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F. Bayart, UFR de Mathématiques, UMR AGAT CNRS, Université des Sciences et Technologies de Lille, F-59665, Villeneuve d'Ascq cedex, France. email: `bayart@agat.univ-lille1.fr`

S. V. Konyagin, Department of Mechanics and Mathematics, Moscow State University, Moscow 119992, Russia. email: `konyagin@ok.ru`

H. Queffélec, UFR de Mathématiques, UMR AGAT CNRS, Université des Sciences et Technologies de Lille, F-59665, Villeneuve d'Ascq cedex, France. email: `queff@agat.univ-lille1.fr`

CONVERGENCE ALMOST EVERYWHERE AND DIVERGENCE EVERYWHERE OF TAYLOR AND DIRICHLET SERIES

Abstract

Recent results concerning the convergence almost everywhere or divergence everywhere of Dirichlet series $\sum a_n n^{it}$ appeared in the literature, revealing significant differences with the case of trigonometric series $\sum a_n e^{int}$. In this work, we prove in several cases the optimality of these results. We also discuss the statistical effect of a change of signs, by considering $\sum \pm a_n n^{it}$. According to the way (probabilistic or topological) this change of signs is made, the properties of the resulting series are quite different, and can also be applied to the theory of power series.

1 Introduction

Dvoretzky and Erdös ([DE]) proved the following.

(1) If $\sum_0^{+\infty} |a_n|^2 = \infty$, and if $|a_n|$ is non-increasing, then there exists a choice of signs $\varepsilon_n = \pm 1$ such that the series $\sum_0^{\infty} \varepsilon_n a_n e^{int}$ diverges for each $t \in \mathbb{R}$. We may even have unbounded divergence everywhere.

In a second paper, they gave an idea of the size of "admissible signs" in (1).

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(2) If $\overline{\lim}\frac{1}{\log N}\sum_1^N |a_n|^2 > 0$, then for almost all choices of signs $\varepsilon_n = \pm 1$, the series $\sum_0^\infty \varepsilon_n a_n e^{int}$ diverges everywhere, and the condition on (a_n) is optimal.

On the other hand, Carleson ([Ca]) proved that, if $\sum_0^\infty |a_n|^2 < \infty$, then $\sum_0^\infty a_n e^{int}$ converges for almost all t (with respect to Lebesgue measure). Later, Hedenmalm and Saksman ([HS]) extended Carleson's result to Dirichlet series as follows.

(3) If $\sum_0^\infty |a_n|^2 < \infty$, then $\sum_1^\infty a_n n^{-1/2+it}$ converges for almost all t.

A simplified proof of (3) was given in [KQ], as well as an extension of (1) to Dirichlet series, as follows.

(4) If $\sum_1^\infty |a_n|^2 = \infty$, and if $n^{1/2}|a_n|$ is non-increasing, then there exists a choice of signs $\varepsilon_n = \pm 1$ such that the series $\sum_1^\infty \varepsilon_n a_n n^{-1/2+it}$ diverges unboundedly everywhere.

The aims of this paper are the following: First, we extend (2) to Dirichlet series, with a change of scale, and show the optimality of the result. Second, we show that (3) is optimal in two respects: denoting by $\mathcal{H}$ the space of Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s}$ such that $\sum_1^\infty |a_n|^2 = \|f\|^2 < \infty$, we show that, for some $f \in \mathcal{H}$, the series $\sum_1^\infty a_n n^{-1/2+it}$ may diverge for $t \in E$, a prescribed set of measure zero on the line. Moreover, even if f belongs to the smaller space $\mathcal{H}^\infty$ (to be defined later), the translation $1/2$ cannot be completely dispensed with. Third, we extend (1) to show that $\sum_0^\infty \varepsilon_n a_n e^{int}$ diverges everywhere for quasi-all choices of signs, and give a similar extension for Dirichlet series. This topological point of view turns out to be better adapted if we want to exhibit Taylor series with a strange behavior (see Theorem 4.3). Finally, part 5 is devoted to some concluding remarks and questions.

A word of explanation is necessary for the terms "almost all" and "quasi-all" which we used. Let $\Omega = \{-1,1\}^{\mathbb{N}}$ be the set of all choices of signs $\omega = (\varepsilon_n(\omega))_{n\geq 0}$, equipped with its natural topology (product of the discrete topology on $\{-1,1\}$) and with its natural probability (product of the probabilities $\frac{1}{2}(\delta_{-1}+\delta_1)$ on each factor), for which the variables $\omega \mapsto \varepsilon_n(\omega)$ appear as independent random variables such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$. Ω is then a Baire space (since it is compact) and a probability space. A property (Q) will be said to hold:

a) For almost all choices of signs if it holds for $\omega \in A$, with $P(A) = 1$.

b) For quasi-all choices of signs if it holds for $\omega \in A$, a dense G_δ subset of Ω.

2 Almost Sure Divergence Everywhere of Dirichlet Series

In this section, we shall prove the following result.

Theorem 2.1. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers.*

a) If $\overline{\lim}_{N\to\infty} \dfrac{1}{\log\log N}\displaystyle\sum_1^N |a_n|^2 = \gamma > 0$ (with $\gamma = \infty$ allowed), then for almost all choices of signs $\varepsilon_n = \pm 1$, the series $\displaystyle\sum_1^\infty \varepsilon_n a_n n^{it}$ diverges for each $t \in \mathbb{R}$.

b) The result of a) is optimal in the following sense : if $\delta_N \xrightarrow{>} 0$, there exists a sequence (a_n) such that $\overline{\lim}_{N\to\infty} \dfrac{1}{\delta_N \log\log N}\displaystyle\sum_1^N |a_n|^2 > 0$, but for each $\omega \in \Omega$, the series $\displaystyle\sum_1^\infty \varepsilon_n(\omega) a_n n^{it}$ converges for at least one $t \in \mathbb{R}$.

Preliminary comment: In this statement, the translation 1/2 of (3) and (4) has disappeared, which is predictable for the following reason: Suppose that the hypotheses are fulfilled, but that $\displaystyle\sum_1^\infty n^{-1}|a_n|^2 < \infty$ (e.g., $a_n = \sqrt{j}$ if $n = 2^{2^j}$, $a_n = 0$ otherwise). Then, for each $t \in \mathbb{R}$, the series $\displaystyle\sum_1^\infty \varepsilon_n(\omega) a_n n^{-1/2+it}$ converges for almost all $\omega \in \Omega$ by the three series Theorem [Ka]; now, by Fubini's Theorem, this implies that, almost surely with respect to ω, the series $\displaystyle\sum_1^\infty \varepsilon_n(\omega) a_n n^{-1/2+it}$ converges for almost all t. Divergence for every t cannot take place for the series $\displaystyle\sum_1^\infty \varepsilon_n(\omega) a_n n^{-1/2+it}$.

Proof. (of Theorem 2.1) We begin with b); M. Weiss [W] proved that, if $b_n \to 0$, then $\displaystyle\sum_0^\infty b_n e^{i2^n t}$ converges for some real t. We will use this result as follows. Set $N_j = 2^{2^j}$, $\delta_j^* = \sup_{n\geq N_j} \delta_n$, $b_j = \sqrt{\delta_j^*}$, and take

$$a_n = b_j \text{ if } n = N_j;\ \ a_n = 0 \text{ if } n \notin \{N_0, N_1, \dots\}.$$

Observe that the hypothesis of b) holds for a_n; in fact, if $N_j \leq N < N_{j+1}$, one has

$$\frac{1}{\delta_N \log\log N} \sum_1^N |a_n|^2 \geq \frac{C}{\delta_j^* \times j} \sum_{k=1}^j \delta_k^* \geq \frac{C}{j\delta_j^*} j\delta_j^* = C,$$

where $C > 0$ is a numerical constant. And if $\varepsilon_n = \pm 1$, setting $\theta_j = \varepsilon_{N_j}$, we have

$$\sum_{n=1}^\infty \varepsilon_n a_n n^{it} = \sum_{j=0}^\infty \theta_j b_j e^{i2^j (t \log 2)},$$

and since $\theta_j b_j \to 0$, this series converges for some t, by the result of M. Weiss.

