Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 193–200

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ON INTEGRALS WITH INTEGRATORS IN BV_p

Abstract

In 1936, L. C. Young proved that the Riemann-Stieltjes integral $\int_a^b f\,dg$ exists, if $f \in BV_p$, $g \in BV_q$, $\frac{1}{p} + \frac{1}{q} > 1$ and f, g do not have common discontinuous points. In this note, using Henstock's approach, we prove that $\int_a^b f\,dg$ still exists without assuming the condition on discontinuous points. Some convergence theorems are also proved.

1 Henstock-Stieltjes Integrals.

In this section, we shall introduce Henstock-Stieltjes integrals and state the Cauchy Criterion, see [3]. In this note, $\mathbb{R}$ denotes the real line.

Definition 1.1. A finite collection $\{I_i\}_{i=1}^n$ of nonoverlapping closed subintervals of $[a,b]$ is said to be a partition of $[a,b]$ if $\cup_{i=1}^n I_i = [a,b]$. Let δ be a positive function on $[a,b]$ and I be a closed subinterval of $[a,b]$. An interval-point pair (I,ξ) is said to be δ-fine if $\xi \in I \subset (\xi - \delta(\xi), \xi + \delta(\xi))$. A finite collection of interval-point pairs, $D = \{(I_i,\xi_i)\}_{i=1}^n$, is called a *$\delta$-fine division* of $[a,b]$ if

(i) each (I_i,ξ_i) is δ-fine and

(ii) $\{I_i\}_{i=1}^n$ is a partition of $[a,b]$.

Definition 1.2. Let $f, g : [a,b] \to \mathbb{R}$. Then f is said to be *Henstock-Stieltjes integrable* (or HS-integrable) to A on $[a,b]$ with respect to g if for every $\epsilon > 0$, there exists a positive function δ such that for every δ-fine division $D = \{([t_i,t_{i+1}],\xi_i)\}_{i=1}^n$ of $[a,b]$, we have

$$|S(f,\delta,D) - A| < \epsilon,$$

Key Words: Henstock,Stieltjes,Young integral,p-variation
Mathematical Reviews subject classification: 26A21, 28B16
Received by the editors February 27, 2004
Communicated by: B. S. Bullen

where

$$S(f,\delta,D)=\sum_{i=1}^{n} f(\xi_i)(g(t_{i+1})-g(t_i)).$$

We denote A by $(HS)\int_a^b f(t)\,dg(t)$ or $(HS)\int_a^b f\,dg$.

Proposition 1.3 (Cauchy Criterion). *Let $f,g:[a,b]\to\mathbb{R}$. Then f is HS-integrable on $[a,b]$ with respect to g if and only if for every $\epsilon>0$, there exists a positive function δ such that for any two δ-fine divisions of $[a,b]$, $D=\{([t_i,t_{i+1}],\xi_i)\}$ and $D'=\{([t'_i,t'_{i+1}],\xi'_i)\}$, we have*

$$|S(f,\delta,D)-S(f,\delta,D')|<\epsilon.$$

2 Young-Love Inequality.

In this section we shall present some results proved by L. C. Young in 1936, see [4, 5].

Definition 2.1. Let $f:[a,b]\to\mathbb{R}$ and let $0<p<\infty$. Given a partition $D=\{[t_i,t_{i+1}]\}_{i=1}^{n}$ of $[a,b]$, let

$$V_p(f,D;[a,b])=\left[\sum_{i=1}^{n}\big|f(t_{i+1})-f(t_i)\big|^p\right]^{1/p}.$$

The p-variation of f is defined by

$$V_p(f;[a,b])=\sup_D V_p(f,D;[a,b]).$$

In this paper, we always denote $V_p(f;[a,b])$ by $V_p(f)$. We say that $f\in BV_p[a,b]$ if $V_p(f)<\infty$.

