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Krzysztof Ciesielski, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA. email: `K_Cies@math.wvu.edu`,

Janusz Pawlikowski*, Department of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.
email: `pawlikow@math.uni.wroc.pl`

CONTINUOUS IMAGES OF BIG SETS AND ADDITIVITY OF s_0 UNDER $\mathrm{CPA}_{\mathrm{prism}}$

Abstract

We prove that the Covering Property Axiom $\mathrm{CPA}_{\mathrm{prism}}$, which holds in the iterated perfect set model, implies the following facts:

- There exists a family $\mathcal{G}$ of uniformly continuous functions from $\mathbb{R}$ to $[0,1]$ such that $|\mathcal{G}| = \omega_1$ and for every $S \in [\mathbb{R}]^{\mathfrak{c}}$ there exists a $g \in \mathcal{G}$ with $g[S] = [0,1]$.
- The additivity of the Marczewski's ideal s_0 is equal to $\omega_1 < \mathfrak{c}$.

1 Preliminaries and Axiom $\mathrm{CPA}_{\mathrm{prism}}$

Our set theoretic terminology is standard and follows that of [1]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use the term *Polish space* for a complete separable metric space `without isolated points`. For a Polish space X the symbol $\mathrm{Perf}(X)$ will stand for the collection of all subsets of X homeomorphic to the Cantor set $\mathfrak{C}$. For a fixed $0 < \alpha < \omega_1$ and $0 < \beta \le \alpha$ the symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$.

Axiom $\mathrm{CPA}_{\mathrm{prism}}$ was introduced by the authors in [3], where it is shown that it holds in the iterated perfect set model. Also, $\mathrm{CPA}_{\mathrm{prism}}$ is a simpler version of the axiom CPA which is described in a monograph [4]. (See also [2].) For the reader's convenience, we will restate the axiom in the next few paragraphs.

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The main notions needed for the axiom are that of *prism* and *prism-density*. Let $0<\alpha<\omega_1$ and let $\Phi_{\rm prism}(\alpha)$ be the family of all continuous injections $f\colon \mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ with the property that

$$f(x)\restriction\beta=f(y)\restriction\beta \ \Leftrightarrow\ x\restriction\beta=y\restriction\beta \text{ for all } \beta\in\alpha \text{ and } x,y\in\mathfrak{C}^\alpha$$

or, equivalently, such that for every $\beta<\alpha$

$$f\restriction\restriction\beta\stackrel{\rm def}{=}\{\langle x\restriction\beta,y\restriction\beta\rangle\colon\langle x,y\rangle\in f\}$$

is a one-to-one function from $\mathfrak{C}^\beta$ into $\mathfrak{C}^\beta$. Functions f from $\Phi_{\rm prism}(\alpha)$ were first introduced, in more general setting, in [7] where they are called *projection-keeping homeomorphisms*. Note that $\Phi_{\rm prism}(\alpha)$ is closed under compositions and that for every $0<\beta<\alpha$ if $f\in\Phi_{\rm prism}(\alpha)$, then $f\restriction\restriction\beta\in\Phi_{\rm prism}(\beta)$. Let

$$\mathbb{P}_\alpha=\{{\rm range}(f)\colon f\in\Phi_{\rm prism}(\alpha)\}$$

and note that if $f\in\Phi_{\rm prism}(\alpha)$ and $E\in\mathbb{P}_\alpha$, then $f[E]\in\mathbb{P}_\alpha$. We will also define $\mathbb{P}_{\omega_1}=\bigcup_{0<\alpha<\omega_1}\mathbb{P}_\alpha$. We will refer to elements of $\mathbb{P}_{\omega_1}$ as *iterated perfect sets*. (In [12] the elements of $\mathbb{P}_{\omega_1}$ are called I-perfect, where I is the ideal of countable sets.)

The simplest elements of $\mathbb{P}_\alpha$ are *perfect cubes*; that is, the sets of the form $C=\prod_{\beta<\alpha}C_\beta$, where $C_\beta\in{\rm Perf}(\mathfrak{C})$ for each $\beta<\alpha$. (This is justified by a function $f=\langle f_\beta\rangle_{\beta<\alpha}\in\Phi_{\rm prism}(\alpha)$, where each f_β is a homeomorphism from $\mathfrak{C}$ onto C_β.)

One of the most important properties of iterated perfect sets, distinguishing them from perfect cubes, is the following fact, which is a particular case of [7, thm. 20]. In its current form it has been used in [3]. It proof can be also found in [4, Lemma 3.2.2].

