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MB-REPRESENTATIONS OF ALGEBRAS GENERATED BY INTERVALS

Abstract

We show various generalizations of the theorem from [3] concerning MB-representation of the interval algebra on $[0,1)$.

Let X be a nonempty set and let $\mathcal{F}$ be a nonempty family of nonempty subsets of X. Following the idea of Burstin and Marczewski we define:

$$S(\mathcal{F}) = \{A \subset X : (\forall\, P \in \mathcal{F})(\exists\, Q \in \mathcal{F})(Q \subset A \cap P \text{ or } Q \subset P \setminus A)\}$$

and

$$S_0(\mathcal{F}) = \{A \subset X : (\forall\, P \in \mathcal{F})(\exists\, Q \in \mathcal{F})(Q \subset P \setminus A)\}.$$

Then $S(\mathcal{F})$ is an algebra (more precisely, a field) of subsets of X, and $S_0(\mathcal{F})$ is an ideal on X (See [3]).

We say that an algebra $\mathcal{A}$ (a pair $\langle \mathcal{A}, \mathcal{I} \rangle$, where $\mathcal{I}$ is an ideal contained in $\mathcal{A}$, respectively) of subsets of X has a Marczewski-Burstin representation if there exists a nonempty family $\mathcal{F}$ of nonempty subsets of X, such that $\mathcal{A} = S(\mathcal{F})$ ($\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$, respectively). If in addition $\mathcal{F} \subset \mathcal{A}$ we say that $\langle \mathcal{A}, \mathcal{I} \rangle$ is inner MB-representable. MB-representations of algebras and ideals were recently considered in the papers [3], [1], [5], [6], [7].

In [3] it was proved (by the idea of S. Wroński) that the interval algebra $\mathcal{A}$ generated by the intervals $[x, y)$ where $x, y \in [0,1)$ has outer MB-representation (i.e., $\mathcal{A} = S(\mathcal{F})$ and $\mathcal{F} \cap \mathcal{A} = \emptyset$). The construction used in the proof of this theorem was generalized in [4], [8], [2].

The aim of this paper is the modification of Wroński's construction which can be used to describe MB-representations of the following algebras on $\mathbb{R}$:

– the algebra generated by all intervals (or, what equivalent, by open intervals);

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– the algebras generated by the intervals of some type (open or one-side-closed) with the endpoints in the fixed set X_0;

– the algebras generated by the intervals considered relative to a fixed set X dense in some interval in $\mathbb{R}$.

Fix an interval $[a,b]$ in the real line with $-\infty \le a < b \le +\infty$, and fix two sets X and X_0 where $X_0 \subset X$ and X is a dense subset of $[a,b]$. To simplify notation we write $\overline{X}$ instead of $[a,b]$ and $[x,y)$, (x,y) instead of $[x,y) \cap X$, $(x,y) \cap X$, respectively. The algebras of sets considered in the paper will always constitute subfields of $\mathcal{P}(X)$.

Let $\mathcal{D}$ be a fixed countable dense subset of X (consequently of $\overline{X}$).

Lemma 1. *There exists a family of almost disjoint sets $\mathcal{D}_\alpha \subset \mathcal{D}, \alpha < 2^\omega$, such that for any $\alpha, \mathcal{D}_\alpha$ is a dense subset of X.*

PROOF. We need a simple modification of the well known construction. Let $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \ldots$ be an enumeration of open intervals with endpoints in $\mathcal{D}$. For any $n < \omega$ choose inductively distinct points $d_\xi \in \mathcal{D} \cap \mathcal{J}_n, \xi \in \{0,1\}^n$. Then put

$$\mathcal{D}_\alpha = \{d_\xi : \xi = \alpha|n,\, n < \omega\},\ \alpha < 2^\omega \qquad \square$$