Theorem 2.2. *[5, p. 256, (6.2)] Let $f\in BV_p[a,b]$ and $g\in BV_q[a,b]$, with $p,q>0$ and $\frac{1}{p}+\frac{1}{q}>1$. Then, for any partition $D=\{[t_i,t_{i+1}]\}_{i=1}^{n}$ of $[a,b]$ and $\xi=t_i$, for some i,*

$$\left|\sum_{i=1}^{n} f(t_{i+1})\big(g(t_{i+1})-g(t_i)\big)-f(\xi)\big(g(b)-g(a)\big)\right|\le\left\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\right\}V_p(f)V_q(g),$$

where $\zeta\big(\frac{1}{p}+\frac{1}{q}\big)=\sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$.

Corollary 2.3. *[5, p. 257, (6.4)] Let $f \in BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} > 1$. Then, for any two partitions, $D = \{[t_i, t_{i+1}]\}, D' = \{[s_j, s_{j+1}]\}$ of $[a,b]$, with any $\xi_i \in [t_i, t_{i+1}], \eta_j \in [s_j, s_{j+1}]$, we have*

$$\begin{aligned}&\Big|(D)\sum f(\xi_i)\big(g(t_{i+1}) - g(t_i)\big) - (D')\sum f(\eta_j)\big(g(s_{j+1}) - g(s_j)\big)\Big|\\ \leq& 2\Big\{1 + \zeta\big(\frac{1}{p} + \frac{1}{q}\big)\Big\}V_p(f)V_q(g),\end{aligned}$$

where $\zeta\big(\frac{1}{p} + \frac{1}{q}\big) = \sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$.

From Jensen's inequality, we have

$$\big[\sum |f(v_i) - f(u_i)|^{p_1}\big]^{1/p_1} \leq \big[\sum |f(v_i) - f(u_i)|^{p}\big]^{1/p}, \text{ if } 0 < p < p_1.$$

Thus, we have the following consequence.

Proposition 2.4. *If $f \in BV_p[a,b]$ and $0 < p < p_1$, then $f \in BV_{p_1}[a,b]$.*

3 Integrable Functions.

In this section, we shall prove that $(HS)\int_a^b f\,dg$ exists if $f \in BV_p, g \in BV_q$, $\frac{1}{p} + \frac{1}{q} > 1$ and $p, q \geq 1$.

Definition 3.1. Let $f : [a,b] \to \mathbb{R}$. Then f is said to be regulated if f has one-sided limits at every point of $[a,b]$; i.e., $\lim_{t\to c^+} f(t)$, $\lim_{t\to c^-} f(t)$ exist, for each $c \in [a,b]$. The set of all regulated functions defined on $[a,b]$ is denoted by $RF[a,b]$.

The following result is known; for example, see [1, p. 24]. However, we shall give a proof.

Theorem 3.2. *If s is a step function and $g \in RF[a,b]$, then s is HS-integrable with respect to g on $[a,b]$.*

Proof. We only prove the following case. Let s be a step function defined by

$$s(t) = \begin{cases} C_1\ , & \text{if } t = a; \\ C_2\ , & \text{if } a < t < c; \\ C_3\ , & \text{if } t = c; \\ C_4\ , & \text{if } c < t < b; \\ C_5\ , & \text{if } t = b. \end{cases}$$

We shall prove that s is HS-integrable with respect to g on $[a, b]$ and

$$\begin{aligned}(HS)\int_a^b s\,dg =& C_1(g(a^+)-g(a))+C_2(g(c^-)-g(a^+))+C_3(g(c^+)-g(c^-))\\ &+C_4(g(b^-)-g(c^+))+C_5(g(b)-g(b^-)),\end{aligned}$$

where

$$g(a^+)=\lim_{t\to a^+} g(t) \text{ and } g(a^-)=\lim_{t\to a^-} g(t).$$

Let $\epsilon > 0$ be given. Then there exists $\delta_1 > 0$, such that

$$|g(t)-g(a^+)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < t-a < \delta_1,$$

$$|g(t)-g(c^-)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < c-t < \delta_1,$$

$$|g(t)-g(c^+)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < t-c < \delta_1,$$

$$|g(t)-g(b^-)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < b-t < \delta_1,$$

where $C_m = \max\{|C_1|, |C_2|, |C_3|, |C_4|\}$.