Lemma 1.1. *For every $0<\alpha<\omega_1$, $E\in\mathbb{P}_\alpha$, a Polish space X, and a continuous function $f\colon E\to X$ there exist $0<\beta\le\alpha$ and $P\in\mathbb{P}_\alpha$, $P\subset E$, such that $f\circ\pi_\beta^{-1}$ is a function on $\pi_\beta[P]\in\mathbb{P}_\beta$ which is either one-to-one or constant.*

To state ${\rm CPA}_{\rm prism}$ we need a few more definitions. For a fixed Polish space X let $\mathcal{F}_{\rm prism}(X)$ stand for the family of all continuous injections from an $E\in\mathbb{P}_{\omega_1}$ onto perfect subsets of X. Each such injection f is called a *prism* and is considered as a coordinate system imposed on $P={\rm range}(f)$.[1] We will usually abuse this terminology and refer to P itself as a *prism* (in X) and

[1]In a language of forcing a coordinate function f is simply a nice name for an element from X.

to f as a *witness function* for P. A function $g \in \mathcal{F}_{\rm prism}(X)$ is *subprism* of f provided $g \subset f$. In the above spirit we call $Q = \mathrm{range}(g)$ a *subprism of a prism* P. Thus, when we say that Q *a subprism of a prism* $P \in \mathrm{Perf}(X)$ we mean that $Q = f[E]$, where f is a witness function for P and $E \subset \mathrm{dom}(f)$ is an iterated perfect set. A family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *prism-dense in* X provided every prism in X contains a subprism $Q \in \mathcal{E}$. It is easy to see (using the fact that $\Phi_{\rm prism}(\alpha)$ is closed under the composition) that we can assume that a witness function of a prism is always defined on the entire space $\mathfrak{C}^\alpha$ for an appropriate α.

Now we are ready to state the axiom.

CPA$_{\rm prism}$**:** $\mathfrak{c} = \omega_2$ and for every Polish space X and every prism-dense family $\mathcal{E} \subset \mathrm{Perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \le \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$.

If in the definition above we restrict our attention only to prisms whose domains are perfect cubes in $\mathfrak{C}^\omega$, we get a notion of cube-density which is stronger than that of prism-density. This naturally leads to a weaker version of CPA$_{\rm prism}$, known as CPA$_{\rm cube}$, obtained from CPA$_{\rm prism}$ by replacing the word “prism” with “cube.” Thus, any consequence of axiom CPA$_{\rm cube}$, which has been studied in [5, 2, 10, 4], follows also from CPA$_{\rm prism}$.

Next, let us consider the following ideals on $\mathfrak{C}$:

$$s_0^{\rm prism} = \left\{\mathfrak{C} \setminus \bigcup \mathcal{E}\colon\ \mathcal{E} \text{ is prism-dense in } \mathrm{Perf}(\mathfrak{C})\right\}$$

and

$$s_0^{\rm cube} = \left\{\mathfrak{C} \setminus \bigcup \mathcal{E}\colon\ \mathcal{E} \text{ is cube-dense in } \mathrm{Perf}(\mathfrak{C})\right\}.$$

Clearly they are the variants of the Marczewski ideal s_0 of subsets of $\mathfrak{C}$; that is, the family of all sets $S \subset \mathfrak{C}$ such that for every $P \in \mathrm{Perf}(\mathfrak{C})$ there exists a $Q \in \mathrm{Perf}(P)$ disjoint from S. It is not difficult to see that

$$[X]^{<\mathfrak{c}} \subset s_0^{\rm cube} \subset s_0^{\rm prism} \subset s_0.$$

(The proof that $[X]^{<\mathfrak{c}} \subset s_0^{\rm cube} \subset s_0$ can be found in [5, Fact 1.3] or [4]. The inclusion $s_0^{\rm cube} \subset s_0^{\rm prism}$ follows immediately from the fact that any cube-dense family is also prism-dense. The proof that $s_0^{\rm prism} \subset s_0$ is identical to that of $s_0^{\rm cube} \subset s_0$.)

Obviously CPA$_{\rm prism}$, used with $X = \mathfrak{C}$, implies that $s_0^{\rm prism} \subset [\mathfrak{C}]^{\le\omega_1}$. So, we get the following consequence.

Proposition 1.2. *If* CPA$_{\rm prism}$ *holds, then* $s_0^{\rm prism} = [\mathfrak{C}]^{\le\omega_1}$.