Denote by Σ the family of sets $\{\mathcal{D}_\alpha : \alpha < 2^\omega\}$ from Lemma 1. Let us divide Σ into two disjoint sets Σ_1 and Σ_2 with $|\Sigma_1| = |\Sigma_2| = 2^\omega$. Let $F_i\,(i = 1,2)$ be bijections from $\overline{X}$ onto Σ_i such that $x \notin F_i(x)$ for any $x \in \overline{X}$. For any $x \in \overline{X} \setminus \{a,b\}$ we define the following families of sets:

$$\begin{aligned}
\mathcal{F}_{ir}(x) &= \{[x,y) \setminus F_i(x) : y > x;\ y \in X\} \quad \text{for } i = 1,2\\
\mathcal{F}_r(x) &= \mathcal{F}_{1r}(x) \cup \mathcal{F}_{2r}(x)\\
\mathcal{G}_{ir}(x) &= \{(x,y) \setminus F_i(x) : y > x;\ y \in X\} \quad \text{for } i = 1,2\\
\mathcal{G}_r(x) &= \mathcal{G}_{1r}(x) \cup \mathcal{G}_{2r}(x)\\
\mathcal{G}_{il}(x) &= \{(y,x) \setminus F_i(x) : y < x;\ y \in X\} \quad \text{for } i = 1,2\\
\mathcal{G}_l(x) &= \mathcal{G}_{1l}(x) \cup \mathcal{G}_{2l}(x)\\
\mathcal{K}_i(x) &= \{(y,t) \setminus F_i(x) : y < x < t;\ y,t \in X\} \quad \text{for } i = 1,2\\
\mathcal{K}(x) &= \mathcal{K}_1(x) \cup \mathcal{K}_2(x).
\end{aligned}$$

For $x = a$ we define only the families of sets with the index "r", and for $x = b$, only with the index "l".

Consider a family $\mathcal{F} \subset \mathcal{P}(X)$ with the following properties:

(1) $\mathcal{F}$ contains only sets of types $\mathcal{F}_r(x)$, $\mathcal{G}_r(x)$, $\mathcal{G}_l(x)$ and $\mathcal{K}(x)$ where $x \in \overline{X}$.

(2) For any $x \in \overline{X} \setminus \{a, b\}$ exactly one of the following possibilities is satisfied:

(a) $\mathcal{G}_l(x) \cup \mathcal{G}_r(x) \subset \mathcal{F}$, $(\mathcal{F}_r(x) \cup \mathcal{K}(x)) \cap \mathcal{F} = \emptyset$

(b) $\mathcal{G}_l(x) \cup \mathcal{F}_r(x) \subset \mathcal{F}$, $(\mathcal{G}_r(x) \cup \mathcal{K}(x)) \cap \mathcal{F} = \emptyset$

(c) $\mathcal{K}(x) \subset \mathcal{F}$, $(\mathcal{G}_l(x) \cup \mathcal{G}_r(x) \cup \mathcal{F}_r(x)) \cap \mathcal{F} = \emptyset$.

(3) For the point a we have

$$\mathcal{G}_r(a) \subset \mathcal{F} \text{ and } \mathcal{F}_r(a) \cap \mathcal{F} = \emptyset \text{ or}$$
$$\mathcal{F}_r(a) \subset \mathcal{F} \text{ and } \mathcal{G}_r(a) \cap \mathcal{F} = \emptyset,$$

and for the point b, always $\mathcal{G}_l(b) \subset \mathcal{F}$.

Lemma 2. *Assume that a family $\mathcal{F} \subset \mathcal{P}(X)$ has properties (1)–(3). Let $P, Q \in \mathcal{F}$ and $Q \subset P$. Then both the sets P and Q belong to the same of the classes $\mathcal{F}_{ir}(x)$, $\mathcal{G}_{ir}(x)$, $\mathcal{G}_{il}(x)$, $\mathcal{K}_i(x)$ with the same parameters i, x.*

Proof. This follows immediately from (1)–(3) and from the fact that the sets $F_i(x)$ are almost disjoint. □

We assume that a family $\mathcal{F}$ considered in the next lemmas satisfies conditions (1)–(3), and consequently, it has the property described in the assertion of Lemma 2.