Choose

$$\delta(\xi)=\begin{cases}\delta_1 & \text{if } \xi=a;\\ \min\{\xi-a, c-\xi\} & \text{if } a<\xi<c;\\ \delta_1 & \text{if } \xi=c;\\ \min\{\xi-c, b-\xi\} & \text{if } c<\xi<b;\\ \delta_1 & \text{if } \xi=b.\end{cases}$$

From the choice of δ, a, b and c are associated points of any δ-fine division of $[a, b]$. Let $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ be δ-fine division of $[a, b]$. Hence,

$$\begin{aligned}&\Big|\sum_{i=1}^n s(\xi_i)(g(v_i)-g(u_i)) - C_1(g(a^+)-g(a)) - C_2(g(c^-)-g(a^+))\\ &\quad - C_3(g(c^+)-g(c^-)) - C_4(g(b^-)-g(c^+)) - C_5(g(b)-g(b^-))\Big|\\ =&\Big|C_1(g(v_1)-g(a^+)) + C_2(g(v_1)-g(a^+)) + C_2(g(c^-)-g(u_j))\\ &\quad + C_3(g(c^-)-g(u_j)) + C_3(g(v_j)-g(c^+)) + C_4(g(v_j)-g(c^+))\\ &\quad + C_4(g(u_n)-g(b^-)) + C_5(g(u_n)-g(b^-))\Big|\\ \le& 2C_m\Big|(g(v_1)-g(a^+)) + (g(c^-)-g(u_j)) + (g(v_j)-g(c^+)) + (g(u_n)-g(b^-))\Big|\\ <& 2C_m 4(\frac{\epsilon}{8C_m}) = \epsilon.\end{aligned}$$

□

We remark that in the above proof, δ is a function, it is impossible to choose a constant δ.

Lemma 3.3. *[2, p. 83] If $f \in BV_p[a,b]$, $p > 0$, then $f \in RF[a,b]$.*

Lemma 3.4. *[4, p. 7] Let $f \in BV_p[a,b]$, $p \geq 1$. Then, for any $p_1 > p$ and $\epsilon > 0$, there is a step function s, such that $V_{p_1}(f-s) < \epsilon$.*

Now, we shall prove the main result of this paper.

Theorem 3.5. *Let $f \in BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p,q \geq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$. Then f is HS-integrable with respect to g.*

PROOF. Let $f \in BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p,q \geq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$. Let $\epsilon > 0$ be given. From Lemma 3.4, there is a step function s and $p_1 > p$ with $\frac{1}{p_1}+\frac{1}{q} > 1$, such that $V_{p_1}(f-s) < \epsilon$. By Theorem 3.2 and Lemma 3.3, $(HS)\int_a^b s\,dg$ exists. Then, there is a positive function δ such that for any two δ-fine divisions $D_1 = \{(\xi_l, [t_l, t_{l+1}])\}$ and $D_2 = \{(\xi_{l'}, [t_{l'}, t_{l'+1}])\}$ of $[a,b]$ we have

$$|(D_1)\sum_l s(\xi_l)(g(t_{l+1})-g(t_l)) - (D_2)\sum_{l'} s(\xi_{l'})(g(t_{l'+1})-g(t_{l'}))| < \epsilon.$$

