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ON PARAMETRIC LIMIT SUPERIOR OF A SEQUENCE OF ANALYTIC SETS

Abstract

Let A_x stand for x-section of a set $A \subset 2^\omega \times 2^\omega$. We prove that any sequence $A^j \subset 2^\omega \times 2^\omega$, $j \in \omega$, of analytic sets, with uncountable $\limsup_{j\in H} A^j_x$ for all $x \in 2^\omega$ and $H \in [\omega]^\omega$, admits a perfect set $P \subset 2^\omega$ and $H \in [\omega]^\omega$ with uncountable $\bigcap_{j\in H} A^j_x$ for all $x \in P$. This is a parametric version of the Komjáth theorem [2].

1 Main Result.

In [2] Komjáth proved that if the sets $A^0, A^1, \ldots$ are analytic sets in a Polish space, and $\limsup_{j\in H} A^j$ is uncountable for each $H \in [\omega]^\omega$, then there exists a set $G \in [\omega]^\omega$ for which the intersection $\bigcap_{j\in G} A^j$ is uncountable. The previous version of this statement was proved by Laczkovich in [3] for a sequence of Borel sets. Komjáth, assuming $MA(\omega_1)$, proved that this statement holds if the analicity of sets A^j is skipped, but assuming the axiom of constructibility, he proved that it is false for a sequence of coanalytic sets.

In this paper we prove a parametric version of the Komjáth result. The parametrized Ellentuck theorem due to Pawlikowski [4] is our basic tool in the proof. We discuss examples which show that some stronger versions of our theorem are impossible.

We use standard set theoretical notation (see [1]). A subset P of a Polish space is called *perfect* if it is nonempty, closed, and dense in itself. For $\alpha \in [\omega]^{<\omega}$ and $H \in [\omega]^\omega$, let $[\alpha, H]$ be an *Ellentuck neighbourhood*; i.e., a set of the form $\{G \in [\omega]^\omega : \alpha \subset G \subset \alpha \cup (H \setminus \max(\alpha))\}$. A set $A \subset 2^\omega \times [\omega]^\omega$ is called *perfectly Ramsey* if for any perfect set $P \subset 2^\omega$ and any Ellentuck neighbourhood $[\alpha, H]$, there exists a perfect set $Q \subset P$ and an infinite set $G \subset H$ such that either $Q \times [\alpha, G] \subset A$ or $(Q \times [\alpha, G]) \cap A = \emptyset$. For $A \subset 2^\omega \times 2^\omega$ and $x \in 2^\omega$, put $A_x = \{y \in 2^\omega : (x, y) \in A\}$; this is called x–section of A.

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Theorem 1. *Let $(A^j)_{j\in\omega}$ be a sequence of analytic subsets of $2^\omega \times 2^\omega$ such that*

$$\forall x \in 2^\omega \ \forall H \in [\omega]^\omega \ \ card(\limsup_{j\in H} A^j_x) > \omega.$$

Then there exist a perfect set $P \subset 2^\omega$ and $H \in [\omega]^\omega$ such that

$$\forall x \in P \ \ card(\bigcap_{j\in H} A^j_x) > \omega.$$

Proof. We treat $[\omega]^\omega$ as a Polish subspace of 2^ω, identifying $H \in [\omega]^\omega$ with its characteristic function. Define

$$A = \{(x,H) \in 2^\omega \times [\omega]^\omega : card(\bigcap_{j\in H} A^j_x) > \omega\}.$$

Consider

$$B = \{(x,H,y) \in 2^\omega \times [\omega]^\omega \times 2^\omega : (x,y) \in \bigcap_{j\in H} A^j\} =$$

$$\{(x,H,y) \in 2^\omega \times [\omega]^\omega \times 2^\omega : \forall j \in \omega \ \ (j \notin H \text{ or } (x,y) \in A^j)\}$$

and note that B is analytic. Thus

$$A = \{(x,H) \in 2^\omega \times [\omega]^\omega : card(B_{(x,H)}) > \omega\}$$

is analytic, by the Mazurkiewicz–Sierpiński theorem [1, 29.19]. Now by [4], the set A is perfectly Ramsey. Hence, there exist a perfect set $P \subset 2^\omega$ and $H \in [\omega]^\omega$ such that either $P \times [\emptyset, H] \subset A$ or $(P \times [\emptyset, H]) \cap A = \emptyset$. This last case is impossible since for each $x \in P$, there is $G \in [H]^\omega$ such that $card(\bigcap_{j\in G} A^j_x) > \omega$ (see [2, Theorem 1]). Finally we obtain that

$$\forall x \in P \ \ card(\bigcap_{j\in H} A^j_x) > \omega.$$

$\square$

2 Examples.

Note that it is impossible to improve Theorem 1 (in ZFC) assuming that sets A^j are coanalytic (see [2, Theorem 4]). The following examples show that we also can not improve it assuming only that all sections of A^j are analytic (even clopen).