This distinguishes the ideal $s_0^{\rm prism}$ from s_0, since there exist ZFC examples of s_0-sets of cardinality $\mathfrak{c}$. (See e.g. [9, thm. 5.10].) The cube analog of Proposition 1.2 was proved in [5].

2 Continuous Images of Sets of Cardinality Continuum

In [8] A. Miller proved the following in the iterated perfect set model.

(A) for every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a (uniformly) continuous function $f\colon \mathbb{R} \to [0,1]$ such that $f[S] = [0,1]$.

This result was refined by the authors in [5] by showing that (A) follows already from $\mathrm{CPA}_{\mathrm{cube}}$. The main goal of this section is to show that $\mathrm{CPA}_{\mathrm{prism}}$ implies the following stronger version of (A).

Theorem 2.1. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that*

(A^*) *there exists a family* $\mathcal{G}$ *of uniformly continuous functions from* $\mathbb{R}$ *to* $[0,1]$ *such that* $|\mathcal{G}| = \omega_1$ *and for every* $S \in [\mathbb{R}]^{\mathfrak{c}}$ *there exists a* $g \in \mathcal{G}$ *with* $g[S] = [0,1]$.

This also constitutes a version of a remark due to Miller [8, p. 581], who noticed that in the Sacks model functions coded in the ground model can be taken as a family $\mathcal{G}$.

To prove the theorem we need some auxiliary results. For a fixed Polish space X and $0 < \alpha < \omega_1$ let $\mathcal{F}^\alpha$ denote the family of all continuous injections from $\mathfrak{C}^\alpha$ into X. Note that if we consider $\mathcal{F}^\alpha$ with the topology of uniform convergence, then

$$\mathcal{F}^\alpha \text{ is a Polish space.} \tag{1}$$

To prove (1) it is enough to show that $\mathcal{F}^\alpha$ is a G_δ subset of the space $\mathcal{C} = \mathcal{C}(\mathfrak{C}^\alpha, X)$ of all continuous functions from $\mathfrak{C}^\alpha$ into X. But $\mathcal{F}^\alpha$ is the intersection of the open sets G_n, $n < \omega$, where the sets G_n are constructed as follows. Fix a finite partition $\mathcal{P}_n$ of $\mathfrak{C}^\alpha$ into clopen sets each of the diameter less than 2^{-n}, and let $\mathcal{H}_n$ be the family of all mappings h from $\mathcal{P}_n$ into the topology of X such that $h(P) \cap h(P') = \emptyset$ for distinct $P, P' \in \mathcal{P}_n$. We put

$$G_n = \bigcup_{h \in \mathcal{H}_n} \{f \in \mathcal{C}\colon (\forall P \in \mathcal{P}_n)(\forall x \in P)\ f(x) \in h(P)\}.$$

This completes the argument for (1).

Lemma 2.2. *Let* X *be a Polish space and* $0 < \alpha < \omega_1$. *Then every map* $f\colon \mathfrak{C}^\beta \to \mathcal{F}^\alpha$ *from* $\mathcal{F}_{\mathrm{prism}}(\mathcal{F}^\alpha)$ *has a restriction* $f^* \in \mathcal{F}_{\mathrm{prism}}(\mathcal{F}^\alpha)$ *for which there exists an* $\hat{f} \in \mathcal{F}_{\mathrm{prism}}(X)$ *defined on a subset of* $\mathfrak{C}^{\beta+\alpha}$ *such that:*

(a) $\hat{f}(s,t) = f^*(s)(t)$ *for all* $\langle s,t\rangle \in \left(\mathfrak{C}^\beta \times \mathfrak{C}^\alpha\right) \cap \mathrm{dom}(\hat{f})$, *and*

(b) for each $s \in \mathrm{dom}(f^*)$ the function $\hat{f}(s,\cdot)\colon \{t \in \mathfrak{C}^\alpha\colon \langle s,t\rangle \in \mathrm{dom}(\hat{f})\} \to X$ is a restriction of $f^*(s)$ and belongs to $\mathcal{F}_{\mathrm{prism}}(X)$.