The proof of a) is more difficult, although we follow more or less closely the trigonometric case; we first assume that $\gamma < \infty$. Let $l > 0$ to be adjusted later; define an increasing sequence (p_k) of integers by $p_0 = 0$, and by the condition that p_{k+1} is the first integer $> p_k$ such that

$$\sum_{p_k < n \leq p_{k+1}} |a_n|^2 > l. \tag{2.1}$$

This is possible since $\sum_1^\infty |a_n|^2 = \infty$. Il follows that $\frac{1}{k} \sum_1^{p_k} |a_n|^2 \to l$ (without loss of generality, we can assume that $a_n \to 0$). Let now (N_s) be an increasing sequence of integers such that $\frac{1}{\log\log N_s} \sum_1^{N_s} |a_n|^2 \to \gamma$ as $s \to \infty$, and K_s be defined by $p_{K_s} \leq N_s < p_{1+K_s}$. Observe that

$$\begin{aligned} \frac{1}{\log\log p_{K_s}} \sum_1^{p_{K_s}} |a_n|^2 &\geq \frac{1}{\log\log N_s} \left(\sum_1^{N_s} |a_n|^2 - \sum_{p_{K_s} < n < p_{1+K_s}} |a_n|^2 \right) \\ &\geq \frac{1}{\log\log N_s} \left(\sum_1^{N_s} |a_n|^2 - l \right), \end{aligned}$$

so that

$$\frac{1}{K_s} \sum_1^{p_{K_s}} |a_n|^2 \geq \frac{\log\log p_{K_s}}{K_s} \frac{1}{\log\log N_s} \left(\sum_1^{N_s} |a_n|^2 - l \right).$$

Letting s tend to infinity now gives $l \geq \gamma \overline{\lim} \frac{\log\log p_{K_s}}{K_s}$. One thus can find an infinite set J of positive integers such that

$$j \in J \implies \frac{\log\log p_j}{j} \leq \frac{2l}{\gamma} \implies \log p_j \leq \exp\left(\frac{2jl}{\gamma}\right). \tag{2.2}$$

Now, we fix $T>0$, and will prove that, almost surely, $\sum_1^{\infty} \varepsilon_n(\omega) a_n n^{it}$ diverges for each $t \in [-T,T]$; this will clearly imply a), and the proof will be based on the following crucial Lemma of minimax.

Lemma 2.2. *Set* $q_j = \left(\sum_{p_j < n \leq p_{j+1}} |a_n|^2 \right)^{1/2}$ *and*

$$B_j(t) = B_j(t,\omega) = \sum_{p_j < n \leq p_{j+1}} \varepsilon_n(\omega) a_n n^{it}.$$

Suppose that $\mu + 1 \in J$. *Then, one has*

$$P\left(\inf_{|t|\leq T} \left(\sup_{\frac{\mu}{2} < j \leq \mu} \frac{|B_j(t)|}{q_j/4} \right) \leq 1 \right) \leq \lambda_T C_l^{\mu} (1-B)^{\mu}, \tag{2.3}$$

where $0 < B < 1$ *is a numerical constant,* $C_l > 1$ *is a constant that tends to one as* $l \overset{>}{\longrightarrow} 0$, *and* $\lambda_T > 0$ *only depends on* T.

Proof of the Lemma. We shall discretize the infimum over t, and shall need for that an estimate for the derivative of B_j, under the form of the following sublemma.

Sublemma. *Let* $Q(t) = \sum_{n=1}^{\nu} \varepsilon_n b_n n^{it}, \nu \geq 2$, *be a random Dirichlet polynomial, and let* $T \geq 1$; *set* $\|f\|_T$ *for* $\sup_{|t|\leq T} |f(t)|$, *and* $\|b\|_2 = \left(\sum_1^{\nu} |b_n|^2\right)^{1/2}$ *for* $b = (b_1, \ldots, b_\nu)$. *Then (E denoting expectation) one has*

$$E\left(\|Q\|_T\right) \leq C_T' \|b\|_2 \sqrt{\log \nu}, \tag{2.4}$$

where the constant C_T' *only depends on* T.

Proof of the Sublemma. Let $N \geq 2$ be an integer to be chosen later, and $t_k = \frac{kT}{N}$, $-N \leq k \leq N$; (t_k) is a $\frac{T}{N}-$net of $[-T,T]$. According to a well-known estimate of Salem and Zygmund [SaZ], we have that

$$E\left(\sup_{|k|\leq N} |Q(t_k)| \right) << \|b\|_2 \sqrt{\log N},$$

where $A_N << B_N$ means that there exists a constant κ, which does not depend on N, such that $A_N \leq \kappa B_N$. Moreover, if $t \in [-T,T]$ and $|t - t_k| \leq \frac{T}{N}$, one has $|Q(t)-Q(t_k)| \leq |t-t_k| \|Q'\|_\infty \leq \frac{T}{N}\sum_1^\nu |b_n|\log n \leq \frac{T}{N}\nu^{1/2}\log\nu\|b\|_2$, by the Cauchy-Schwarz inequality. Therefore,

$$\|Q\|_T \leq \sup_{|k|\leq N} |Q(t_k)| + \frac{T\nu^{1/2}\log\nu}{N}\|b\|_2,$$

and

$$E\left(\|Q\|_T\right) << \|b\|_2 \left(\sqrt{\log N} + \frac{T\nu^{1/2}\log\nu}{N}\right).$$

The choice $N = \left[T\nu^{1/2}\log\nu\right] + 1$ (where [.] stands for the integer part) gives the result, observing that $\log(T\nu) \leq \log(\nu^T) \leq T\log\nu$. $\square$

Let us return to the proof of the Lemma. We abbreviate $\inf_{|t|\leq T}$ as $\inf_t$, and denote by E_μ the event $\inf_t \left(\sup_{\frac{\mu}{2}<j\leq\mu} \frac{|B_j(t)|}{q_j/4}\right) \leq 1$ appearing in the left hand-side of (2.3). Let $N \geq 2$ to be chosen later, $t_k = \frac{kT}{N}$, $-N \leq k \leq N$ and (μ being kept fixed) F_N be the event

$$\inf_{|k|\leq N} \left(\sup_{\frac{\mu}{2}<j\leq\mu} \frac{|B_j(t_k)|}{q_j/2}\right) \leq 1$$

and G_N be the event

$$\sup_{\frac{\mu}{2}<j\leq\mu} \frac{\|B_j'\|_T}{q_j/4} > \frac{N}{T}.$$

First, we observe that

$$E_\mu \subset F_N \cup G_N. \tag{2.5}$$

In fact, if $\omega \in F_N^c \cap G_N^c$, and if $-T \leq t \leq T$, let k be such that $|t-t_k| \leq \frac{T}{N}$, and $j \in]\frac{\mu}{2},\mu]$ be such that $|B_j(t_k)| > \frac{q_j}{2}$. Then

$$\begin{aligned}|B_j(t)| &\geq |B_j(t_k)| - |B_j(t)-B_j(t_k)| \\ &\geq |B_j(t_k)| - |t-t_k|\|B_j'\|_T \\ &> \frac{q_j}{2} - \frac{T}{N}\frac{q_j}{4}\frac{N}{T} = \frac{q_j}{4},\end{aligned}$$

so that $\omega \in E_\mu^c$. Second, we will show that

$$P(G_N) \leq \frac{C_T''}{N}\mu C_l^\mu, \text{ where } C_l = \exp\left(\frac{4l}{\gamma}\right). \tag{2.6}$$

In fact, setting $I_j =]p_j, p_{j+1}]$, we have $B'_j(t) = \sum_{n\in I_j} (i\log n) a_n \varepsilon_n n^{it}$, and the sublemma gives

$$E\left(\|B'_j\|_T\right) \le C'_T \sqrt{\log p_{j+1}} \left(\sum_{n\in I_j} \log^2 n |a_n|^2\right)^{1/2} \le C'_T \left(\log p_{j+1}\right)^{3/2} q_j,$$

so that by Markov's inequality

$$\begin{aligned} P(G_N) &\le \sum_{\frac{\mu}{2}<j\le\mu} P\left(\|B'_j\|_T > \frac{q_j}{4}\frac{N}{T}\right) \\ &\le \sum_{\frac{\mu}{2}<j\le\mu} \frac{4T}{Nq_j} C'_T q_j \left(\log p_{j+1}\right)^{3/2} \\ &\le \frac{4T}{N} C'_T \mu \left(\log p_{\mu+1}\right)^{3/2} \\ &\le \frac{4TC'_T}{N} \mu \exp\left(\frac{3(\mu+1)l}{\gamma}\right), \end{aligned}$$

where we used (2.2) since $\mu + 1 \in J$. This gives (2.6), with $C''_T = 4TC'_T$.

