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RECURSIVE SET RELATIONS

Abstract

Let $A(x)$, $B(x)$, $C(x)$ be characteristic functions of three measurable sets of real numbers. We determine necessary and sufficient conditions for which $A(x+a_n)+B(x+b_n)+C(x+c_n)=A(x)+B(x)+C(x)$ almost everywhere, where $\{a_n\},\{b_n\},\{c_n\}$ are sequences of nonzero shifts approaching zero.

1 Introduction

Linear recursive relations on the integers (Fibonacci relations) have been extensively studied and are well-understood. Recursion on the real number line is more complicated and in this article we will deal only with some very basic cases. We begin with an exercise in Walter Rudin's book ([1], pg 156), which calls for a proof of the following theorem. A short proof outline follows.

Theorem 1. *Let $A\subset\mathbb{R}$ be a measurable set satisfying the relation*

$$A(x+a_n)=A(x)\ a.e.$$

for some sequence $\{a_n\}$ of non-zero numbers approaching 0. Then A is either a null set or a set of full measure.

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Proof. If $A(x+a_n)=A(x)$ *a.e.* then $F_n(x)=\int_x^{x+a_n} A(t)\,dt$ has derivative 0 everywhere and is therefore constant. Thus A has the same density at almost every point so A is null or full measure. □

This result says that except in trivial cases (i.e., A is null or full) a measurable set cannot be almost periodic with arbitrarily small periods. The converse is also easy. If A is either null or full, then for any given value of a_n, $A(x+a_n)=A(x)$ *a.e.* A natural question then is whether two or more sets can satisfy the corresponding system of equations in some non-obvious way.

Question 1. For which measurable sets $A_1,\dots,A_K$ do there exist sequences $\{a_{k,n}\}_{n=1}^{\infty}$, $k=1,\dots,K$, of non-zero numbers converging to 0 such that

$$A_1(x+a_{1,n})+\cdots+A_K(x+a_{K,n})=A_1(x)+\cdots+A_K(x) \text{ a.e.?} \tag{1}$$

We will refer to such a system of sets, shifts, and equation as a *recursive system of degree K*.

Theorem 1 answers our question for systems of degree one. In this article we will answer it for degrees two and three. The solution for $K=2$ is relatively short but for $K=3$ is lengthened by the need to check several cases. We did not see an easy way to extend our results to $K>3$ without an explosion in the number of cases to consider.

Let us begin with some obvious examples. One type of easy solution occurs when the sum $\sum_{k=1}^{K} A_k(x)$ is constant for almost all x. In the case $K=1$, this means that A_1 is either null or full. When $K=2$, this will occur if both of the sets are null or full, or if they are complements almost everywhere. Similar, but more complicated relationships hold for larger values of K. For example, if almost every x is in exactly two of the sets A_1, A_2, A_3, then $A_1(x)+A_2(x)+A_3(x)=2$ *a.e.* We will call solutions of this type *elementary.* Theorem 1, paraphrased, says that the only recursive systems of degree one are the elementary ones. Elementary recursive systems will trivially satisfy (1) whenever the shifts are equal, that is, whenever $a_{1,n}=a_{2,n}=\cdots=a_{K,n}$. However, as we shall see next, this *equality of shifts* is not a necessary condition.

Another way to form a recursive system is to add together two systems of smaller degree. For example, if $A(x+a_n)+B(x+b_n)+C(x+c_n)=A(x)+B(x)+C(x)$ *a.e.* and $D(x+d_n)+E(x+e_n)=D(x)+E(x)$ *a.e.* then certainly $A(x+a_n)+B(x+b_n)+C(x+c_n)+D(x+d_n)+E(x+e_n)=A(x)+B(x)+C(x)+D(x)+E(x)$ *a.e.* We will call this new system *reducible.*

More precisely, the recursive system (1) will be called *reducible* if under some reordering of the sets, we have

$$\begin{aligned} A_1(x+a_{1,n})+\cdots+A_m(x+a_{m,n}) &= A_1(x)+\cdots+A_m(x) \text{ a.e.} \\ A_{m+1}(x+a_{m+1,n})+\cdots+A_K(x+a_{K,n}) &= A_{m+1}(x)+\cdots+A_K(x) \text{ a.e.,} \end{aligned}$$

for some $1 \le m < K$ and for all sufficiently large n. Otherwise, we will call the system *irreducible*. Note that adding together elementary systems gives another elementary system, but equality of shifts is not necessarily preserved.

These easy examples together with Theorem 1 motivate the following conjectures, which together say that the only recursive systems are the obvious ones.

Conjecture 1. Every recursive system is elementary.