PROOF. Let $f\colon \mathfrak{C}^\beta \to \mathcal{F}^\alpha$, $f \in \mathcal{F}_{\mathrm{prism}}(\mathcal{F}^\alpha)$, and define a function g from a set $\mathfrak{C}^\beta \times \mathfrak{C}^\alpha = \mathfrak{C}^{\beta+\alpha}$ into X by $g(s,t) = f(s)(t)$ for $\langle s,t\rangle \in \mathfrak{C}^\beta \times \mathfrak{C}^\alpha$. It is easy to see that g is continuous. Apply Lemma 1.1 to $E = \mathfrak{C}^{\beta+\alpha} \in \mathbb{P}_{\beta+\alpha}$ and to the function g to find a $\gamma \le \beta+\alpha$ and a subset $P \in \mathbb{P}_{\beta+\alpha}$ of E such that $g \circ \pi_\gamma^{-1}$ is a function on $\pi_\gamma[P] \in \mathbb{P}_\gamma$ which is either one-to-one or constant. Let $f^* = f \restriction \pi_\beta[P]$. We will show that it is as desired.

First note that $\gamma = \beta+\alpha$ and g is one-to-one on P. Indeed, if $z \in \mathrm{range}(f^*) \cap \mathcal{F}_{\mathrm{prism}}(X)$ and $z = f^*(s)$, then for every different $t_0, t_1 \in \mathfrak{C}^\alpha$ with $\langle s,t_0\rangle, \langle s,t_1\rangle \in P$ we have $g(s,t_0) = f(s)(t_0) = z(t_0) \neq z(t_1) = g(s,t_1)$. So, g cannot be constant and if $\gamma < \beta+\alpha$, then we can find t_0 and t_1 such that $\pi_\gamma(\langle s,t_0\rangle) = \pi_\gamma(\langle s,t_1\rangle)$ contradicting the above calculation.

It is easy to see that $\hat{f} = g \restriction P$ is as desired. □

Lemma 2.2 implies the following useful fact.

Proposition 2.3. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that for every Polish space X there exists a family $\mathcal{H}$ of continuous functions from compact subsets of X onto $\mathfrak{C}\times\mathfrak{C}$ such that $|\mathcal{H}| \le \omega_1$ and*

- *for every prism P in X there are $h \in \mathcal{H}$ and $c \in \mathfrak{C}$ such that $h^{-1}(\{c\}\times\mathfrak{C})$ and $h^{-1}(\langle c,d\rangle)$ are subprisms of P for every $d \in \mathfrak{C}$.*

In particular, $\mathcal{F} = \{h^{-1}(\{c\}\times\mathfrak{C})\colon h \in \mathcal{H}\ \&\ c \in \mathfrak{C}\}$ is prism-dense in X.

PROOF. Let $0 < \alpha < \omega_1$. We use the notation as in Lemma 2.2. Since the family of all sets $\mathrm{range}(f^*)$ is prism-dense in $\mathcal{F}^\alpha$, by $\mathrm{CPA}_{\mathrm{prism}}$ we can find a family $\mathcal{G}_\alpha = \{f^*_\xi\colon \xi < \omega_1\}$ such that $R_\alpha = \mathcal{F}^\alpha \setminus \bigcup_{\xi<\omega_1} \mathrm{range}(f^*_\xi)$ has cardinality less than or equal to ω_1. If $f^* \in \mathcal{G}_\alpha$, then $\hat{f}$ maps injectively a $P = P_f \in \mathbb{P}_{\beta+\alpha}$ onto $Q = Q_f \subset X$. Moreover, for every $z \in \mathcal{F}^\alpha \setminus R_\alpha$ there are $f^* \in \mathcal{G}_\alpha$ and $s \in \mathrm{dom}(f^*)$ such that $z = f^*(s)$ and $\hat{f}(s,\cdot) \in \mathcal{F}_{\mathrm{prism}}(X)$ is a restriction of z.

Now, let $H_f \in \Phi_{\mathrm{prism}}(\beta+\alpha)$ be from $\mathfrak{C}^{\beta+\alpha}$ onto P and consider the composition $\hat{f} \circ H_f\colon \mathfrak{C}^{\beta+\alpha} \to Q$. Then functions $(\hat{f} \circ H_f)^{-1}\colon Q_f \to \mathfrak{C}^{\beta+\alpha}$ are our desired functions modulo some projections. More precisely, let $k_0\colon \mathfrak{C}^\beta \to \mathfrak{C}$ be a homeomorphism and let $k_1\colon \mathfrak{C} \to \mathfrak{C}$ be such that $k_1^{-1}(c) \in \mathrm{Perf}(\mathfrak{C})$ for every $c \in \mathfrak{C}$. Define $h^\alpha_f\colon Q_f \to \mathfrak{C}\times\mathfrak{C}$ by

$$h^\alpha_f(x) = \langle (k_0 \circ \pi_\beta)((\hat{f} \circ H_f)^{-1}(x)), k_1([(\hat{f} \circ H_f)^{-1}(x)](\beta))\rangle.$$

