Real Analysis Exchange
Vol. 29(2), 2003/2004, pp. 979–981

V. Sinha* and Inder K. Rana, Department of Mathematics, Indian Institute of Technology, Bombay, India. email: `vivek@math.iitb.ac.in` and `ikr@math.iitb.ac.in`

ON THE CONTINUITY OF ASSOCIATED INTERVAL FUNCTIONS

Abstract

The aim of this note is to show that for a given continuous function F on a set $E \subset \mathbb{R}$, the associated interval function need not be continuous. We also give an example to show that the associated interval function can be continuous even if F is not continuous.

1 Introduction

Let $E \subset \mathbb{R}$ be a bounded set and $F : E \to \mathbb{R}$. The *associated interval function* of F is the function, again denoted by F, defined on the set of all closed bounded intervals included in E as follows :

$$F([a,b]) = \begin{cases} F(b) - F(a) & \text{if } a \leq b, \\ 0 & \text{if } a > b. \end{cases}$$

In [1, exercise 11.2.6] it is claimed that F is continuous if and only if the associated interval function is continuous in the sense of the definition given below. The aim of this note is to correct the above statement, which is false in general (see Theorem 3.2).

2 Preliminaries

We follow the definitions and notations of [1]. All the intervals considered are closed and bounded intervals. A non-degenerate interval is called a *cell.* A finite, possibly empty, union of cells is called a *figure.* Let $E \subset \mathbb{R}$ and F be a function defined on the class of all intervals which are subsets of E.

Key Words: Interval functions, continuity
Mathematical Reviews subject classification: Primary: 26A99
Received by the editors December 16, 2003
Communicated by: Peter Bullen
*Supported by the Council of Scientific and Industrial Research, India.

We say that F is *additive* if $F(B \cup C) = F(B) + F(C)$ for all nonoverlapping subintervals $B, C \subset E$ for which $B \cup C$ is also an interval. An additive function $F : E \to \mathbb{R}$, extends uniquely to a function, still denoted by F, defined on the class of all figures which are subsets of E such that F is also additive: $F(B \cup C) = F(B) + F(C)$ for all nonoverlapping figures $B, C \subset E$, see [1, page 212]. The *perimeter* of a figure $B \subset \mathbb{R}$, denoted by $||B||$, is the number of points in ∂B, the boundary of B. Equivalently, $||B||$ is twice the number of the connected components of B. In particular, $||B|| = 2$ when B is a cell, and $||B|| = 0$ when $B = \emptyset$. The following definition is given in[1, Definition 11.2.5].

3 Main Results

Definition 3.1. Let F be an additive function defined on all figures in a set $E \subset \mathbb{R}$. We say F is *continuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|F(B)| < \epsilon$ for each figure $B \subset E$ with $||B|| < 1/\epsilon$ and $|B| < \delta$.

For a function f defined on a set E, we define an additive function F on all intervals (and hence on all figures) contained in E as follows :

$$F([a,b]) = \begin{cases} f(b) - f(a) & \text{if } a \leq b, \\ 0 & \text{if } a > b. \end{cases}$$

The function F is called the *associated interval function* of f.

Theorem 3.2. *There exist bounded sets E and functions $f : E \to \mathbb{R}$ such that the following hold :*

(i) *f is continuous and the associated interval function F is not continuous.*

(ii) *E is compact, f is discontinuous, and the associated interval function F is continuous.*

Proof. Let $E = (0,1)$ and $f(x) = 1/x$ for each $x \in E$. Then f is continuous, but the associated interval function is not continuous. Indeed, let $\epsilon = 1/3$ and choose a positive $\delta < 1$. If $B_x = [x, x + \delta/2]$, then $B_x \subset E$, $\|B_x\| = 2 < 1/\epsilon$, and $|B_x| < \delta$ for each $x \in (0, 1/2)$. Since

$$F(B_x) = -\frac{\delta}{x(2x + \delta)},$$

we see that $\big|F(B_x)\big| > \epsilon$ for all sufficiently small $x > 0$.

(ii). On the compact set $E = [-1,0] \cup \{1/n : n \in \mathbf{N}\}$, define a function f by

$$f(x) = \begin{cases} 1 & \forall\, x \in [-1,0], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is not continuous at $0 \in E$, yet the associated interval function F is continuous, since it is identically zero. □

Theorem 3.3. *Let $E \subset \mathbb{R}$ be any set, and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Then the associated interval function F is continuous.*

PROOF. For a given $\epsilon > 0$, find $\delta > 0$ so that $\big|f(y) - f(x)\big| < \epsilon^2$ for all $x, y \in E$ with $|y - x| < \delta$. If $B \subset E$ is a figure with $|B| < \delta$ and $\|B\| < 1/\epsilon$, then B is the union of k intervals of length less than δ, where $k < 1/(2\epsilon)$. Thus

$$\big|F(B)\big| < k\epsilon^2 < \frac{1}{2}\epsilon < \epsilon,$$

and the continuity of F is established. □

Theorem 3.4. *Let $E \subset \mathbb{R}$ be a figure and let $f : E \to \mathbb{R}$ be a function. Then f is continuous if and only if the associated interval function F is continuous*

PROOF. Suppose F is continuous. Choose a positive $\epsilon < 1/2$, and a $\delta > 0$ so that $\big|F(B)\big| < \epsilon$ for each figure $B \subset E$ with $|B| < \delta$ and $\|B\| < 1/\epsilon$. Making δ smaller, we may assume that each pair of points $x, y \in E$ with $|y - x| < \delta$ belongs to the same connected component of E. Select $x, y \in E$ with $|y - x| < \delta$, and denote by B the interval whose end points are x and y. Then $B \subset E$, $|B| < \delta$, and $\|B\| = 2 < 1/\epsilon$. Consequently

$$\big|f(y) - f(x)\big| = \big|F(B)\big| < \epsilon,$$

and we see that f is continuous. As E is a compact set, the converse follows from Theorem 3.3.

Acknowledgement. The authors wish to thank a referee for suggestions that led to shorter proofs. □

References

[1] W. F. Pfeffer, *The Riemann Approach to Integration,* Cambridge University Press, 1993.