Hence, from Corollary 2.3 and Lemma 3.4, we can see that

$$\begin{aligned}
&|(D_1)\sum_l f(\xi_l)(g(t_{l+1})-g(t_l)) - (D_2)\sum_{l'} f(\xi_{l'})(g(t_{l'+1})-g(t_{l'}))|\\
=&|(D_1)\sum_l (f(\xi_l)-s(\xi_l))(g(t_{l+1})-g(t_l)) + (D_1)\sum_l s(\xi_l)(g(t_{l+1})-g(t_l))\\
&-(D_2)\sum_{l'} s(\xi_{l'})(g(t_{l'+1})-g(t_{l'})) - (D_2)\sum_{l'}(f(\xi_{l'})-s(\xi_{l'}))(g(t_{l'+1})-g(t_{l'}))|\\
=&|(D_1)\sum_l (f(\xi_l)-s(\xi_l))(g(t_{l+1})-g(t_l))\\
&-(D_2)\sum_{l'}(f(\xi_{l'})-s(\xi_{l'}))(g(t_{l'+1})-g(t_{l'}))|\\
&+|(D_1)\sum_l s(\xi_l)(g(t_{l+1})-g(t_l)) - (D_2)\sum_{l'} s(\xi_{l'})(g(t_{l'+1})-g(t_{l'}))|\\
\leq&2\{1+\zeta\Big(\frac{1}{p_1}+\frac{1}{q}\Big)\}V_{p_1}(f-s)V_q(g)+\epsilon\\
\leq&2\{1+\zeta\Big(\frac{1}{p_1}+\frac{1}{q}\Big)\}\epsilon V_q(g)+\epsilon = (2V_q(g)\{1+\zeta\Big(\frac{1}{p_1}+\frac{1}{q}\Big)\}+1)\epsilon.
\end{aligned}$$

Hence $(HS)\displaystyle\int_a^b f\,dg$ exists. □

Henceforth, if $f \in BV_p[a,b], g \in BV_q[a,b]$, where $\frac{1}{p_1} + \frac{1}{p} > 1$, $p, q \geq 1$, then the Henstock-Stieltjes integral $(HS) \displaystyle\int_a^b f\, dg$ is called the Henstock-Young integral, and denoted by $(HY) \displaystyle\int_a^b f\, dg$.

4 Convergence Theorems.

In this section we will prove some convergence theorems.

Definition 4.1 (Two-norm convergence). A sequence $\{f_n\}$ of functions defined on $[a,b]$ is said to be two-norm convergent to f in $BV_p[a,b]$ if $f_n \in BV_p[a,b]$, for all $n = 1, 2, \ldots$, and

(i) f_n is uniformly convergent to f on $[a,b]$,

(ii) $V_p(f_n) \leq A$ for every $n = 1, 2, \ldots$.

In symbols, we denote the two-norm convergence by $f_n \twoheadrightarrow f$.

It is clear that $BV_p[a,b]$ is complete under two-norm convergence; i.e., if $f_n \in BV_p[a,b]$, $n = 1, 2, \ldots$, and $f_n \twoheadrightarrow f$, then $f \in BV_p[a,b]$.

Theorem 4.2. *If a sequence $\{f^{(n)}\}$ is two-norm convergent to f in $BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then $(HY) \displaystyle\int_a^b f\, dg$ exists and*

$$\lim_{n\to\infty} (HY) \int_a^b f^{(n)}\, dg = (HY) \int_a^b f\, dg.$$

Proof. Let $\epsilon > 0$ be given. Let $\{f^{(n)}\}$ be two-norm convergent to f in $BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. First, $\displaystyle\int_a^b (f^{(n)} - f)\, dg$ exists. Thus, there is a positive function δ_n such that for every δ_n-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a,b]$,

$$\left| \Big(\int_a^b (f^{(n)} - f\Big)\, dg) - (D) \sum (f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i)) \right| < \epsilon. \quad (1)$$

Let $V_p(f^{(n)} - f) \leq A$ for every n and $V_q(g) = B$. Since $f^{(n)} \twoheadrightarrow f$, there is a positive integer N such that for every $n \geq N$, we have

$$\sup_{t\in[a,b]} \{|f^{(n)}(t) - f(t)|\} = ||f^{(n)} - f||_\infty < \frac{\epsilon}{2}. \quad (2)$$