Then family $\mathcal{H}_0 = \{h^\alpha_f\colon \alpha < \omega_1\ \&\ f^* \in \mathcal{G}_\alpha\}$ works for all functions not in $R = \bigcup_{0<\alpha<\omega_1} R_\alpha$. Also, for every function $g \in R$ it is easy to find a continuous function h_g from range(g) onto $\mathfrak{C}\times\mathfrak{C}$ such that $h_g^{-1}(\{c\}\times\mathfrak{C})$ and $h_g^{-1}(\langle c,d\rangle)$ are subprisms of range(g) for every $c,d\in\mathfrak{C}$. Then $\mathcal{H}=\mathcal{H}_0\cup\{h_g\colon g\in R\}$ is as desired. □

PROOF OF THEOREM 2.1. Let $\mathcal{H}$ be as in Proposition 2.3 used with $X=\mathbb{R}$, let $k\colon\mathfrak{C}\to[0,1]$ be continuous surjection, and for every $h=\langle h_0,h_1\rangle\in\mathcal{H}$ let $g_h\colon\mathbb{R}\to[0,1]$ be a continuous extension of a function $h^*\colon \operatorname{dom}(h)\to[0,1]$ defined by $h^*(x)=k(h_1(x))$. We claim that $\mathcal{G}=\{g_h\colon h\in\mathcal{H}\}$ is as desired.

To see it, let $S\in[\mathbb{R}]^{\mathfrak{c}}$and let $\mathcal{E}=\{P\in\operatorname{Perf}(\mathbb{R})\colon P\cap S=\emptyset\}$. Since $\mathbb{R}\setminus\bigcup\mathcal{E}$ contains S, it has cardinality $\mathfrak{c}$. So, from $\mathrm{CPA}_{\mathrm{prism}}$ we conclude that $\mathcal{E}$ is not prism-dense. (Compare with Proposition 1.2.) Thus, there exists a prism P in $\mathbb{R}$ such that S intersects every subprism of P. Let $h\in\mathcal{H}$ and $c\in\mathfrak{C}$ be such that $h^{-1}(\{c\}\times\mathfrak{C})$ and $h^{-1}(\langle c,d\rangle)$ are subprisms of P for every $d\in\mathfrak{C}$. Then S intersects each $h^{-1}(\langle c,d\rangle)$; so $h[S]$ contains $\{c\}\times\mathfrak{C}$. Thus $g_h[S]=[0,1]$. □

3 $\mathrm{CPA}_{\mathrm{prism}}$ Implies That $\operatorname{add}(s_0)=\omega_1$

Recall that the additivity number is defined as

$$\operatorname{add}(s_0)=\min\left\{|F|\colon F\subset s_0\ \&\ \bigcup F\notin s_0\right\}.$$

Numbers $\operatorname{add}(s_0)$, $\operatorname{cov}(s_0)$, $\operatorname{non}(s_0)$, and $\operatorname{cof}(s_0)$ has been intensively studied. (See e.g. [6].) It is known that $\operatorname{cof}(s_0)>\mathfrak{c}$ (see e.g. [6, thm. 1.3]) and that $\operatorname{non}(s_0)=\mathfrak{c}$ since there are s_0-sets of cardinality $\mathfrak{c}$. There are models of ZFC+MA with $\mathfrak{c}=\omega_2$ and $\operatorname{cov}(s_0)=\omega_1$, while the Proper Forcing Axiom implies that $\operatorname{add}(s_0)=\mathfrak{c}$. Here we prove that $\mathrm{CPA}_{\mathrm{prism}}$ implies $\operatorname{add}(s_0)=\omega_1$. Note also that a stronger form of CPA implies that $\operatorname{cov}(s_0)=\omega_2$. (See [4, prop. 6.1.1].)

In the proof we will use the following fact in which the assumption that $\mathcal{D}$ is an open subset of $\operatorname{Perf}(\mathfrak{C})$ means that $\operatorname{Perf}(P)\subset\mathcal{D}$ for every $P\in\mathcal{D}$.