Finally, we will show that

$$P(F_N) \le 3N(1-\delta)^\mu, \text{ where } 0<\delta<1 \text{ is a numerical constant.} \tag{2.7}$$

In fact, $F_N = \bigcup_{|k|\le N} \left(\bigcap_{\frac{\mu}{2}<j\le\mu} F_{j,k}\right)$, where $F_{j,k}$ is the event

$$F_{j,k} = \left\{\omega;\ |B_j(t_k)| \le \frac{q_j}{2}\right\}.$$

For fixed k, the $F_{j,k}$ are independent, since the blocks B_j have pairwise disjoint supports I_j. Therefore we have

$$P(F_N) \le \sum_{|k|\le N} \prod_{\frac{\mu}{2}<j\le\mu} P(F_{j,k}).$$

We shall majorize $P(F_{j,k})$ by minorizing $P(F^c_{j,k})$ with the help of the Paley-Zygmund inequality ([Ka])

$$P(X \ge \lambda E(X)) \ge (1-\lambda)^2 \frac{(E(X))^2}{E(X^2)},$$

for $0<\lambda<1$ and X a positive random variable. This gives

$$\begin{aligned}P(F_{j,k}^c) =&P\left(|B_j(t_k)|>\frac{1}{2}q_j\right)=P\left(|B_j(t_k)|^2>\frac{1}{4}E(|B_j(t_k)|^2)\right)\\ \geq&\frac{9}{16}\frac{\left(E(|B_j(t_k)|^2)\right)^2}{E(|B_j(t_k)|^4)}\geq\frac{1}{16}=\delta',\end{aligned}$$

by Khintchine's inequality ([Dis]). Therefore

$$P(F_N)\leq\sum_{|k|\leq N}(1-\delta')^{\mu/2}\leq 3N(1-\delta)^\mu,$$

where δ is defined by $\sqrt{1-\delta'}=1-\delta$. It now follows from (2.5), (2.6) and (2.7) that

$$P(E_\mu)\leq P(F_N)+P(G_N)\leq 3N(1-\delta)^\mu+\frac{C''_T}{N}\mu C_l^\mu.$$

We optimize this inequality with the choice

$$N=\left[\left(C''_T\mu C_l^\mu(1-\delta)^{-\mu}\right)^{1/2}\right]+1,$$

and this gives (2.3) of Lemma 2.2 with an appropriate numerical constant $B\in]0,1[$, and with an appropriate λ_T. □

Lemma 2.2 easily gives the conclusion in a) of Theorem 2.1. Let us first fix $l>0$ so small that $C_l(1-B)=q<1$; this is possible since $C_l\to 1$ as $l\xrightarrow{>}0$. Then, (2.3) gives $P(E_\mu)\leq\lambda_T q^\mu$ (once and for all, we restrict ourselves to $\mu+1\in J$), so that $\sum_\mu P(E_\mu)<\infty$, implying $P\left(\underline{\lim}_\mu E_\mu^c\right)=1$, by the Borel-Cantelli Lemma. Equivalently, there exists $\Omega_0\subset\Omega$, with $P(\Omega_0)=1$, such that for $\omega\in\Omega_0$ and for μ large enough (depending on ω), one has

$$\inf_{|t|\leq T}\left(\sup_{\frac{\mu}{2}<j\leq\mu}\frac{|B_j(t,\omega)|}{q_j/4}\right)>1.$$

Fix $t\in[-T,T]$, and $\omega\in\Omega_0$. We can find arbitrarily large μ, and arbitrarily large $j\in]\mu/2,\mu]$, such that $|B_j(t,\omega)|>\frac{q_j}{4}>\frac{\sqrt{l}}{4}$ (in view of (2.1)). Since B_j is a block of the series $\sum\varepsilon_n(\omega)a_n n^{it}$, this proves that the Cauchy criterion fails for this series, which is therefore divergent.

If we now suppose that $\overline{\lim}_{N\to\infty}\dfrac{1}{\log\log N}\displaystyle\sum_1^N|a_n|^2=+\infty$, then for almost all choices of signs $\varepsilon_n=\pm1$, the series $\displaystyle\sum_1^\infty\varepsilon_n a_n n^{it}$ is unboundedly divergent for each $t \in \mathbb{R}$. Indeed, in this case, the previous proof gives, for any $0 < \gamma < \infty$, a subset Ω_γ of Ω with $P(\Omega_\gamma) = 1$ such that, for each $t \in [-T, T]$ and each $\omega \in \Omega_\gamma$, there exists arbitrarily large j such that $B_j(t,\omega) > \frac{\sqrt{l}}{4}$, with $l = \kappa\gamma$, where κ is a numerical constant (l just depends on γ by the condition $\exp\left(\frac{4l}{\gamma}\right)(1 - B) < 1$, where B is a numerical constant). Setting $\gamma_n = n$, and $\Omega_\infty = \bigcap_n \Omega_{\gamma_n}$, for each t in $[-T, T]$ and each ω in Ω_∞, the series $\sum_1^\infty \varepsilon_n(\omega) a_n n^{it}$ is unboundedly divergent. □

3 Optimality of the Hedenmalm-Saksman Theorem; the Case of $\mathcal{H}^\infty$.

In this section, we are going to prove the optimality of (3) in the introduction and, answering a question of Hedenmalm ([He]), we will show that, even for the smaller space $\mathcal{H}^\infty$, the translation factor $n^{-1/2}$, or at least some translation factor $n^{-\varepsilon}$, cannot be dispensed with.

Theorem 3.1. *Let $E \subset \mathbb{R}$ be a set of Lebesgue-measure zero. Then, there exists a Dirichlet series $\sum_1^\infty a_n n^{-s}$ such that:*

a) $\sum_1^\infty |a_n|^2 < \infty$.

b) $\sum_1^\infty a_n n^{-1/2+it}$ *diverges for $t \in E$.*

PROOF. In [KQ] (see c) of Theorem 2.1), to prove a Dvoretzky-Erdös type result, we had successively built:

i) a sequence of blocks $B_k = \sum_{p_k < n \leq p_{k+1}} |a_n| n^{-1/2}$ of the divergent series $\sum |a_n| n^{-1/2}$, such that $1 < B_k \leq 2$, and that the sequence of integers (p_k) is not too lacunary: $\displaystyle\sum_1^\infty \frac{1}{\log \frac{p_{k+1}}{p_k}} = \infty$.

ii) a sequence (l_k) of "lengths" adapted to the the p_k's: $l_k = \dfrac{1}{4 \log \frac{p_{k+1}}{p_k}}$, so that $\sum_1^\infty l_k = \infty$.

iii) a sequence (Δ_k) of intervals of length $2l_k$, covering $\mathbb{R}$ infinitely many times.

iv) a sequence of blocks $D_k(t) = \displaystyle\sum_{n \in I_k} \varepsilon_n a_n n^{-1/2+it}$, with $I_k =]p_k, p_{k+1}]$, $\varepsilon_n = \pm 1$, $|D_k|$ big at the center of Δ_k and oscillating little on Δ_k.

Here, we use exactly the same ingredients, but in a different order. Since E is negligible, we can find a sequence of closed intervals $\Delta_k = [t_k - l_k; t_k + l_k]$ such that $0 < l_k < 1/4$ and that:

$$\sum_1^\infty l_k < \infty \tag{3.1}$$

$$\text{Each point of } E \text{ belongs to infinitely many } \Delta_k\text{'s.} \tag{3.2}$$

(This should be thought of as a converse of the Borel-Cantelli Lemma). Now, motivated by ii), we define inductively an increasing sequence of positive integers by

$$p_1 = 4; \quad p_{k+1} = \left[p_k e^{1/4l_k}\right], \ k \geq 1. \tag{3.3}$$

We denote by I_k the block of integers $]p_k, p_{k+1}]$ and define a sequence (b_n) of positive numbers (the moduli of the a_n's) by

$$b_n = 0 \text{ for } n = 1, \dots, 4; \quad b_n = l_k n^{-1/2} \text{ for } n \in I_k, k \geq 1.$$

We claim that:

$$\sum_1^\infty b_n^2 < \infty \tag{3.4}$$

$$\sum_{n \in I_k} n^{-1/2} b_n \geq \frac{1}{16} \text{ for } k \text{ large enough } (k \geq k_0). \tag{3.5}$$

In fact,

$$\sum_{n \in I_k} b_n^2 = l_k^2 \sum_{p_k < n \leq p_{k+1}} \frac{1}{n} \leq l_k^2 \log \frac{p_{k+1}}{p_k} \leq \frac{l_k}{4}$$

in view of (3.3), and (3.4) follows since $\sum_1^\infty l_k < \infty$. Moreover

$$\sum_{n \in I_k} n^{-1/2} b_n = l_k \sum_{n \in I_k} \frac{1}{n} \geq \frac{l_k}{2} \log \frac{p_{k+1}}{p_k} \geq \frac{l_k}{2} \frac{1}{8 l_k} = \frac{1}{16}$$

for $k \geq k_0$. We can now adjust blockwise the real signs $\varepsilon_n = \pm 1$ in such a way that, setting $D_k(t) = \sum_{n \in I_k} \varepsilon_n b_n n^{-1/2+it}$, we have

$$|D_k(t_k)| \geq \frac{1}{2} \sum_{n \in I_k} b_n n^{-1/2} \geq \frac{1}{32}, \tag{3.6}$$

for $k \geq k_0$. Due to the fact that $|D_k|$ oscillates little on the small interval Δ_k around t_k, it follows that

$$|D_k(t)| \geq \frac{1}{64} \text{ for } t \in \Delta_k \text{ and } k \geq k_0. \tag{3.7}$$