Conjecture 2. If a recursive system is irreducible, the shifts must be equal for all large enough n.

By induction, using Theorem 1 as the base case, the first conjecture need only be proved for irreducible systems. Our main result is that these two conjectures hold for recursive systems of degree two or three (see Theorems 5 and 14 below). This answers a question raised at the Fifteenth Annual Auburn Miniconference in Real Analysis in 2002.

Although we did not see an easy way to generalize to systems of higher degree, there is an important lemma that does generalize and at least gives a start. This lemma (see Theorem 3 below) says that in any recursive system there exist constants α_k and δ, not all zero, such that $\sum_{k=1}^{K} \alpha_k A_k(x) = \delta$ *a.e.* Although this functional identity gives Theorem 1 as a corollary, it does not seem to be enough to settle either of the above conjectures in general.

We are primarily concerned with recursive systems on the whole real line, but they can also be naturally restricted to open intervals. Thus, given a non-empty open interval I, a *recursive system on I* is a collection of measurable sets and nonzero sequences approaching zero, such that (1) holds whenever $x, x+a_{1,n}, \dots, x+a_{K,n}$ are all in I. The previous conjectures then have natural generalizations replacing "recursive system" with "recursive system on I". At first glance, one might think that such a generalization is a mere triviality. However, we were not able to prove the following.

Conjecture 3. Any recursive system on a non-empty open interval can be extended to a recursive system on the entire real line.

If this is true, it would make relativized versions of the first two conjectures unnecessary. We believe it to be true mainly because it follows from the relativized version of Conjecture 1. Since we have no proof, we will present our results on the first two conjectures for the seemingly stronger relativized case. To start with, notice that the proof of Theorem 1 goes through unscathed when we restrict our attention to an open interval.

2 Notation and Conventions

Throughout this paper we will be dealing with a recursive system on a fixed non-empty open interval I and will consider only Lebesgue measurable subsets of I. Thus we will say that a set A is *null* if $A \cap I$ has measure zero. We will say that A is *full* if $I \setminus A$ is null. If A is neither null nor full then we will call it an *intermediate* set. We will freely write operations and relations involving the sets of the system with the understanding that these are relative to I ignoring sets of measure zero. For example, if A, B are two sets of the recursive system, then A' will denote the complement of A relative to I and $A = B$ will mean that $A = B$ almost everywhere on I. Likewise, $A(x) = B(x)$ and similar statements involving the characteristic functions will mean that the statement holds for almost all $x \in I$.

We will also be dealing with sequences $\{a_{k,n}\}_{n=1}^{\infty}$, $k = 1, \ldots, K$, of non-zero numbers converging to 0, which will be referred to as *shifts*.

Remark 1. We will often assume some property of these shifts and will justify the assumption by passing to a subsequence, if necessary. These assumptions will be automatically justified in a proof of Conjecture 1, since it does not say anything about the shifts themselves.

For example, using Remark 1 we will assume that each sequence is either always positive or always negative. Letting $\delta_n = \max\{|a_{1,n}|, |a_{2,n}|, \ldots, |a_{K,n}|\}$ we will further assume that each of the limits,

$$\alpha_k = \lim_{n\to\infty} a_{k,n}/\delta_n$$

exist with one of them being ± 1. These constants play an important role and will be referred to frequently throughout this article.

Remark 2. Conjecture 2 says something very specific about the eventual behavior of the shifts. Therefore, whenever we prove a case of Conjecture 2, we will be sure to return to the original sequences with only finitely many elements left out.

Finally, when we consider a recursive system of degree two or three, we will want to drop some of the subscripts. Thus, A_1, A_2, A_3 will be replaced with A, B, C, $a_{1,n}, a_{2,n} a_{3,n}$ will be replaced with a_n, b_n, c_n, and $\alpha_1, \alpha_2, \alpha_3$ will be replaced with α, β, γ.

3 The Basic Functional Identity

The following lemma will give as a corollary the functional identity mentioned in the introduction. The more general form presented here will be convenient in subsequent proofs.

Lemma 2. *Let I be an open interval and let $\{F_k\colon k = 1, \ldots, K\}$ be a collection of Lebesgue integrable functions on I. Let $\{a_{k,n}\}_{n=1}^{\infty}$, $k = 1, \ldots, K$, be sequences of non-zero numbers converging to 0. Suppose that there is a sequence of continuous functions $\{f_n(x)\}_{n=1}^{\infty}$ such that for all n and for almost every $x \in I$,*

$$\sum_{k=1}^{K} [F_k(x + a_{k,n}) - F_k(x)] = f_n(x) \text{ whenever } x + a_{k,n} \in I.$$

Then $\Psi_n(x) = \sum_{k=1}^{K} \int_{x}^{x+a_{k,n}} F_k(t)\,dt$ is differentiable everywhere on I and $\Psi_n'(x) = f_n(x)$.