Choose a fixed $p_1 > p$ such that $\frac{1}{p_1} + \frac{1}{q} > 1$. Then $f^{(n)} - f \in BV_{p_1}[a,b]$. Furthermore for $n \geq N$ and a δ_n-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a,b]$, by Corollary 2.2, inequalities (1) and (2).

$$\begin{aligned}
&\left|\int_a^b f^{(n)}\,dg - \int_a^b f\,dg\right| \\
\leq&\left|(f^{(n)}(a) - f(a))(g(b) - g(a))\right| \\
&+\left|\int_a^b (f^{(n)} - f)\,dg - (f^{(n)}(a) - f(a))(g(b) - g(a))\right| \\
\leq&\frac{\epsilon}{2}B + \left|\int_a^b (f^{(n)} - f)\,dg - (D)\sum(f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i))\right| \\
&+\Big|(D)\sum(f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i)) \\
&\quad - \sum(f^{(n)}(t_{i+1}) - f(t_{i+1}))(g(t_{i+1}) - g(t_i))\Big| \\
&+\left|\sum(f^{(n)}(t_{i+1}) - f(t_{i+1}))(g(t_{i+1}) - g(t_i)) - (f^{(n)}(a) - f(a))(g(b) - g(a))\right| \\
\leq&\frac{\epsilon}{2}B + \epsilon + 2\left\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\right\}V_{p_1}(f^{(n)} - f)V_q(g) \\
&+\left\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\right\}V_{p_1}(f^{(n)} - f)V_q(g) \\
\leq&\frac{\epsilon}{2}B + \epsilon + 3\left\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\right\}\epsilon^{(p_1 - p/p_1)}V_p^{p/p_1}(f^{(n)} - f)V_q(g) \\
=&\frac{\epsilon}{2}B + \epsilon + 3\left\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\right\}\epsilon^{(p_1 - p/p_1)}A^{p/p_1}B.
\end{aligned}$$

Hence, $\lim_{n\to\infty}(HY)\int_a^b f^{(n)}\,dg = (HY)\int_a^b f\,dg.$ □

Using the idea of the above proof, we have the following assertion.

Theorem 4.3. *If $f \in BV_p[a,b]$ and $\{g^{(n)}\}$ is two-norm convergent to g in $BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then $(HY)\int_a^b f\,dg$ exists and*

$$\lim_{n\to\infty}(HY)\int_a^b f\,dg^{(n)} = (HY)\int_a^b f\,dg.$$

Again following the proof of Theorem 4.2, we have

$$\lim_{n\to\infty}\big((HS)\int_a^b f^{(n)}\,dg^{(n)} - (HS)\int_a^b f\,dg^{(n)}\big) = 0.$$

Hence, we have the following theorem.

Theorem 4.4. *If $\{f^{(n)}\}$ and $\{g^{(n)}\}$ are two-norm convergent to f and g in $BV_p[a,b]$ and $BV_q[a,b]$, respectively, with $p,q \geq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$, then $(HY)\int_a^b f\,dg$ exists and*

$$\lim_{n\to\infty}(HY)\int_a^b f^{(n)}\,dg^{(n)} = (HY)\int_a^b f\,dg.$$

Similar Convergence theorems have been proved by K. K. Aye in [1, p. 71] under stronger conditions.

Acknowledgement. We would like to thank Professor Lee Peng-Yee for his constructive comments on this paper.

References

[1] K. K. Aye, *The Duals of Some Banach Spaces*, Ph.D. Thesis, National Institute of Education, Singapore, 2002.

[2] R. M. Dudley and R. Norvaisa, *Differentiability of Six Operators on Nonsmooth Functions and p-Variation*, Springer-Verlag, Berlin Heidelberg, 1999.

[3] R. Henstock, *Lectures on the Theory of Integration*, World Scientific, 1988.

[4] E. R. Love and L. C. Young, *On fractional integration by parts*, Proc. London Math. Soc., (Ser 2), **44** (1938), 1–35.

[5] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math., **67** (1936), 251–282.