_1, O_2, \dots, O_{n+1}$ in $\mathbb{R}^d$ so that if $u \in O_k$, then $\langle u, e_i\rangle \in \{0, \pm 2^{-\frac{m}{2}(4k-3)}\}$ for all $i = 1, 2, \dots, d$.*

Proof. Let $m \in \mathbb{N}$. We use induction on n. If $n = 0$, let O_1 be the Walsh basis in $\mathbb{R}^{2^m}$. (It consists of orthonormal vectors of the form $2^{-m/2}\sum_{i=1}^{2^m} \pm e_i$.) Now assume $S_1, S_2, \dots, S_{n+1}$ are the required orthonormal bases in $\mathbb{R}^{d_0}$, where $d_0 = 2^{m(4n+1)}$. Let $d = 2^{m(4(n+1)+1)} = 2^{4m}d_0$. We write $\mathbb{R}^d$ as a product of 2^{4m} copies of $\mathbb{R}^{d_0}$ and for $k = 1, \dots, n+1$, $j = 1, \dots, 2^{4m}$, denote the basis S_k in the j-th copy of $\mathbb{R}^{d_0}$ by S_k^j. For $k = 1, \dots, n+1$ let $O_k = \bigcup_{j=1}^{2^{4m}} S_k^j$, and let O_{n+2} be the Walsh basis of $\mathbb{R}^d$. ☐

In the next proposition we strengthen an example from [M]. For a suitable dimension d we choose $n \approx \varepsilon \ln d$ orthonormal bases $O_1, \dots, O_n$ in $\mathbb{R}^d$ as it

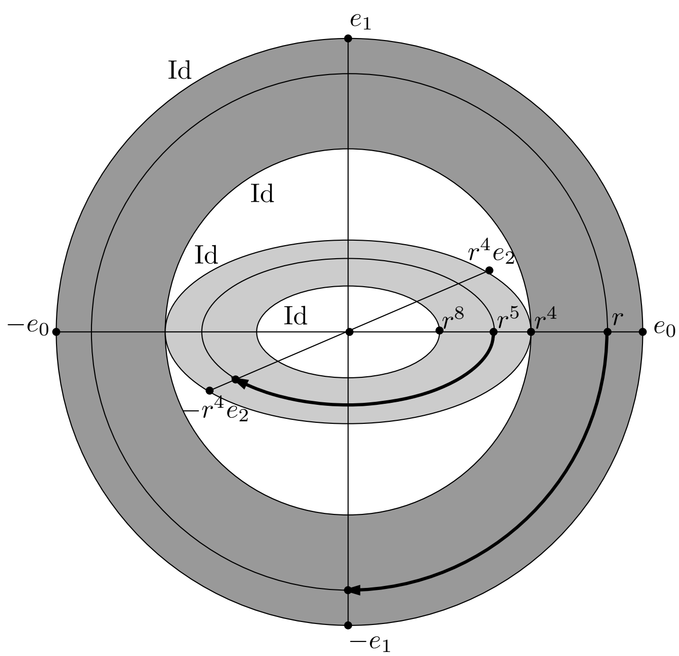

Figure 3: Die Korfsche Uhr [Mo].

was done in Lemma 6.2. The mapping H from Lemma 6.1 applied to each of the vectors $e_1, \dots, e_d$ in place of e_0 produces new n copies $X_1, \dots, X_n$ of $\mathbb{R}^d$ and "twists" each O_k out of $\mathbb{R}^d$ to become a basis of X_k.

Proposition 6.3. *Let $\varepsilon > 0$ be given. For every $K \in \mathbb{N}$ there exists $d > K$ with the following property. There exists $N \geq c\varepsilon d \ln d$ and a norm-preserving $(1+\varepsilon)$-bilipschitz mapping f of $\mathbb{R}^N$ onto itself such that $f(x) = -f(-x)$ for $x \in \mathbb{R}^N$ and $f(B_{\mathbb{R}^d})$ contains an orthonormal basis of $\mathbb{R}^N$.*

Proof. Let $0 < \varepsilon < 1$ be given. Choose the smallest $m \in \mathbb{N}$ so that $2^{-m/2} \leq e^{-\frac{\pi}{2\varepsilon}}$; that is, $m = \lceil \frac{\pi}{\varepsilon \ln 2} \rceil$. Let ε' satisfy $2^{-m/2} = e^{-\pi/2\varepsilon'}$. Then $\varepsilon/2 \leq \varepsilon' \leq \varepsilon$. Choose $n \in \mathbb{N}$ so that $d = 2^{m(4n+1)} > K$. Write $\mathbb{R}^{d(n+2)} = X_0 \oplus X_1 \oplus \dots \oplus X_{n+1}$, where $\mathbb{R}^d \cong X_k = \operatorname{span}\{e_{1+kd}, \dots, e_{d+kd}\}$. Let $O_0 = \{e_1, \dots, e_d\}$ and let $O_1, \dots, O_{n+1}$ be the orthonormal bases in X_0 which exist according to Lemma 6.2. Let S_k be a copy of O_k in X_k, $k = 1, \dots, n+1$. Define $f : \mathbb{R}^{d(n+2)} \to \mathbb{R}^{d(n+2)}$ "block-wise"; namely let f act on each of the blocks $Y_j = \operatorname{span}\{e_{j+0}, e_{j+d}, \dots, e_{j+(n+1)d}\}$ as H, where H is the mapping from Lemma 6.1 (with $r = 2^{-m/2} = e^{-\pi/2\varepsilon'}$). The mapping f is norm-preserving and $(1+\varepsilon)$-bilipschitz with $f(x) = -f(-x)$, since H is such a mapping. It follows that $f = \mathrm{Id}$ on $\pm O_0$ and $f(\pm O_k) = \mp S_k$ if $k = 1, \dots, n+1$. Finally,

$$n + 2 = \frac{\ln d}{4m \ln 2} + \frac{7}{4} = \frac{\varepsilon' \ln d}{4\pi} + \frac{7}{4} > \frac{1}{8\pi} \varepsilon \ln d. \qquad \square$$

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