PROOF. For real numbers x and h we can write

$$\begin{aligned} \int_{x+h}^{x+a_{k,n}+h} F_k(t)\,dt - \int_{x}^{x+a_{k,n}} F_k(t)\,dt &= \int_{x+a_{k,n}}^{x+a_{k,n}+h} F_k(t)\,dt - \int_{x}^{x+h} F_k(t)\,dt \\ &= \int_{x}^{x+h} F_k(t + a_{k,n})\,dt - \int_{x}^{x+h} F_k(t)\,dt. \end{aligned}$$

Using this to compute the difference quotient, we get

$$\frac{\Psi_n(x+h) - \Psi_n(x)}{h} = \frac{\int_{x}^{x+h} \sum_{k=1}^{K} [F_k(t + a_{k,n}) - F_k(t)]\;dt}{h} = \frac{1}{h} \int_{x}^{x+h} f_n(t)\,dt.$$

Letting $h \to 0$ we obtain $\Psi_n'(x) = f_n(x)$. □

Theorem 3. *For any recursive system on I with sets $A_1, \ldots, A_K$, there is a constant δ such that,*

$$\sum_{k=1}^{K} \alpha_k A_k(x) = \delta. \tag{2}$$

PROOF. The functions $F_k(x) = A_k(x)$ and $f_n(x) = 0$ satisfy the conditions of Lemma 2, and we get that $\Psi_n(x) = \Psi_n$ is constant for each n. By the Lebesgue Density Theorem, for almost every x the limit as $n \to \infty$ of the left side of the identity

$$\sum_{k=1}^{K} \frac{a_{k,n}}{\delta_n} \cdot \frac{1}{a_{k,n}} \int_{x}^{x+a_{k,n}} A_k(t)\, dt = \frac{\Psi_n}{\delta_n}$$

exists and equals $\sum_{k=1}^{K} \alpha_k A_k(x)$. Thus the limit of the right side also exists and (2) is satisfied by setting $\delta = \lim_{n\to\infty} \dfrac{\Psi_n}{\delta_n}$. □

Remark 3. Since $\alpha_k = \pm 1$ for some k, the identity (2) is non-trivial. Also, if $K = 1$ then this implies that A is either null or full, so Theorem 3 generalizes Theorem 1.

4 Recursive Systems of Degree Two

This section will establish Conjectures 1 and 2 for systems of degree two.

Lemma 4. *Let A and B be two intermediate sets and suppose there exist constants δ, α, β, not all zero, such that $\delta = \alpha A(x) + \beta B(x)$. Then $A = B$ or $A = B'$*

PROOF. We have that

$$\delta = \alpha A(x) + \beta B(x) = \begin{cases} 0 & \text{if } x \in A' \cap B' \\ \alpha & \text{if } x \in A \cap B' \\ \beta & \text{if } x \in A' \cap B \\ \alpha + \beta & \text{if } x \in A \cap B \end{cases}$$

is a constant. If $A \neq B$ then one of the sets $A \cap B'$, $A' \cap B$ is not null. Therefore, δ must be either α or β. If $A \neq B'$ then the one of the sets $A' \cap B'$, $A \cap B$ is not null. Therefore, δ must be either 0 or $\alpha + \beta$. This leaves four possibilities, $\alpha = 0$, $\beta = 0$, $\alpha = \alpha + \beta$, or $\beta = \alpha + \beta$. In all four cases, one of α, β is zero, and by assumption, one of them is not zero. Then one of $A(x)$ or $B(x)$ is constant contradicting that they are both intermediate sets. □

Theorem 5. *Suppose we have an irreducible recursive system on I of degree two with sets A, B. Then $A = B'$ and for all sufficiently large n, $a_n = b_n$.*

Proof. Since the system is irreducible, we can assume that both A and B are intermediate sets. By Theorem 3, there are constants δ, α, β, not all zero, such that $\delta = \alpha A(x) + \beta B(x)$. Hence $A = B$ or $A = B'$. Consider the original (see Remark 2) relations

$$A(x + a_n) + B(x + b_n) = A(x) + B(x). \tag{3}$$

If $A = B$ this becomes $A(x+a_n)+A(x+b_n) = 2A(x)$. Since $A(x)$ can only take on values 0 or 1, this requires that $A(x+a_n) = A(x)$, and the system reduces. On the other hand, if $A = B'$ then (3) becomes $A(x + a_n) - A(x + b_n) = 0$. Replacing x with $x - a_n$, we obtain $A(x + b_n - a_n) = A(x)$. If $a_n \neq b_n$ for infinitely many n, then by Theorem 1, A is null or full, a contradiction. □