Indeed, setting

$$C_k(t) = D_k(t)p_k^{-it} = \sum_{n \in I_k} \varepsilon_n b_n n^{-1/2} \left(\frac{n}{p_k}\right)^{it},$$

and $S_k = \sum_{n \in I_k} b_n n^{-1/2}$, we have for $t \in \Delta_k$ and $k \geq k_0$

$$\begin{aligned}
&\big||D_k(t)| - |D_k(t_k)|\big| = \big||C_k(t)| - |C_k(t_k)|\big| \\
&\leq |C_k(t) - C_k(t_k)| \\
&\qquad \leq \sum_{n \in I_k} b_n n^{-1/2} |t - t_k| \log \frac{n}{p_k} \leq l_k \log \frac{p_{k+1}}{p_k} S_k \leq \frac{S_k}{4},
\end{aligned}$$

so that

$$|D_k(t)| \geq |D_k(t_k)| - \frac{S_k}{4} \geq \frac{S_k}{2} - \frac{S_k}{4} = \frac{S_k}{4} \geq \frac{1}{64},$$

in view of (3.5) and (3.6). Finally, set $a_n = \varepsilon_n b_n$; (3.2), (3.4) and (3.7) show that the sequence (a_n) has the required properties. □

Remark. Let now E be a null-set of the circle. It is known ([KK]) that there is a continuous function whose Fourier series diverges on E. The method of proof of the previous Theorem can be used to obtain in a simple way such a function f, maybe not continuous, but in H^2. Just define arcs Δ_k on the circle satisfying (3.1) and (3.2). Inductively define integers p_k by $p_1 = 1$ and $p_{k+1} = p_k + \left[\frac{1}{4l_k}\right]$. Set $I_k =]p_k, p_{k+1}]$, $b_n = 0$ for $n = 0, \dots, 4$, $b_n = l_k$ for $n \in I_k$. Adjust real signs $\varepsilon_n = \pm 1$ blockwise so that $\left|\sum_{n \in I_k} \varepsilon_n b_n e^{int_k}\right| \geq \frac{1}{8}$ for k large enough. This implies, as for Dirichlet series, that

$$\left|\sum_{n \in I_k} \varepsilon_n b_n e^{int}\right| \geq \frac{1}{16} \text{ for } t \in \Delta_k,$$

so that $f(t) = \sum_0^\infty \varepsilon_n b_n e^{int}$ has the required properties.

Now let us turn to the case of $\mathcal{H}^\infty$, the space of Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s}$, with convergence and boundedness of f in the half-plane $\mathbb{C}_0 = \{s \in \mathbb{C};\ \Re(s) > 0\}$. We have the inclusion $\mathcal{H}^\infty \subset \mathcal{H}$, with contraction of norms: $f \in \mathcal{H}^\infty \implies \|f\|_{\mathcal{H}} \leq \|f\|_\infty = \sup_{\Re(s)>0} |f(s)|$. More precisely (see [HLS]), $\mathcal{H}^\infty$ is the set of multipliers of $\mathcal{H}$. So, the space $\mathcal{H}^\infty$ should be considered as a much smaller space than $\mathcal{H}$. In spite of this, we have the following result, which answers in the negative a question of H.Hedenmalm ([He]).

Theorem 3.2. *There exists a Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s} \in \mathcal{H}^\infty$, such that $\sum_1^\infty a_n n^{it}$ diverges for each $t \in \mathbb{R}$. We even have that f is continuous on the closed half-plane $\overline{\mathbb{C}_0}$.*

Proof. We modify a construction due to Lick ([Li]). The following well-known inequality ([B, vol.1, p.90]) is crucial:

$$\left|\sum_{n=1}^{N} \frac{\sin nt}{n}\right| \leq A, \quad \forall t \in \mathbb{R},\ \forall N \geq 1, \tag{3.8}$$

A being a constant. We will use three sequences (h_k), (H_k), and (n_k) having the following properties:

1. (h_k) is an increasing sequence of integers such that $\sum_{k\geq 1} \frac{1}{\log h_k} < \infty$ (one can take $h_k = 2^{k^2}$).
2. (H_k) is a sequence of positive real numbers such that
   a) $\exp\left(\frac{1}{H_k}\right)$ is rational.
   b) $H_k \geq 4kh_k$.

   Such a sequence clearly exists.
3. (n_k) is a sequence of integers such that:
   a) $n_k \exp\left(\frac{l}{H_k}\right)$ is an integer for $1 \leq l \leq 2h_k$.
   b) $n_{k+1} > n_k \exp\left(\frac{2h_k}{H_k}\right)$.
   c) $\sum_{k=1}^{\infty} n_k^{-\sigma} < \infty$ for each $\sigma > 0$.

If $\exp\left(\frac{1}{H_k}\right) = \frac{p_k}{q_k}$, one can for example take $n_k = c_k (q_k)^{2h_k}$, the integers c_k being chosen so large that b) and c) hold. Those three sequences being fixed, let us set, for $k = 1, 2, \ldots$:

$$\begin{aligned} P_k(s) =& \frac{1}{h_k} + \frac{e^{-s}}{h_k - 1} + \cdots + \frac{e^{-(h_k-1)s}}{1} - \frac{e^{-(h_k+1)s}}{1} - \cdots - \frac{e^{-2h_k s}}{h_k} \\ =& \sum_{j=0}^{h_k-1} \frac{e^{-js}}{h_k - j} - \sum_{j=0}^{h_k-1} \frac{e^{-(2h_k-j)s}}{h_k - j} \\ =& \sum_{n=1}^{h_k} \frac{e^{-(h_k-n)s} - e^{-(h_k+n)s}}{n}. \end{aligned} \tag{3.9}$$

For $s = it$ purely imaginary, one clearly has

$$P_k(s) = 2ie^{-ih_k t} \sum_{n=1}^{h_k} \frac{\sin nt}{n},$$

so that (3.8) and the maximum modulus principle give

$$|P_k(s)| \leq 2A, \text{ for each } s \text{ such that } \Re(s) > 0. \tag{3.10}$$

We now modify the definition of P_k to obtain a Dirichlet polynomial, by setting

$$Q_k(s) = \frac{1}{\log(h_k)} P_k\left(\frac{s}{H_k}\right) n_k^{-s}.$$

Equivalently,

$$\begin{aligned} Q_k(s) =& \frac{1}{\log(h_k)} \left(\sum_{j=0}^{h_k-1} \frac{1}{h_k - j} \left(\exp\left(\frac{j}{H_k}\right) n_k \right)^{-s} \right. \\ & \left. - \sum_{j=0}^{h_k-1} \frac{1}{h_k - j} \left(\exp\left(\frac{2h_k - j}{H_k}\right) n_k \right)^{-s} \right). \end{aligned}$$

Condition 3.a) guarantees that the Q_k's are Dirichlet polynomials, while condition 3.b) implies that their spectra are disjoint. We can therefore consider the Dirichlet series

$$f(s) = \sum_{k=1}^{\infty} Q_k(s) = \sum_{1}^{\infty} a_n n^{-s},$$

which will be the required counterexample. In fact:

- If $s=\sigma+it$, with $\sigma>0$, $\sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent.
  To prove this, denote by $Q_k^*(s)$ the sum of the moduli of the terms appearing in $Q_k(s)$. We have

$$
\begin{aligned}
Q_k^*(s) =& \frac{1}{\log(h_k)}\left(\sum_{j=0}^{h_k-1}\left|\frac{1}{h_k-j}\left(\exp\left(\frac{j}{H_k}\right)n_k\right)^{-s}\right|\right.\\
&\left.+\sum_{j=0}^{h_k-1}\left|\frac{1}{h_k-j}\left(\exp\left(\frac{2h_k-j}{H_k}\right)n_k\right)^{-s}\right|\right)\\
\leq& \frac{2}{\log h_k} n_k^{-\sigma}\sum_{j=1}^{h_k}\frac{1}{j}\leq 4n_k^{-\sigma},
\end{aligned}
$$

  and condition 3.c) ensures that

$$
\sum_{n=1}^{\infty}|a_n n^{-s}|=\sum_{k=1}^{\infty}Q_k^*(s)\leq 4\sum_{k=1}^{\infty}n_k^{-\sigma}<\infty.
$$

- $f\in\mathcal{H}^\infty$ and f is continuous on $\overline{\mathbb{C}_0}$.
  To prove this, simply note that, in view of (3.10):

$$
\Re(s)\geq 0 \implies |Q_k(s)|\leq \frac{2A}{\log h_k},
$$

  so that the series defining f converges normally on $\overline{\mathbb{C}_0}$, and that, for $s\in\overline{\mathbb{C}_0}$, one has

$$
|f(s)|\leq\sum_{k=1}^{\infty}\frac{2A}{\log h_k}=M<\infty.
$$

- $\displaystyle\sum_{1}^{\infty} a_n n^{it}$ diverges for each $t\in\mathbb{R}$.
  To prove this, take $t\in\mathbb{R}$. For k large enough, one has $\dfrac{h_k|t|}{H_k}\leq 1$, in view

of 2.b). So that, considering the "first half" of Q_k, one has

$$\begin{aligned}
\Big|\frac{1}{\log h_k}\sum_{j=0}^{h_k-1}\frac{1}{h_k-j}\left(\exp\left(\frac{j}{H_k}\right)n_k\right)^{it}\Big|
&=\Big|\frac{1}{\log h_k}\sum_{j=0}^{h_k-1}\frac{1}{h_k-j}\exp\left(\frac{ijt}{H_k}\right)\Big| \\
&\geq \frac{1}{\log h_k}\sum_{j=0}^{h_k-1}\frac{1}{h_k-j}\cos\left(\frac{jt}{H_k}\right) \\
&\geq \frac{1}{\log h_k}\sum_{j=0}^{h_k-1}\frac{1}{h_k-j}\cos 1 \geq \delta > 0,
\end{aligned}$$

where δ is an absolute constant (we used the fact that, for $0 \leq j \leq h_k - 1$, one has $\left|\frac{jt}{H_k}\right| \leq \frac{h_k|t|}{H_k} \leq 1$). This shows that $\sum_1^\infty a_n n^{it}$ fails to verify the Cauchy criterion and therefore diverges. □

Remarks.