Example 1. We will construct a sequence $(A^j)_{j\in\omega}$ of subsets of $2^\omega \times 2^\omega$ such that

- $\forall H \in [\omega]^\omega \ \forall x \in 2^\omega \ \ card(\limsup_{j\in H} A^j_x) > \omega$,
- A^j_x is clopen for all $x \in 2^\omega$,

and there is no perfect set $P \subset 2^\omega$, and no $H \in [\omega]^\omega$ such that

$$\forall x \in P \ \ card(\bigcap_{j\in H} A^j_x) > \omega.$$

Let $\{N_i : i \in \omega\}$ be a family of almost disjoint infinite subsets of ω such that for every $s \in [\omega]^{<\omega}$, there exists $i, j \in \omega$ with $N_i \cap N_j = s$. Let $\{B_i : i \in \omega\}$ be a partition of 2^ω into pairwise disjoint Bernstein subsets. Fix two disjoint clopen sets $C^0, C^1 \subset 2^\omega$. Put

$$A^j = \bigcup_{i\in\omega} B_i \times C^{\chi_{N_i}(j)},$$

where χ_{N_i} is the characteristic function of N_i. Immediately from the definition of A^j we obtain that for every $x \in 2^\omega$, A^j_x is a clopen set, and for any $x \in 2^\omega$ and $H \in [\omega]^\omega$, there exists $G \in [H]^\omega$ such that $card(\bigcap_{j\in G} A^j_x) > \omega$.

Suppose that there exists a perfect set P and $H \in [\omega]^\omega$ such that

$$\forall x \in P \ \ card(\bigcap_{j\in H} A^j_x) > \omega.$$

Then P intersects every set B_i, $i \in \omega$. For each i, since P intersects B_i, it follows that either $H \subset N_i$ or $H \subset \omega \setminus N_i$. Since $N_i \cap N_j$ is finite for $i \neq j$, there exists $i_0 \in \omega$ such that $H \subset \omega \setminus N_i$ for all $i \neq i_0$. Let $s \subset H$ be finite and nonempty. There exists $i, j \in \omega$ such that $N_i \cap N_j = s$. Then $H \cap N_i \neq \emptyset$, and $H \cap N_j \neq \emptyset$. This implies that $i = j = i_0$, which is a contradiction.

Remark. If the axiom of constructibility holds that there is a partition $\{B_i : i \in \omega\}$ of 2^ω into pairwise disjoint Bernstein sets, such that $B_i \in \Delta^1_2(2^\omega)$ for all $i \in \omega$. Hence, we can construct a sequence $(A^j)_{j\in\omega}$ of Δ^1_2 sets in 2^ω with the same properties as in Example 1.

Now we will show that under CH there is a more pathological example.

Lemma 2. *Assume CH and list all sets in* $[\omega]^\omega$ *as* H_α, $\alpha < \omega_1$. *Then there are sets* $G_\alpha \in [\bigcup_{\beta<\alpha} H_\beta]^\omega$, $\alpha < \omega_1$, *such that for each* $\alpha < \omega_1$ *we have*

$$\forall \beta < \alpha \ \ (G_\alpha \cap H_\beta \neq \emptyset \text{ and } H_\beta \setminus G_\alpha \neq \emptyset).$$

PROOF. Let $G_0 \in [\omega]^\omega$ be such that $G_0 \subset H_0$ and $H_0 \setminus G_0 \neq \emptyset$. For $\alpha < \omega_1$, let $(F_n)_{n\in\omega}$ be an enumeration of $\{H_\beta : \beta < \alpha\}$. For $n \in \omega$, let $F_n = \{a_n^0, a_n^1, a_n^2, ...\}$ and fix $m_n \in F_n \setminus \{a_i^j : i, j < 2n\}$. Put $G_\alpha = \{m_n : n \in \omega\}$ and notice that

$$\forall n \in \omega \ \ (G_\alpha \cap F_n \neq \emptyset \text{ and } F_n \setminus G_\alpha \neq \emptyset).$$

$\square$

Example 2. Assume CH. Let G_α, $\alpha < \omega_1$, be sets from Lemma 2. Let $\{r_\alpha : \alpha < \omega_1\}$ be an enumeration of 2^ω. Fix two disjoint clopen sets $C^0, C^1 \subset 2^\omega$. Put

$$A^j = \bigcup_{\alpha<\omega_1} \{r_\alpha\} \times C^{\chi_{G_\alpha}(j)}.$$

Suppose that there is an uncountable set $E \subset \omega_1$ and $H \in [\omega]^\omega$ such that for all $\alpha \in E$, we have

$$card(\bigcap_{j\in H} A^j_{r_\alpha}) = \omega_1.$$

Since $A_r^j = C^0$ or $A_r^j = C^1$, we obtain

$$[\forall \alpha \in E \, \forall j \in H \ \ (A^j_{r_\alpha} = C^0)] \text{ or } [\forall \alpha \in E \, \forall j \in H \ \ (A^j_{r_\alpha} = C^1)].$$

There exists $\alpha_0 < \omega_1$ for which $H = H_{\alpha_0}$. Let $\alpha \in E$ be such that $\alpha_0 < \alpha$. Since $G_\alpha \cap H \neq \emptyset$ and $H \setminus G_\alpha \neq \emptyset$, pick $j_1 \in G_\alpha \cap H$ and $j_0 \in H \setminus G_\alpha$. See that $A^{j_1}_{r_\alpha} = C^1$ and $A^{j_0}_{r_\alpha} = C^0$, which yields a contradiction. Hence for every uncountable $P \subset 2^\omega$ and $H \in [\omega]^\omega$, there is $r \in P$ with $\bigcap_{j\in H} A_r^j = \emptyset$.

The Referee claims that "it might be interesting to have a direct forcing proof of the Theorem 1. For example, is it true that if $\mathbb{S}$ denotes Sacks forcing and $\mathbb{M}$ denotes Mathias forcing, and if (x, H) is generic for $\mathbb{S} \times \mathbb{M}$, then $\bigcap_{j\in H} A_x^j$ is uncountable?"

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