5 Recursive Systems of Degree Three

For the remainder of this article we will be dealing with a recursive system of degree three on a non-empty open interval I with sets A, B, C. Our first task will be to narrow down the functional identity to just a few possibilities. To start, we are confronted with eight possible combinations of positive or negative values for the shift sequences $\{a_n\}, \{b_n\}, \{c_n\}$. The values of α, β, and γ will either be zero or of the same sign as the corresponding sequence. In order to combine some of the arguments we define $\widetilde{A} = \begin{cases} A & \text{if } a_n > 0 \\ A' & \text{if } a_n < 0 \end{cases}$ and similarly for $\widetilde{B}$ and $\widetilde{C}$. With this notation Equation (2) can be written as

$$|\alpha|\, \widetilde{A}(x) + |\beta|\, \widetilde{B}(x) + |\gamma| \widetilde{C}(x) = \delta \tag{4}$$

for some, perhaps new, positive constant δ. For example, if $\alpha < 0$ while $\beta \geq 0$ and $\gamma \geq 0$, we can rewrite Equation 2 as $\alpha(1 - A'(x)) + \beta B(x) + \gamma C(x) = \delta$ which becomes $|\alpha|\, A'(x) + \beta B(x) + \gamma C(x) = \delta - \alpha$. This version has nonnegative coefficients on the left and $\delta - \alpha$ can be replaced with a new constant δ. We may also assume temporarily (only for the purposes of the next lemma), that the sets are reordered so that $0 \leq |\alpha| \leq |\beta| \leq |\gamma| = 1$.

Lemma 6. *Let A, B and C be any three intermediate subsets of I. Let α, β and γ be three constants, with $0 \leq \alpha \leq \beta \leq \gamma = 1$, such that $\alpha A(x) + \beta B(x) + \gamma C(x) = \delta$ is a constant. Then $A(x)+B(x)+C(x) \equiv 1$ or $A(x)+B(x)+C(x) \equiv 2$ or $B(x) + C(x) \equiv 1$.*

Proof. Since C is not null, we must have $\gamma \leq \delta$. Since C is not full, we must have $\delta \leq \alpha + \beta$. We distinguish three cases.

Case 1: $\gamma = \delta = \alpha + \beta$. It follows from the ordering on α, β, γ that $\beta > 0$. Then $C \cap B$ must be null and $C' \cap B'$ must be null. Therefore, $C = B'$ and hence $B(x) + C(x) = 1$.

Case 2: $\gamma = \delta < \alpha + \beta$. Then at most one of the sets can be occupied at a time. But at least one must be occupied all the time since $\delta = \gamma = 1$. So $A(x) + B(x) + C(x) = 1$.

Case 3: $\gamma < \delta \leq \alpha + \beta$. Then δ must take on the value $\alpha + \beta$. We must always have at least two of the sets occupied, but we can never have all three occupied. Therefore, $A(x) + B(x) + C(x) = 2$. $\square$

When combined with (4) this lemma immediately gives the following.

Corollary 7. *In an irreducible recursive system of degree three, either two of the sets are equal, two are complements, the sum of two of the characteristic functions differs from the third by a constant, or all three characteristic functions add to a constant.*

The proof of the two main conjectures will be broken into steps.

Step 1: In Section 5.2 we show that if two of the sets are either equal or complements then the system reduces.

Step 2: In Section 5.3 we show that if two of the characteristic functions differ from the third by a constant, then the system reduces.

Step 3: In Section 5.4 we show that $A(x) + B(x) + C(x)$ constant implies that the shifts are eventually equal.

Note that together with Corollary 7, these three steps settle Conjectures 1 and 2 are for systems of degree three.

5.1 Preliminaries

Before we dive into the heart of the proof, we will define some auxiliary functions and develop some of their properties. Consider the following functions:

$$F(x) = \int_0^x A(t)\,dt, \quad G(x) = \int_0^x B(t)\,dt, \quad H(x) = \int_0^x C(t)\,dt\ .$$

where 0 represents some distinguished element of I. Several important properties of these three functions are established in Lemma 8. First notice that, by Lemma 2, for each n the sum

$$\int_x^{x+a_n} A(t)\,dt + \int_x^{x+b_n} B(t)\,dt + \int_x^{x+c_n} C(t)\,dt$$

has derivative zero everywhere on I and is thus constant, say d_n. This is equivalent to the following. For every $x \in I$ the functions F, G, and H satisfy the recurrence relation,

$$F(x+a_n) - F(x) + G(x+b_n) - G(x) + H(x+c_n) - H(x) = d_n\ .$$

Applying the same lemma to this new system, we get another sequence of constants $\{e_n\}$ such that

$$\Psi_n(x) = \int_x^{x+a_n} F(t)\,dt + \int_x^{x+b_n} G(t)\,dt + \int_x^{x+c_n} H(t)\,dt = d_n x + e_n \tag{5}$$

for all $x \in I$. Equation (5) is crucial to the proofs in the next section.