1. In our example, the series $\sum a_n n^{it}$ is boundedly divergent at each real number t. In fact, for each integer N, there exists an integer r such that

$$\sum_1^N a_n n^{it} = Q_1(-it) + \cdots + Q_r(-it) + \alpha_r,$$

where α_r is a partial sum of $Q_{r+1}(-it)$, so that

$$\begin{aligned}
\left|\sum_1^N a_n n^{it}\right| &\leq |Q_1(-it)| + \cdots + |Q_r(-it)| + Q_{r+1}^*(-it) \\
&\leq 2A\sum_{k=1}^\infty \frac{1}{\log h_k} + 4.
\end{aligned}$$

This is in contrast with what happens for Fourier series. According to a Theorem of Marcinkiewicz ([Kö]), if $a_n \to 0$, $\sum_0^\infty a_n e^{int}$ cannot be boundedly divergent everywhere. We shall return to this in the last section.

2. If one considers general Dirichlet series $f(s) = \sum_{n=1}^\infty a_n e^{-\lambda_n s}$, convergent and with bounded sum f in the half-plane $\mathbb{C}_0$, and if one looks at

the convergence on the boundary $i\mathbb{R}$ of $\mathbb{C}_0$, three fairly distinct behaviors appear:

a) If $\lambda_n = n$, we have convergence almost everywhere on the boundary by Carleson's Theorem.

b) If $\lambda_n = \log n$, we may have everywhere divergence on $i\mathbb{R}$, as indicated by Theorem 3.2.

c) If $\lambda_n = \log p_n$, where p_n is the n^{th} prime number, Kronecker's Theorem implies that we have absolute convergence everywhere on the boundary. (More generally, Bohr's Theorem implies that, in all the examples of Theorem 3.2, we necessarily have $\sum_{n=1}^{\infty} |a_{p_n}| < \infty$.

4 Quasi-Sure Divergence Everywhere of Taylor and Dirichlet Series; an Application

In this section, we will consider the set of "admissible" signs $\omega = (\varepsilon_n(\omega))$ in $(1), \ldots, (4)$ from the topological, i.e. quasi-sure, point of view, which turns out to be better adapted to some counterexamples. Let X be a $\sigma-$compact topological space; i.e., $X = \cup_{k=1}^{\infty} X_k$, where (X_k) is an increasing sequence of compact subsets of X. We first prove a general theorem (Recall that a series is said to be unboundedly divergent if its partial sums are unbounded.) for a series $(f_n)_{n\geq 1}$ of continuous functions $:X \to \mathbb{C}$. It is convenient to set

$$S_n = f_1 + \cdots + f_n,\ \delta_N(t) = \sup_{n>N} |S_n(t) - S_N(t)| \text{ and } \delta(t) = \liminf_{N\to\infty} \delta_N(t).$$

Theorem 4.1. *Let $(f_n)_{n\geq 1}$ be a sequence of continuous functions $:X \to \mathbb{C}$. Then we have the following:*

a) If $\delta(t) = \infty$ for all t, then for quasi-all choices of signs ω, the series $\sum_{1}^{\infty} \varepsilon_n(\omega) f_n(t)$ is unboundedly divergent everywhere on X.

b) If there is a continuous function φ such that $\delta(t) > \varphi(t) > 0$ for all t, then for quasi-all choices of signs ω, the series $\sum_{1}^{\infty} \varepsilon_n(\omega) f_n(t)$ is divergent everywhere on X.

PROOF.

a) We shall construct an increasing sequence $(N_k)_{k\geq 0}$ of integers such that

$$\forall t \in X_k, \quad \sup_{N_{k-1}<n\leq N_k} \left|S_n(t) - S_{N_{k-1}}(t)\right| > k. \tag{4.1}$$

Take $N_0 = 0$. We assume that $N_0, \dots, N_{k-1}$ have been defined, and describe how to choose N_k. Let

$$\delta_N^*(t) = \sup_{N_{k-1}<n\leq N} \left|S_n(t) - S_{N_{k-1}}(t)\right|.$$

By our assumption, the continuous functions $(\delta_N^*(t)+1)^{-1}$ decrease to 0 for each $t \in X_k$, and Dini's Theorem for the compact set X_k implies the existence of $N > N_{k-1}$ such that $(\delta_N^*(t)+1)^{-1} < (k+1)^{-1}$ for each $t \in X_k$. Just take $N_k = N$ to get (4.1). Now, set

$$O_k = \{\omega \in \Omega;\ \varepsilon_n(\omega) = 1 \text{ if } N_{k-1} < n \leq N_k\},$$

and

$$\Omega_k = \bigcup_{l\geq k} O_l.$$

O_k is an open set, and Ω_k is a dense open set. In fact, if $\alpha \in \Omega$ and if N is a positive integer, take $l \geq k$ such that $N_{l-1} \geq N$, and define $\omega \in \Omega$ by $\varepsilon_n(\omega) = \varepsilon_n(\alpha)$ if $n \leq N$, $\varepsilon_n(\omega) = 1$ if $n > N$, then $\omega \in \Omega_k$, which shows that $\alpha \in \overline{\Omega_k}$. The set

$$A = \bigcap_{k\geq 1} \Omega_k = \overline{\lim}\, O_k$$

is therefore quasi-sure and for $\omega \in A$, and $t \in \mathbb{R}$, one has

$$\sup_{N_{k-1}<n\leq N_k} \Big|\sum_{N_{k-1}+1}^{n} \varepsilon_n(\omega) f_n(t)\Big| = \sup_{N_{k-1}<n\leq N_k} \left|S_n(t) - S_{N_{k-1}}(t)\right| > k$$

for arbitrarily large values of k such that $t \in X_k$, which proves the unbounded divergence of $\displaystyle\sum_1^\infty \varepsilon_n(\omega) f_n(t)$.

b) We will now construct the sequence (N_k) so that

$$\forall t \in X_k, \quad \sup_{N_{k-1}<n\leq N_k} \left|S_n(t) - S_{N_{k-1}}(t)\right| \geq \frac{\varphi(t)}{2}. \tag{4.2}$$

Take $N_0 = 0$ and having chosen $N_0, \dots, N_{k-1}$, construct N_k by a variant of Dini's Theorem. For each t, there exists $N(t) > N_{k-1}$ such that

$$\delta^*_{N(t)}(t) > \frac{\delta(t)}{2} > \frac{\varphi(t)}{2},$$

and by continuity

$$\delta^*_{N(t)}(t') > \frac{\varphi(t')}{2} \text{ for } t' \in U(t), \text{ an open neighbourhood of } t.$$

Take a finite cover $U(t_1), \dots, U(t_r)$ of X_k and define $N_k = \max_{j \le r} N(t_j)$ to get (4.2); the rest of the proof is unchanged. Conclusion b) no longer holds under the sole assumption that $\sum f_n(t)$ diverges everywhere. See the recent work of Keleti and Matrai ([KM]) for a counterexample.

$\square$

If we specialize Theorem 4.1 to Taylor and Dirichlet series and to $X = \mathbb{R}$, we get the following.

Theorem 4.2. *Let $(a_n)_{n\ge 0}$ be a sequence of complex numbers with $\sum_0^\infty |a_n|^2 = \infty$.*

a) If $(|a_n|)$ is non-increasing, then for quasi-all choices of signs ω, the series $\sum_0^\infty \varepsilon_n(\omega) a_n e^{int}$ is unboundedly divergent everywhere.

b) If $(n^{1/2}|a_n|)_{n\ge 1}$ is non-increasing, then for quasi-all choices of signs ω, the series $\sum_1^\infty \varepsilon_n(\omega) a_n n^{-1/2+it}$ is unboundedly divergent everywhere.

Proof. Let $(f_n(t))$ be either $a_n e^{int}$ or $a_n n^{-1/2} n^{it}$. In both cases, it follows respectively from [DE] or [KQ] that there exists a choice of real signs $\theta_n = \pm 1$ such that the series $\sum \theta_n f_n(t)$ is unboundedly divergent everywhere. Now, Theorem 4.1 (applied to $\theta_n f_n$) shows that, for quasi-all ω, the series $\sum \varepsilon_n(\omega)\theta_n f_n(t)$ is unboundedly divergent everywhere (in t); and we have the same conclusion for the series $\sum \varepsilon_n(\omega) f_n(t)$, since the "translation" $(\varepsilon_n(\omega)) \mapsto (\varepsilon_n(\omega)\theta_n)$ is a homeomorphism of Ω onto itself. $\square$

Theorem 4.2 will be applied to give examples of Taylor series with pathological (although generic!) properties, which would probably be very difficult to obtain explicitly.