Lemma 3. *Assume that for some $x \in \overline{X}$ we have $\mathcal{F}_r(x) \subset \mathcal{F}$. Let $x \in A \in S(\mathcal{F})$ for some set $A \in \mathcal{P}(X)$. Then there exists $y \in X$ such that $y > x$ and $[x, y) \subset A$.*

Proof. Consider $U_1 \in \mathcal{F}_{1r}(x)$. Since $A \in S(\mathcal{F})$, there is a $V_1 \in \mathcal{F}$ such that either $V_1 \subset A \cap U_1$ or $V_1 \subset U_1 \setminus A$. By Lemma 2 we have $V_1 \in \mathcal{F}_{1r}(x)$. Since $x \in A \cap V_1$, we have $V_1 \subset U_1 \cap A \subset A$. Let $V_1 = [x, y_1) \setminus F_1(x)$. Taking $U_2 \in \mathcal{F}_{2r}$, in the same way we obtain a set $V_2 = [x, y_2) \setminus F_2(x) \subset U_2 \cap A \subset A$. Let $y_0 = \min\{y_1, y_2\}$. Because $V_1 \cup V_2 \subset A$, we have $[x, y_0) \setminus (F_1(x) \cap F_2(x)) \subset A$. By the almost disjointedness of $F_1(x)$ and $F_2(x)$, there exists a $y \leq y_0$ for which $[x, y) \subset A$. □

Lemma 4. *Assume that $\mathcal{G}_r(x) \subset \mathcal{F}$ and $A \in S(\mathcal{F})$. Then there exists a $y \in X$ such that $y > x$ and either $(x, y) \subset A$ or $(x, y) \subset A^c$.*

Proof. Consider a $U_1 \in \mathcal{G}_{1r}(x)$. So, there exists $V_1 \in \mathcal{F}$ such that either $V_1 \subset A \cap U_1$ or $V_1 \subset U_1 \setminus A$. By Lemma 2 we have $V_1 \in \mathcal{G}_{1r}(x)$. Assume the first possibility, i.e. $V_1 \subset A \cap U_1 \subset A$. Put $V_1 = (x, y_1) \setminus F_1(x)$ and let $U_2 \in \mathcal{G}_{2r}(x)$. By the same considerations, there exists $V_2 \in \mathcal{G}_{2r}(x)$ such that either $V_2 \subset A \cap U_2$ or $V_2 \subset U_2 \setminus A$. By our assumption $V_1 \subset A \cap U_1$, only the first condition is possible (we have $V_1 \cap V_2 \neq \emptyset$). Hence there exists y_2 such that $(x, y_2) \setminus F_2(x) \subset U_2 \cap A \subset A$. As in the previous lemma we obtain that for some y we have $(x, y) \subset A$. By the same reasoning, if $V_1 \subset U_1 \setminus A$ we obtain $(x, y) \subset A^c$ for some $y \in X$. □

Lemma 5. *Assume that $\mathcal{G}_l(x) \subset \mathcal{F}$ and let $A \in S(\mathcal{F})$. Then there exists a $y \in X$ such that $y < x$ and either $(y, x) \subset A$ or $(y, x) \subset A^c$.*

Proof. Similar as for Lemma 4. □

Lemma 6. *Assume that $\mathcal{K}(x) \subset \mathcal{F}$ and let $A \in S(\mathcal{F})$. Then there exist $y, t \in X$ such that $y < x < t$ and either $(y, t) \subset A$ or $(y, t) \subset A^c$.*

Proof. Consider a $U_1 \in \mathcal{K}_1(x)$. Then by Lemma 2 there exists a $V_1 \in \mathcal{K}_1(x)$ such that either $V_1 \subset U_1 \cap A$ or $V_1 \subset U_1 \setminus A$. If the first possibility holds, then similarly as in the proof of the Lemma 4 we obtain, using $U_2 \in \mathcal{K}_2(x)$ and $V_2 \subset U_2 \cap A$, that for some y, t we have $(y, t) \subset A$. In the second case we can construct an interval $(y, t) \subset A^c$. □

Theorem 1. *Let X be a dense subset of $\overline{X} = [a, b] \subset \overline{R}$ and X_0 be an arbitrary fixed subset of X. Consider the following algebras $\mathcal{A}', \mathcal{A}" \subset \mathcal{P}(X)$:*

(i) $\mathcal{A}'$ generated by all intervals $[x, y)$ in X, with $x, y \in X_0$, and

(ii) $\mathcal{A}"$ generated by all intervals (x, y) in X, with $x, y \in X_0$.