Lemma 8. *Let A be a measurable set and $F(x) = \int_0^x A(t)\,dt$. Then F possesses the following properties:*

1. F is increasing, continuous and $|F(x) - F(y)| \le |x - y|$ for all $x, y \in I$.

2. F is differentiable almost everywhere and $F' = A$ a.e.

3. If $\{J_n\}$ is a sequence of intervals converging to x, (meaning that both endpoints converge to x) then

$$\lim_{J_n \to x} \frac{1}{|J_n|} \int_{J_n} F(t)\,dt = F(x).$$

4. If u, $u+h$, $u+z$, $u+h+z$, v, $v+h$, $v+z$, $v+h+z$, are all in I then

$$|hz| \geq \int\limits_{u}^{u+h} [F(t+z) - F(t)]\, dt - \int\limits_{v}^{v+h} [F(t+z) - F(t)]\, dt\,.$$

5. If u is a density point of $A \cap I$ and v is a density point of $A' \cap I$, then

$$\lim_{h,z\to 0} \frac{1}{hz} \left(\int\limits_{u}^{u+h} [F(t+z) - F(t)]\, dt - \int\limits_{v}^{v+h} [F(t+z) - F(t)]\, dt \right) = 1\,.$$

Proof. A proof of the first property is straightforward. Property 2 is just the Lebesgue Density Theorem. Property 3 follows from the fact that F is continuous. To prove Property 4 rewrite the right side of the inequality as

$$\int\limits_{0}^{h} [F(t+u+z) - F(t+u)] - [F(t+v+z) - F(t+v)]\, dt\,.$$

By Property 1, each of the terms in brackets is between 0 and z, so their difference is between z and $-z$.

To prove Property 5, first notice that

$$\int\limits_{u}^{u+h} [F(t+z) - F(t)]\, dt = \int\limits_{u}^{u+z} [F(t+h) - F(t)]\, dt\,.$$

Because of this symmetry in h and z, we can assume that $|h| \leq |z|$. Let t be between u and $u+h$. Since $|h| \leq |z|$, u is between $t-z$ and $t+z$ so the interval between t and $t+z$ is at least half of an interval containing u. Since u is a density point of A, the ratio of $[F(t+z) - F(t)]$ and z becomes arbitrarily close to 1 as $z \to 0$. Thus

$$\lim_{z\to 0} \frac{1}{hz} \int_{u}^{u+h} [F(t+z) - F(t)]\, dt = 1\,. \tag{6}$$

Similarly, using that v is a density point of A', we get that

$$\lim_{z\to 0} \frac{1}{hz} \int_{v}^{v+h} [F(t+z) - F(t)]\, dt = 0\,. \tag{7}$$

Subtracting (7) from (6) gives Property 5. □

5.2 The Case Where Two Sets Are Equal Or Complementary

In this section we will carry out Step 1 in our proof. We have an irreducible recursive system of degree three where two of the sets are equal or complementary. We may assume without loss of generality that $B = C$ or $B = C'$. We will first show that either of these implies that $A = C$ or $A = C'$. In other words, all three sets are equal or two of them are equal to the complement of the third. By re-labelling the sets, if necessary, we can assume that either $A = B = C$ or $A' = B = C$. We will finish by showing that both of these possibilities lead to a reduction in the system.

Lemma 9. *In an irreducible recursive system on a non-empty open interval I, if $B = C$ then $A = C$ or $A = C'$.*

Proof. In this case, Equation (5) becomes

$$\begin{aligned} d_n x + e_n &= \int_x^{x+a_n} F(t)\,dt + \int_x^{x+b_n} H(t)\,dt + \int_x^{x+c_n} H(t)\,dt \\ &= \int_x^{x+a_n} F(t)\,dt + \int_x^{x+c_n+b_n} H(t)\,dt + \int_{x+b_n}^{x} \left[H(t+c_n) - H(t)\right]\,dt\,. \end{aligned}$$