Theorem 4.3. *There exists a Taylor series* $\sum_0^\infty b_n z^n = f(z)$*, with radius of convergence 1, having the following three properties:*

a) The series converges at $z = 1$ *and diverges unboundedly for* $|z| = 1$ *and* $z \neq 1$.

b) f *is unbounded on any circle* $\gamma_\rho(\theta) = 1 - \rho + \rho e^{2i\pi\theta}$, $0 < \rho < 1$, $0 \leq \theta \leq 1$, *internally tangent to* $\mathbb{D}$ *at 1. In particular, the series* $\sum_0^\infty b_n z^n$ *converges uniformly on no circle* γ_ρ.

c) The unit circle $|z| = 1$ *is a natural boundary for* f.

Proof. Let $(\rho_j)_{j\geq 1}$ be a sequence of values of ρ which decreases to zero, and let

$$I_{n,j} = \int_0^1 |\gamma_{\rho_j}(\theta)|^n |1 - \gamma_{\rho_j}(\theta)| d\theta.$$

We claim that

$$\sum_{n=0}^{\infty} I_{n,j} = \infty \text{ for every } j. \tag{4.3}$$

In fact, setting $e(\theta) = e^{2i\pi\theta}$, one has

$$\begin{aligned}\sum_{n=0}^{\infty} I_{n,j} &= \int_0^1 \frac{|1 - \gamma_{\rho_j}(\theta)|}{1 - |\gamma_{\rho_j}(\theta)|} d\theta \geq \int_0^1 \frac{|1 - \gamma_{\rho_j}(\theta)|}{1 - |\gamma_{\rho_j}(\theta)|^2} d\theta \\ &\geq \int_0^1 \frac{\rho_j |1 - e(\theta)|}{2\rho_j(1-\rho_j)(1 - \cos 2\pi\theta)} d\theta \geq \delta_j \int_0^{1/2} \frac{|\theta|}{\theta^2} d\theta = \infty\end{aligned}$$

(δ_j depending only on ρ_j).

Lemma 4.4. *There exists a sequence* $(w_n)_{n\geq 0}$*, non-increasing and tending to zero, and such that*

$$\sum_{n=0}^{\infty} w_n^2 = \infty \textit{ and } \sum_{n=0}^{\infty} w_n I_{n,k} = \infty \textit{ for } k \geq 1. \tag{4.4}$$

Proof of Lemma. In view of (4.3), we can find a sequence $0 = n_1 < n_2 < \dots$ of integers such that

$$\sum_{n_j \leq n < n_{j+1}} I_{n,k} \geq j \text{ for } k = 1, \dots, j. \tag{4.5}$$

Define (w_n) by $w_n = j^{-1/2}$ for $n_j \le n < n_{j+1}$. We have

$$\sum_0^\infty w_n^2 = \sum_1^\infty \frac{n_{j+1}-n_j}{j} \ge \sum_1^\infty \frac{1}{j} = \infty.$$

And, for fixed k, as soon as $j \ge k$, (4.5) implies that

$$\sum_{n_j \le n < n_{j+1}} w_n I_{n,k} \ge j^{-1/2} \text{ and } \sum_{n_j \le n < n_{j+1}} I_{n,k} \ge j^{1/2},$$

so that $\sum_{n=0}^\infty w_n I_{n,k} = \infty$. □

Now, for $\omega \in \Omega$, we define

$$f_\omega(z) = (1-z)\sum_0^\infty \varepsilon_n(\omega) w_n z^n = \sum_0^\infty b_n z^n. \tag{4.6}$$

We claim that, quasi-surely, f_ω has the three properties a), b), c) of Theorem 4.3. First of all (omitting ω), one has

$$b_0 = \varepsilon_0 w_0, \quad b_n = \varepsilon_n w_n - \varepsilon_{n-1} w_{n-1} \text{ for } n \ge 1,$$

so that $b_0 + \cdots + b_n = \varepsilon_n w_n \to 0$, and that the series converges at $z = 1$. Second, since $\sum_0^\infty w_n^2 = \infty$ and since (w_n) is non-increasing, Theorem 4.1 implies the following.

$$\text{Quasi-surely, } \sum_0^\infty b_n z^n \text{ diverges unboundedly for } |z| = 1 \text{ and } z \ne 1. \tag{4.7}$$

In fact, we have

$$\begin{aligned}\sum_0^N b_n z^n =& (1-z)\sum_0^{N-1} \varepsilon_n w_n z^n + \varepsilon_N w_N z^N \\ =& (1-z)\sum_0^{N-1} \varepsilon_n w_n z^n + o(1),\end{aligned}$$

and $\sum_0^\infty \varepsilon_n w_n z^n$ is quasi-surely unboundedly divergent. We also have

$$\text{Quasi-surely, the unit circle is a natural boundary for } \sum_0^\infty b_n z^n. \tag{4.8}$$

In fact, the result is well-known ([Ka]) for $\sum_0^\infty \varepsilon_n(\omega) w_n z^n$, which has the radius of convergence 1, and the multiplication by $(1-z)$ does not affect the result. Let us now fix $j \geq 1$. We have

$$\text{Quasi-surely, } f_\omega \text{ is unbounded on } \gamma_{\rho_j}. \tag{4.9}$$

In fact, if it were not the case, the topological zero-one law (see for example [Le]) would imply the existence of a constant C such that

$$|1-z| \sum_0^\infty w_n |z|^n \leq C \text{ for } z \in \gamma_{\rho_j};$$

i.e.,

$$|1-\gamma_{\rho_j}(\theta)| \sum_0^\infty w_n |\gamma_{\rho_j}(\theta)|^n \leq C \text{ for } 0 \leq \theta \leq 1.$$

Integrating with respect to θ and permuting, we would get

$$\sum_0^\infty w_n I_{n,j} \leq C,$$

contradicting Lemma 4.4. By the stability of quasi-sure properties, (4.7), (4.8) and (4.9) hold simultaneously for $\omega \in A$, where A is quasi-sure. But then, properties a), b), c) of Theorem 4.3 hold for f_ω, $\omega \in A$. Indeed, fix $0 < \rho < 1$ and $\omega \in A$. If f_ω were bounded, say by M, on the circle γ_ρ, by the maximum modulus principle, it would be bounded by M on γ_{ρ_j} for any j such that $\rho_j < \rho$, since γ_{ρ_j} is interior to γ_ρ; and this would contradict (4.9). This ends the proof of Theorem 4.3. $\square$

Comment. Among other things, Theorem 4.3 shows the optimality of Abel's non-tangential Theorem (see also part 5). There are explicit examples showing this optimality, but one needs some luck to find them. For example, let $a > 0$ and let

$$\sum_0^\infty b_n z^n = \exp\left(\frac{-az}{1-z}\right).$$

One then has

$$\sum_0^\infty (b_0 + \cdots + b_n) z^n = (1-z)^{-1} \exp\left(\frac{-az}{1-z}\right),$$

and the right hand-side is the generating series $\sum_0^\infty L_n(a)z^n$ of the Laguerre polynomials at a ([L]), so that $b_0+\cdots+b_n = L_n(a)$. But it follows for example from the integral representation ([L])

$$L_n(a) = \frac{e^a}{n!}\int_0^\infty t^n J_0(2\sqrt{at})e^{-t}dt$$

(where J_0 is the Bessel function of order zero) that $L_n(a) \to 0$ as $n \to \infty$. Therefore, $\sum_0^\infty b_n$ converges and $\sum_0^\infty b_n = 0$. Now, the fractional linear transformation $z \mapsto \dfrac{az}{1-z} = w$ takes the circle γ_ρ, tangent to $\mathbb{D}$ at 1, to the vertical line $\Re(w) = \dfrac{a(1-2\rho)}{2\rho}$. Therefore

$$\left|\sum_0^\infty b_n z^n\right| = e^{-\Re w} = e^{-\frac{a(1-2\rho)}{2\rho}}$$

for $z \in \gamma_\rho$ and $z \neq 1$, while $\sum_0^\infty b_n = 0$, which prevents the uniform convergence of $\sum_0^\infty b_n z^n$ on γ_ρ. On the other hand, it is easy to see that the previous series converges at z if $|z| = 1$ and $z \neq 1$ (it is the Fourier series of a C^∞ function in the neighborhood of z on the circle), and moreover the sum $\exp\left(\frac{-az}{1-z}\right)$ of the series can be analytically continued at all the points of the circle different from one; so we recover only part of the properties in Theorem 4.3.