Then $\mathcal{A}'$ and $\mathcal{A}"$ are MB-representable.

Proof. Let $\mathcal{F}'$ consist of the sets belonging to $\mathcal{G}_l(x) \cup \mathcal{F}_r(x)$ for $x \in X_0$, and of the sets belonging to $\mathcal{K}(x)$ for $x \in \overline{X} \setminus X_0$ (for $x = a$, we take $\mathcal{G}_r(a)$ if $a \notin X_0$ and $\mathcal{F}_r(a)$ if $a \in X_0$; for $x = b$ we take $\mathcal{G}_l(b)$).

We claim that $\mathcal{A}' = S(\mathcal{F}')$. Indeed, any interval $[x, y)$, for $x, y \in X_0$, belongs to $S(\mathcal{F}')$. By the component of a set $A \in \mathcal{P}(X)$ we understand any maximal interval (maybe degenerated to one point) contained in A. From Lemmas 3 and 6 it follows that each component of a set $A \in S(\mathcal{F}')$ is of the form $[x, y)$ or $[x, b]$ where $x \in \{a\} \cup X_0$, $y \in X_0$. So we only need to show that if $A \in S(\mathcal{F}')$, then A is a union of at most a finite family of disjoint intervals. Suppose that $\bigcup_{n=1}^{\infty} [x_n, y_n) \subset A$ where $[x_n, y_n)$ are pairwise disjoint. Let us consider a sequence $\{z_n\}$ such that $z_n \in [x_n, y_n)$. Pick a strictly monotonic subsequence $\{z_{n_k}\}$ of $\{z_n\}$. Let $z \in \overline{X}$ be the supremum (in the case of increasing subsequence) or the infimum (in the opposite case) of $\{z_{n_k}\}$. Using Lemma 3, Lemma 5 or Lemma 6 (the last one in the case $z \notin X_0$) we obtain a contradiction. So, the case (i) of our theorem has been proved. Let now

$\mathcal{F}$" consist of the sets belonging to $\mathcal{G}_l(x) \cup \mathcal{G}_r(x)$ for $x \in X_0$ and to $\mathcal{K}(x)$ for $x \in \overline{X} \setminus X_0$. For $x = a$ we take $\mathcal{G}_r(a)$, and for $x = b$ we take $\mathcal{G}_l(b)$. We claim that $\mathcal{A}" = S(\mathcal{F}")$. By similar reasoning as previously, we verify that $(x, y) \in S(\mathcal{F}")$ for each $x, y \in X_0$, $x < y$, and that each component of $A \in S(\mathcal{F}")$ is an interval (maybe degenerated) with endpoints in $\{a\} \cup \{b\} \cup X_0$. In a similar way we obtain that A is a union of at most a finite family of disjoint components. So we have (ii). □

Corollary 1. *The algebra generated by intervals* $[x, y)$ *and the algebra generated by intervals* (x, y) *with rational endpoints, considered in the space of the real numbers or of the rational numbers in some interval* $[a, b]$*, are MB-representable.*

Proof. In our Theorem we put $X_0 = Q \cap [a, b]$, $X = [a, b]$ or $X_0 = X = Q \cap [a, b]$, respectively. □

Remark 1. New ideas that appeared in our paper, in comparison with [3, Theorem 2.1], are the following:

(1) consideration of the family Σ of almost disjoint dense sets contained in some dense countable subset of X, instead of the family of disjoint dense subsets of $[0, 1)$;

(2) division of Σ into two disjoint subfamilies Σ_1, Σ_2.

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