Evaluating this at two points u and v and subtracting, we get

$$d_n(u-v) = K_n + \int_u^{u+a_n} F(t)\,dt - \int_v^{v+a_n} F(t)\,dt + \int_u^{u+b_n+c_n} H(t)\,dt - \int_v^{v+b_n+c_n} H(t)\,dt\,,$$

where K_n equals

$$\int_{u+b_n}^{u} \left[H(t+c_n) - H(t)\right]dt - \int_{v+b_n}^{v} \left[H(t+c_n) - H(t)\right]dt\,.$$

Note that by Property 4 of Lemma 8, $|K_n| \le |b_n c_n|$. Let

$$\gamma_n = \max\{|d_n|, |a_n|, |b_n + c_n|, |b_n c_n|\}\,.$$

We may assume (see Remark 1) that $\{b_n + c_n\}$ is always zero, or else never zero, that each of the limits, $L_0 = \lim \dfrac{d_n}{\gamma_n}$, $L_1 = \lim \dfrac{a_n}{\gamma_n}$, $L_2 = \lim \dfrac{b_n + c_n}{\gamma_n}$,

$L_3 = \lim \dfrac{b_n c_n}{\gamma_n}$ exist, and that one of these limits is ± 1. Assume first that $b_n + c_n \neq 0$. Then we may write

$$\begin{aligned}
\frac{d_n}{\gamma_n}(u-v) \quad = \quad & \frac{K_n}{\gamma_n} + \frac{a_n}{\gamma_n}\frac{1}{a_n}\int_u^{u+a_n} F(t)\,dt - \frac{a_n}{\gamma_n}\frac{1}{a_n}\int_v^{v+a_n} F(t)\,dt \\
& + \frac{b_n + c_n}{\gamma_n}\frac{1}{b_n + c_n}\int_u^{u+b_n+c_n} H(t)\,dt \qquad (8) \\
& - \frac{b_n + c_n}{\gamma_n}\frac{1}{b_n + c_n}\int_v^{v+b_n+c_n} H(t)\,dt\,. \qquad (9)
\end{aligned}$$

Solving for $\dfrac{K_n}{\gamma_n}$ and taking limits, using Property 3 of Lemma 8, we get

$$\lim_{n\to\infty} \frac{K_n}{\gamma_n} = L_0(u-v) - L_1(F(u) - F(v)) - L_2(H(u) - H(v))\,. \qquad (10)$$

If $b_n + c_n = 0$ then (10) is still valid, since in that case the terms (8) and (9) vanish and the limit L_2 is zero. By Property 4 of Lemma 8 we know that $|L_3| \geq \lim_{n\to\infty} \frac{|K_n|}{\gamma_n}$ whereas Property 5 gives us equality in the case that u is a density point of C, and v is a density point of C'. We can assume A and C are intermediate sets, otherwise the system is reducible. Hence there will be such density points arbitrarily close to each other. Combining with (10) and using the continuity of F and H, we get that $L_3 = 0$. Thus, for all u, v in I we have that

$$L_0(u-v) = L_1(F(u) - F(v)) + L_2(H(u) - H(v))\,.$$

Dividing by $u - v$ and taking the limit as $v \to u$ we get from Property 2 of Lemma 8 that

$$L_0 = L_1 A(u) + L_2 C(u)\,.$$

Hence by Lemma 4, $A = C$ or $A = C'$. $\square$

Lemma 10. *In a recursive system on I, if $B = C'$ then $A = C$ or $A = C'$.*

PROOF. Since $B(x)+C(x)=1$, $G(t)=t-H(t)$ and Equation (5) becomes

$$d_n x + e_n = \int_x^{x+a_n} F(t)\,dt + \int_{x+b_n}^{x+c_n} H(t)\,dt + \int_x^{x+b_n} t\,dt\,,$$

where the last integral is $b_n x + b_n^2/2$. Evaluating at u, v and subtracting,

$$(d_n-b_n)(u-v) = \int_u^{u+a_n} F(t)\,dt - \int_v^{v+a_n} F(t)\,dt + \int_{u+b_n}^{u+c_n} H(t)\,dt - \int_{v+b_n}^{v+c_n} H(t)\,dt\,.$$

Let

$$\gamma_n = \max\{|d_n-b_n|, |a_n|, |c_n-b_n|\}\,,$$

and assume that the limits $L_0=\lim \dfrac{d_n-b_n}{\gamma_n}$, $L_1=\lim\dfrac{a_n}{\gamma_n}$, $L_2=\lim\dfrac{c_n-b_n}{\gamma_n}$ all exist, one of them being ± 1 (see Remark 1). We may also assume that c_n-b_n is either always zero or never zero. Suppose $c_n-b_n\neq 0$. Then

$$\begin{aligned}\frac{d_n-b_n}{\gamma_n}(u-v) =& \frac{a_n}{\gamma_n}\frac{1}{a_n}\left(\int_u^{u+a_n} F(t)\,dt - \int_v^{v+a_n} F(t)\,dt\right)\\ &+ \frac{c_n-b_n}{\gamma_n}\frac{1}{c_n-b_n}\left(\int_{u+b_n}^{u+c_n} H(t)\,dt - \int_{v+b_n}^{v+c_n} H(t)\,dt\right)\,.\end{aligned}$$