5 Concluding Remarks and Questions

Let $\mathbb{D}$ be the open unit disk, and let $E \subset \mathbb{D}$ be such that $1 \in \overline{E}$ and $\overline{E} \subset \mathbb{D}\cup\{1\}$. We will say that E is tangent to $\mathbb{D}$ at 1 if we have

$$\limsup_{z\in E, z\to 1} \frac{|1-z|}{1-|z|} = \infty. \tag{5.1}$$

The classical non-tangential Theorem of Abel [Din] states that, if $\sum a_n$ converges and $E \subset \mathbb{D}$ is not tangent to $\mathbb{D}$ at 1 (we assume $\overline{E} \subset \mathbb{D} \cup \{1\}$), then $\sum a_n z^n$ is uniformly convergent for $z \in E$. In the special case when E is a circle γ_ρ, internally tangent to $\mathbb{D}$ at 1, we used in Theorem 4.3 the test-space of choices of signs to get a Taylor series $\sum a_n z^n$, convergent at 1, and not uniformly convergent on γ_ρ. In the general case, we will use the "larger" test-space c_0, and the Banach-Steinhaus Theorem (which is a version of Baire's Theorem and of quasi-sure properties), to revisit classical results of Hardy and Littlewood, showing at the same time that Abel's Theorem is definitely optimal.

Theorem 5.1. *Let $E \subset \mathbb{D}$ be any set which is tangent to $\mathbb{D}$ at 1. Then, there is a Taylor series $\sum_0^\infty b_n z^n = f(z)$ such that:*

1. *The series has radius of convergence 1, and converges at $z = 1$.*
2. *f is not bounded on E. In particular, the series does not converge uniformly on E.*

Proof. Let c_0 be the Banach space of sequences $a = (a_n)_{n\geq 0}$ tending to zero at infinity, with its natural norm $\|a\| = \sup_n |a_n|$. We will search our series $\sum_0^\infty b_n z^n$ in the form

$$\sum_0^\infty b_n z^n = (1-z)\sum_0^\infty a_n z^n = a_0 + \sum_1^\infty (a_n - a_{n-1})z^n,$$

with $a = (a_n) \in c_0$. Let L_z $(z \in E)$ be the linear form on c_0 defined by $L_z(a) = (1-z)\sum_0^\infty a_n z^n$. Since the dual space of c_0 is ℓ^1, we have

$$\|L_z\| = |1-z|\sum_0^\infty |z|^n = \frac{|1-z|}{1-|z|}. \tag{5.2}$$

It follows from (5.1) and (5.2) that $\sup_{z\in E}\|L_z\| = \infty$. The Banach-Steinhaus Theorem now implies the existence of $a \in c_0$ such that $\sup_{z\in E}|L_z(a)| = \infty$. If $b_0 = a_0$ and $b_n = a_n - a_{n-1}$ for $n \geq 1$, we have $b_0 + \cdots + b_n = a_n \to 0$, and the sequence (b_n) fulfills the requirements of Theorem 5.1. $\square$

Remarks.

1. Theorem 5.1 can be applied to produce Dirichlet series with similar properties. Indeed, if $F \subset \mathbb{C}_0$, we will say that F is tangent to $\mathbb{C}_0$ at 0 if $\overline{F} \subset \mathbb{C}_0 \cup \{1\}$ and $\limsup_{s\to 0, s\in F} \frac{s}{\Re(s)} = \infty$. Set $\phi(s) = 2^{-s}$, and $E = \phi(F)$. E is tangent to $\mathbb{D}$ at 1. Let $f(z)$ be the Taylor series given by Theorem 5.1, and set $g(s) = f(2^{-s})$. Then g has abscissa of convergence 0, converges at s=0, and is not bounded on F.
2. Theorem 5.1 can be compared with the following Theorem of Littlewood ([Lit]). For any curve $E \subset \mathbb{D}$, tangent to $\mathbb{D}$ at 1, there exists a bounded, holomorphic function f on $\mathbb{D}$, such that, for almost all θ, the limit of $f(ze^{i\theta})$ as $z \to 1$, $z \in E$, does not exist (while the non-tangential limit of $f(w)$ as $w \to e^{i\theta}$ exists for almost all θ).

3. Of course, if $\sum_0^\infty |b_n| < \infty$, the series $\sum_0^\infty b_n z^n$ has a better behavior; it converges uniformly on $\mathbb{D}$. Hardy and Littlewood [HL] proved that, even if $\sum_0^\infty b_n$ converges, no weaker condition than $\sum_0^\infty |b_n| < \infty$ can force the tangential uniform convergence at 1. More precisely, they proved the following.

Theorem 5.2. *Let $(w_n)_{n\geq 0}$ be a non-increasing sequence of positive numbers such that $w_n \to 0$ and $\sum_0^\infty w_n = \infty$. Then, there exists a Taylor series $\sum_0^\infty b_n z^n = f(z)$, with radius of convergence 1, and a curve $E \subset \mathbb{D}$, tangent to $\mathbb{D}$ at 1, such that:*

a) $\sum_0^\infty b_n$ *converges,*

b) $b_n = O(w_n)$*,*

c) f *is not bounded on* E *and in particular* $\sum_0^\infty b_n z^n$ *does not converge uniformly on* E*.*

Preliminary Comment: The initial proof of Hardy and Littlewood was complicated, and explicitly written only for $w_n = \dfrac{1}{n\log n}$.

PROOF OF THEOREM 5.2. We once more appeal to the Banach-Steinhaus Theorem. Let X be the space of sequences $(a_n)_{n\geq 0}$ such that $a_n = O(w_n)$, equipped with the norm

$$\|a\| = \sup_n \frac{|a_n|}{w_n}. \tag{5.3}$$

$(X, \|.\|)$ is a Banach space, isometric to l^∞. Let L_z $(z \in \mathbb{D})$ be the linear form on X defined by $L_z(a) = (1-z)\sum_0^\infty a_{n+1} z^n$. It is plain that

$$\|L_z\| = |1-z| \sum_0^\infty w_{n+1} |z|^n. \tag{5.4}$$

Let now $0 < \varepsilon_j < 1$, $\varepsilon_j \to 0$. Since $\sum_0^\infty w_n = \infty$, we can choose z_j on the circular arc $|z-1| = \varepsilon_j$, $|z| < 1$, such that $\sum_0^\infty w_{n+1}|z_j|^n \geq \frac{j}{\varepsilon_j}$, and (5.4) shows that $\|L_{z_j}\| \geq \varepsilon_j \dfrac{j}{\varepsilon_j} = j$. By the Banach-Steinhaus Theorem, one can find $a \in X$ such that $\sup_j |L_{z_j}(a)| = \infty$. Join the points z_j by a curve E; this curve will be tangent to $\mathbb{D}$ at 1. In fact, we have

$$j \leq \|L_{z_j}\| \leq |1-z_j| \sum_0^\infty w_1 |z_j|^n = \frac{|1-z_j|}{1-|z_j|} w_1.$$

And set $b_0 = a_1$, $b_n = a_{n+1} - a_n$ if $n \geq 1$, so that

$$|b_n| \leq |a_{n+1}| + |a_n| \leq \|a\|(w_{n+1} + w_n) \leq 2\|a\|w_n.$$

Since we have $\sum_0^\infty b_n z^n = L_z(a)$, and since $b_0 + \cdots + b_n = a_{n+1} = O(w_{n+1})$, the sequence (b_n) fulfills all the requirements of Theorem 5.2. □

We will end up with the following observation. Theorems 2.1 and 4.2 have shown that a change of sign can worsen the behavior of the series, which may become everywhere divergent, but a change of sign may sometimes improve the behavior, as indicated by the following.

Theorem 5.3. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers such that $|a_n|$ decreases to zero. Then:*

1. *There exists a sequence $(\theta_n)_{n\geq 1}$ of complex signs such that $\sum_1^\infty \theta_n a_n n^{it}$ converges for each $t \in \mathbb{R}$.*
2. *If moreover $\displaystyle\sum_1^\infty \frac{|a_n|}{n} < \infty$, one can choose the signs θ_n real ($\theta_n = \pm 1$) in 1.*

PROOF.