Fact 3.1. *For any open dense subset $\mathcal{D}$ of $\operatorname{Perf}(\mathfrak{C})$ (considered as ordered by inclusion) there exists a maximal antichain $\mathcal{A}\subset\mathcal{D}$ consisting of pairwise disjoint sets such that every $P\in\operatorname{Perf}(\bigcup\mathcal{A})$ is covered by less than continuum many sets from $\mathcal{A}$.*

PROOF. Let $\operatorname{Perf}(\mathfrak{C})=\{P_\alpha\colon\alpha<\mathfrak{c}\}$. We will build inductively a sequence $\langle\langle A_\alpha,x_\alpha\rangle\in\mathcal{D}\times\mathfrak{C}\colon\alpha<\mathfrak{c}\rangle$ aiming for $\mathcal{A}=\{A_\alpha\colon\alpha<\mathfrak{c}\}$. At step $\alpha<\mathfrak{c}$, given already $\langle\langle A_\beta,x_\beta\rangle\colon\beta<\alpha\rangle$ we look at P_α.

Choice of x_α: If $P_\alpha\subset\bigcup_{\beta<\alpha}A_\beta$, we take x_α as an arbitrary element of $\mathfrak{C}$; otherwise we pick $x_\alpha\in P_\alpha\setminus\bigcup_{\beta<\alpha}A_\beta$.

Choice of A_α: If there is a $\beta < \alpha$ such that $P_\alpha \cap A_\beta$ is uncountable, we let $A_\alpha = A_\beta$; otherwise pick $A_\alpha \in \mathcal{D}$ below P_α and notice that we can refine it, if necessary, to be disjoint from $\bigcup_{\beta<\alpha} A_\beta \cup \{x_\beta\colon \beta \le \alpha\}$. It is easy to see that $\mathcal{A} = \{A_\alpha\colon \alpha < \mathfrak{c}\}$ is as required. □

Theorem 3.2. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that* $\mathrm{add}(s_0) = \omega_1$.

PROOF. Let $\mathcal{H} = \{h_\xi\colon \xi < \omega_1\}$ be as in Proposition 2.3 with $X = \mathfrak{C}$. For every $\xi < \omega_1$ put $\mathcal{A}^0_\xi = \{h_\xi^{-1}(\{c\} \times \mathfrak{C})\colon c \in \mathfrak{C}\}$. Then each $\mathcal{A}^0_\xi$ is a family of pairwise disjoint sets and $\mathcal{A}^0 = \bigcup_{\xi<\omega_1} \mathcal{A}^0_\xi$ is dense in $\mathrm{Perf}(\mathfrak{C})$.

For each $\xi < \omega_1$ let A^*_ξ be a maximal antichain extending $\mathcal{A}^0_\xi$, define $\mathcal{D}_\xi = \{P \in \mathrm{Perf}(\mathfrak{C})\colon P \subset A$ for some $A \in \mathcal{A}^*_\xi\}$, and let $\mathcal{A}_\xi \subset \mathcal{D}_\xi$ be as in Fact 3.1. Then $\mathcal{A} = \bigcup_{\xi<\omega_1} \mathcal{A}_\xi$ is still dense in $\mathrm{Perf}(\mathfrak{C})$.

For each $\xi < \omega_1$ let $\{P^\alpha_\xi\colon \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{A}_\xi$. (Note that each $\mathcal{A}_\xi$ has cardinality $\mathfrak{c}$, since this was the case for sets $\mathcal{A}^0_\xi$.) Pick x^α_ξ from each P^α_ξ and put $A_\xi = \{x^\alpha_\xi\colon \alpha < \mathfrak{c}\}$. Then $A_\xi \in s_0$ for every $\xi < \omega_1$. However, $A = \bigcup_{\xi<\omega_1} A_\xi \notin s_0$ since it intersects every element of a dense set $\mathcal{A}$. □

It can be also shown that $\mathrm{CPA}_{\mathrm{prism}}$, with a help of Proposition 2.3, implies that the Sacks forcing $\mathbb{P} = \langle \mathrm{Perf}(\mathfrak{C}), \subset\rangle$ collapses $\mathfrak{c}$ to ω_1. However, this also follows immediately from a theorem of P. Simon [11] that $\mathbb{P}$ collapses $\mathfrak{c}$ to $\mathfrak{b}$ while already $\mathrm{CPA}_{\mathrm{cube}}$ implies that $\mathfrak{b} \le \mathrm{cof}(\mathcal{N}) = \omega_1$.

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