Taking limits, using Property 3 of Lemma 8, we get

$$L_0(u-v) = L_1(F(u)-F(v)) + L_2(H(u)-H(v))\,.$$

This is still valid if $c_n-b_n=0$; the integrals over H disappear and $L_2=0$. Dividing by $u-v$ and taking the limit as $v\to u$ using Property 2 of Lemma 8,

$$L_0 = L_1 A(u) + L_2 C(u)\,.$$

If either A or C is null or full, the system reduces. Otherwise, we get from Lemma 4 that either $A=C$ or $A=C'$. $\square$

Lemma 11. *In a recursive system of degree three, if $A=B=C$ or $A'=B=C$ the system reduces.*

PROOF. If $A = B = C$ then the relations become $A(x+a_n) + A(x+b_n) + A(x+c_n) = 3A(x)$. Since $A(x)$ only takes on the values 0 and 1, this implies that $A(x+a_n) = A(x)$. By Theorem 1, A is null or full so the system reduces.

If $A' = B = C$, then $F(t) = t - H(t)$ and Equation (5) becomes

$$\begin{aligned}
d_n x + e_n = & \int_x^{x+a_n} (t - H(t))\,dt + \int_x^{x+b_n} H(t)\,dt + \int_x^{x+c_n} H(t)\,dt \\
= & a_n x + \frac{a_n^2}{2} + \int_{x+a_n}^{x+b_n+c_n} H(t)\,dt + \int_{x+b_n+c_n}^{x+c_n} H(t)\,dt + \int_x^{x+b_n} H(t)\,dt \\
= & a_n x + \frac{a_n^2}{2} + \int_{x+a_n}^{x+b_n+c_n} H(t)\,dt + \int_{x+b_n}^{x} [H(t+c_n) - H(t)]\,dt.
\end{aligned}$$

Evaluating this at two points u and v and subtracting, we get

$$(d_n - a_n)(u - v) = \int_{u+a_n}^{u+b_n+c_n} H(t)\,dt - \int_{v+a_n}^{v+b_n+c_n} H(t)\,dt + K_n$$

where, as before, K_n is an abbreviation for

$$\int_{u+b_n}^{u} [H(t+c_n) - H(t)]\,dt - \int_{v+b_n}^{v} [H(t+c_n) - H(t)]\,dt\,.$$

This time, let $\gamma_n = \max\{|d_n - a_n|, |b_n + c_n - a_n|, |b_n c_n|\}$. We may assume (see Remark 1) that $\{b_n + c_n - a_n\}$ is always zero, or else never zero, that each of the limits, $L_0 = \lim \dfrac{d_n - a_n}{\gamma_n}$, $L_1 = \lim \dfrac{b_n + c_n - a_n}{\gamma_n}$, $L_2 = \lim \dfrac{b_n c_n}{\gamma_n}$ exist, and that one of these limits is ± 1. Assume first that $b_n + c_n - a_n \neq 0$. Then

$$\begin{aligned}
\frac{K_n}{\gamma_n} = & \frac{d_n - a_n}{\gamma_n}(u - v) - \frac{b_n + c_n - a_n}{\gamma_n} \frac{1}{b_n + c_n - a_n} \int_{u+a_n}^{u+b_n+c_n} H(t)\,dt \\
& + \frac{b_n + c_n - a_n}{\gamma_n} \frac{1}{b_n + c_n - a_n} \int_{v+a_n}^{v+b_n+c_n} H(t)\,dt\,.
\end{aligned}$$

Taking limits using Property 3 of Lemma 8, we get

$$\lim_{n\to\infty} \frac{K_n}{\gamma_n} = L_0(u-v) - L_1(H(u) - H(v))\,. \tag{11}$$