1. We will use the following.

Lemma 5.4. *For each $t \in \mathbb{R}$, there exists a constant $C_t > 0$ such that*

$$\left|\sum_{n=1}^{N} (-1)^n n^{it}\right| \leq C_t, \quad \textit{for } N = 1, 2, \ldots \tag{5.5}$$

PROOF OF THE LEMMA. The result is clear for $t = 0$; for $t \neq 0$, it is well-known ([Ha]) that $\sum_{1}^{\infty} n^{it-1}$ is boundedly divergent; therefore,

$$\begin{aligned}\sum_{1}^{2N}(-1)^n n^{it} &= \sum_{1}^{N}\left[(2n)^{it}-(2n-1)^{it}\right]\\ &= \sum_{1}^{N}(2n)^{it}\left(1-\left(1-\frac{1}{2n}\right)^{it}\right)\\ &= \sum_{1}^{N}(2n)^{it}\left(\frac{it}{2n}+O(n^{-2})\right)\\ &= it2^{it-1}\sum_{1}^{N} n^{it-1}+O(1)=O(1),\end{aligned}$$

which proves (5.5). Now, if we adjust θ_n, $|\theta_n|=1$, so as to have $\theta_n a_n = (-1)^n|a_n|$, (5.5) and an Abel's summation by parts show that $\sum \theta_n a_n n^{it}$ converges everywhere. □

2. We will use the following Lemma ([DC]).

Lemma 5.5. *There exists a numerical constant $C>0$ with the following property. If $z_1,\ldots,z_N$ are complex numbers of modulus less than one, there exist real signs $\varepsilon_1,\ldots,\varepsilon_N$ such that*

$$|\varepsilon_1 z_1+\cdots+\varepsilon_n z_n| \leq C \text{ for } 1\leq n\leq N. \tag{5.6}$$

(The result was generalized in [BG]). We can now end the proof of 2. as follows. Let $b_k=|a_{2^k}|$, $k=0,1,\ldots$ Since $|a_n|$ decreases, and since $\sum_1^\infty \frac{|a_n|}{n}<\infty$, we clearly have $\sum_0^\infty b_k<\infty$. Now, Lemma 5.5 allows us to build a sequence (θ_n) of real signs by blocks such that

$$\left|\sum_{2^k\leq n\leq N}\theta_n a_n\right| \leq Cb_k, \text{ for } k\geq 0 \text{ and } 2^k\leq N<2^{k+1}. \tag{5.7}$$

It follows that

$$\left|\sum_{2^k\leq n\leq N} n^{it}\theta_n a_n\right| \leq C'_t b_k, \text{ for } k\geq 0 \text{ and } 2^k\leq N<2^{k+1}. \tag{5.8}$$

In fact, fix k and set $S_n = \sum_{2^k \le j \le n} \theta_j a_j$ if $n \ge 2^k$, $S_{2^k-1} = 0$. We have, for $2^k \le N < 2^{k+1}$

$$\begin{aligned}\sum_{2^k \le n \le N} n^{it}\theta_n a_n &= \sum_{2^k \le n \le N} (S_n - S_{n-1}) n^{it} \\ &= \sum_{2^k \le n \le N-1} S_n \left(n^{it} - (n+1)^{it}\right) + S_N N^{it},\end{aligned}$$

so that

$$\left|\sum_{2^k \le n \le N} n^{it}\theta_n a_n\right| \le \sum_{2^k \le n \le N-1} \frac{Cb_k|t|}{n} + Cb_k \le (C|t| + C)b_k := C'_t b_k.$$

Now, since $\sum b_k < \infty$, the convergence of $\sum \theta_n a_n n^{it}$ clearly follows from (5.8). ☐

The inequality (5.5) is in marked contrast with the case of power series. According to a result of Lesigne and Petersen ([LP]), if a (necessarily bounded) sequence $(a_n)_{n\ge 0}$ of complex numbers satisfies

$$\left|\sum_0^N a_n e^{int}\right| \le C_t \qquad \forall t \in \mathbb{R},\ \forall N \in \mathbb{N}, \tag{5.9}$$

then one must have $\underline{\lim}|a_n| = 0$ and a little bit more, in terms of ergodic theory (see (5.14) below). We will reformulate their result, and give a slightly shorter proof. Recall that a subset E of the natural numbers $\mathbb{N}$ is said to be of uniform density zero if we have

$$\lim_{N\to\infty}\left(\sup_{t\ge 0} \frac{|E \cap [t, t+N]|}{N+1}\right) = 0.$$

Theorem 5.6. *Assume that a complex sequence $(a_n)_{n\ge 0}$ satisfies (5.9). Then there exists a subset E of $\mathbb{N}$, with uniform density zero, such that*

$$\lim_{\substack{n\to\infty \\ n\notin E}} a_n = 0. \tag{5.10}$$

Equivalently,

$$\lim_{N\to\infty}\left(\sup_{j\in\mathbb{N}} \frac{|a_j| + \cdots + |a_{j+N}|}{N+1}\right) = 0. \tag{5.11}$$

PROOF. At the beginning, we follow [LP]. Let K be a compact set of the plane where (a_n) takes its values, τ be the unilateral shift $(b_n) \mapsto (b_{n+1})$ on the cartesian product $K^{\mathbb{N}}$, X be the closed orbit of $a = (a_n)$ under τ, and μ be a τ-invariant measure on X, so that the composition operator $T : L^2(\mu) \to L^2(\mu)$ given by $T(g) = g \circ \tau$ is an isometry ([Wa]). One easily deduces from (5.9) that, for any $x = (x_n) \in X$

$$\left| \sum_0^n x_k e^{-ik\theta} \right| \leq 2C_{-\theta} = D_\theta, \quad \forall \theta,\ \forall n. \tag{5.12}$$

Now, denote by $f \in L^2(\mu)$ the function defined by $f(x) = x_0$, and by σ the spectral measure of f, which is the positive measure on the circle $\mathbb{T}$ defined by (via Bochner's Theorem, see [Kat]):

$$\hat{\sigma}(k) = < T^k f, f > \text{ if } k \geq 0,\ \hat{\sigma}(k) = < f, T^{-k} f > \text{ if } k < 0,$$

where $\hat{\sigma}(k) = \int_{\mathbb{T}} e^{ikt}\, d\sigma(t)$. (5.12) may be rewritten as $\left| \sum_{k=0}^n T^k f(x) e^{-ik\theta} \right| \leq D_\theta$, so that if we square and integrate with respect to $d\mu(x)$, we get

$$\int_{\mathbb{T}} \left| \sum_{k=0}^n e^{ikt} e^{-ik\theta} \right|^2 d\sigma(t) \leq D_\theta^2. \tag{5.13}$$

In fact, we have

$$\begin{aligned} \int_X \left| \sum_{k=0}^n T^k f(x) e^{-ik\theta} \right|^2 d\mu(x) &= \sum_{0 \leq k,l \leq n} e^{i(l-k)\theta} \int_X T^k f(x) \overline{T^l f(x)}\, d\mu(x) \\ &= \sum_{0 \leq k,l \leq n} e^{i(l-k)\theta} \int_{\mathbb{T}} e^{ikt} e^{-ilt}\, d\sigma(t) \\ &= \int_{\mathbb{T}} \left| \sum_{k=0}^n e^{ikt} e^{-ik\theta} \right|^2 d\sigma(t). \end{aligned}$$

We now derive from (5.13) the following property (which implies that $0 \in K$).

$$\mu = \delta_\omega, \text{ where } \omega = (0, 0, \ldots, 0, \ldots). \tag{5.14}$$

Indeed, suppose that this is not the case, and pick a point $u \neq \omega$ in the closed support of μ. We can find an index k, a neighborhood V of u, and a positive number ε such that, if $x = (x_n) \in V$, we have $|x_k| \geq \varepsilon$. This implies that

$$\int |x_k|^2\, d\mu(x) \geq \varepsilon^2 \mu(V) > 0.$$

Since μ is $\tau-$invariant, we have as well $\int |x_0|^2\, d\mu(x) > 0$. Therefore, σ is a non zero-measure, since

$$\|\sigma\| = \hat{\sigma}(0) =< f, f >= \int |x_0|^2 d\mu(x) > 0.$$

Let θ_0 be a density point of σ. Testing (5.13) at θ_0 and using the properties of the Dirichlet kernel, we get

$$\begin{aligned} D_{\theta_0}^2 \geq & \int_{|t-\theta_0|\leq \frac{\pi}{n}} \Big|\sum_{k=0}^{n} e^{ik(t-\theta_0)}\Big|^2 d\sigma(t) \\ \geq & \alpha n^2 \sigma\left(\theta_0 - \frac{\pi}{n}, \theta_0 + \frac{\pi}{n}\right) \geq \alpha\beta n^2 \times \frac{1}{n} = \alpha\beta n, \end{aligned}$$

where α, β are positive constants; but this is clearly impossible for big n, proving (5.14) by contradiction. Here, we slightly diverge from [LP] as follows. Since δ_ω is the unique invariant probability measure for τ, the well-known Theorem of Oxtoby ([Wa]) implies that, for any continuous function $g : X \to \mathbb{C}$, we have

$$\frac{g(x) + g(\tau x) + \cdots + g(\tau^n x)}{n+1} \to g(\omega) = \int g\, d\mu \text{ uniformly on } X. \tag{5.15}$$

Testing (5.15) on the function $g(x) = |x_0|$ and on the point $x = \tau^j a$, we get (5.11). And, since (a_n) is bounded, it is well-known ([Wa]) that (5.11) is equivalent to (5.10). $\square$

Remark. Conclusion (5.10) is not a weakening of the information (5.14), since a routine approximation argument shows that, conversely, (5.10) implies (5.15) and therefore (5.14). We find this formulation (5.10) more suggestive (see Question 2. below).

Question 1. Is the conclusion 2. in Theorem 5.3 still valid without the assumption $\sum_1^\infty n^{-1}|a_n| < \infty$?

Question 2. In Theorem 5.6, can one have the stronger conclusion $a_n \to 0$? If this were not the case, we would have an example of a trigonometric series which is *boundedly* divergent everywhere, making a nice complement to the Theorem of Marcinkiewicz quoted in section 3.

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