If $b_n + c_n - a_n = 0$ then (11) is still valid; the two integrals over H vanish and the limit L_1 is zero. By Property 4 of Lemma 8, $|L_2| \geq \lim_{n\to\infty} \frac{|K_n|}{\gamma_n}$, whereas Property 5 gives us equality in the case that u is a density point of C and v is a density point of C'. Since C is an intermediate set, there will be such density points arbitrarily close to each other, and using the continuity of H, we get from (11) that $L_2 = 0$. Thus, for all u, v in I we have that $L_0(u-v) = L_1(H(u)-H(v))$. Dividing by $u-v$ and taking the limit as $v \to u$ we get from Property 2 of Lemma 8 that

$$L_0 = L_1 C(u)\,.$$

Since one of L_0, L_1 is ± 1, C is null or full so the system reduces. □

5.3 The Case $A'(x) + B(x) + C(x)$ Is Constant

In this section we will carry out step 2 of our proof. We have a recursive system of degree three where two of the characteristic functions differ from the third by a constant. Assume without loss of generality that $A'(x)+B(x)+C(x) = k$. We will need the following generalization of Theorem 1.

Theorem 12. *Suppose there exist arbitrarily small $h \neq 0$ such that for almost every $x \in I$, $A(x+h) - A(x)$ is non-negative (or non-positive). Then I can be split into two subintervals I_0 and I_1 such that A is null on I_0 and full on I_1.*

PROOF. Assume without loss of generality that $A(x+h) \geq A(x)$ for almost all $x \in I$ and for arbitrarily small $h > 0$. For each such h, the continuous function $F_h(x) = \dfrac{1}{h}\int\limits_x^{x+h} A(t)\,dt$ has a non-negative lower derivate everywhere on I, and so is non-decreasing. As h becomes small, $F_h(x)$ becomes arbitrarily close to 1 if x is a density point of A and arbitrarily close to 0 if x is a density point of A'. Hence there are no density points of A below any density point of A'. The theorem is now satisfied by splitting the interval I at the infimum of the density points of A. □

Lemma 13. *In a recursive system of degree three, if $A'(x)+B(x)+C(x) = k$ is constant then the system is reducible.*

Proof. The set relations simplify to

$$A(x+a_n)+B(x+b_n)+C(x+c_n)=2A(x)+k-1. \tag{12}$$

If $k<1$ then A is full and if $k>2$ then A is null. So $k=1$ or 2. If $k=1$ then $A(x+a_n)\le A(x)$. If $k=2$, then $A(x)\le A(x+a_n)$. In either case, by Theorem 12, I can be split into two subintervals I_0, I_1, such that A is null in I_0 and full in I_1. If x, $x+a_n$ are both in I_0, Equation (12) implies

$$B(x+b_n)+C(x+c_n)=k-1=B(x)+C(x).$$

If x and $x+a_n$ are both in I_1 we get

$$B(x+b_n)+C(x+c_n)=k=B(x)+C(x).$$

This gives us a degree two recursive system on both subintervals. Suppose first that both systems are reducible, so that both B and C are either null or full on both pieces. On I_0, $B(x)+C(x)=k-1$ while on I_1, $B(x)+C(x)=k$. Thus, if $k=1$ one of the two sets is null on all of I and if $k=2$ then one of them is full. In ether case, we are done. On the other hand, if the system is irreducible on at least one of the subintervals, then by Theorem 5, $b_n=c_n$ for sufficiently large n. Using the relationship $B(x)+C(x)=A(x)+k-1$ Equation (12) then implies

$$A(x+a_n)+A(x+b_n)=2A(x).$$

This describes a recursive system of degree two. So either it is reducible, in which case A is null or full, or $A=A'$, a contradiction. □

5.4 The Case $A(x)+B(x)+C(x)$ Constant

In this section we will complete step 3 of the proof. The next theorem summarizes our main result.

Theorem 14. *In an irreducible recursive system on I, $A(x)+B(x)+C(x)$ is constant and is equal to 1 or 2, and $a_n=b_n=c_n$ for sufficiently large n.*

Proof. We have already established that $A(x)+B(x)+C(x)$ is constant. We need only show that the shifts are eventually equal. We may assume that each of the sets is intermediate. Replace x with $x-a_n$ in each of the original set relations (see Remark 2) to obtain

$$A(x)+B(x+b_n-a_n)+C(x+c_n-a_n)=A(x-a_n)+B(x-a_n)+C(x-a_n).$$

Since $A(x) + B(x) + C(x)$ is constant, this simplifies to

$$B(x + b_n - a_n) + C(x + c_n - a_n) = B(x) + C(x). \tag{13}$$

Assume towards a contradiction that $b_n - a_n$ or $c_n - a_n$ is not eventually zero. Then (13) describes a recursive system of degree one or two. If it is degree one then we are done by Theorem 1. If it is degree two then since B and C are intermediate sets, we have from Theorem 5 that $B(x) + C(x)$ is constant. But then $A(x)$ is also constant so the system reduces. $\square$

References

[1] W. Rudin, *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York, 1986.