THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAMES

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1. Introduction. Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of maximal length originating in the distinguished vertex x_0. In a *win-lose* game the set S of all plays is divided into two sets S_I and S_{II} such that player I wins the play s if $s \in S_I$ and player II wins it if $s \in S_{II}$. Gale and Stewart have shown that a two-person infinite win-lose game of perfect information with no chance moves (called a GS game here) is strictly determined if S_I belongs to the smallest Boolean algebra containing the open sets of a certain topology for S. Here we answer affirmatively the question posed by them: Is a GS game strictly determined if S_I is a G_δ (or, equivalently, an F_σ)? The notation and results of [1] are used throughout, as well as the partial ordering of X given by: $x > y$ if $f^n(x) = y$ for some $n \geqslant 1$.

2. Alternative description of S_I. Let Γ be the game $(x_0, X_I, X_{II}$ $X, f, S, S_I, S_{II})$, where

$$S_I = \bigcap_{n=1}^{\infty} E_n ,$$

$E_1 \supseteq E_2 \supseteq \cdots$, and E_n is open. Following [3], let the rank $rk(x)$, for $x \in X$, be the unique k such that $f^k(x) = x_0$. As in [1], $\mathfrak{U}(x)$ is the set of all plays passing through x (the topology for S is that in which $\mathfrak{U}(x)$ is a neighborhood of each play in it). Then for each n,

$$E_n = \bigcup \{\mathfrak{U}(y) : \mathfrak{U}(y) \subseteq E_n\} ;$$

and since for any $y \in X$ we have

$$\mathfrak{U}(y) = \bigcup \{\mathfrak{U}(z) : f(z) = y\} ,$$

with

$$rk(z) = 1 + rk(y) ,$$

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there exists for each n a subset Y_n of X such that $rk(y)>n$ for all $y\in Y_n$ and

$$E_n=\bigcup\{\mathfrak{U}(y):y\in Y_n\}\ .$$

Furthermore, since of any two neighborhoods having a non-void intersection, one is contained in the other, each Y_n may be chosen so that $\mathfrak{U}(y)$, $\mathfrak{U}(y')$ are disjoint for different y, y' in Y_n.

Since $s\in S_I$ if and only if $s\in E_n$ for an infinite number of values of n, we have: $s\in S_I$ if and only if for infinitely many n there exists i (dependent on n) such that $s(i)\in Y_n$. Thus, since on the one hand $i=rk(s(i))>n$, and on the other for any n there is at most one i such that $s(i)\in Y_n$, letting

$$Y=\bigcup_{n=1}^{\infty} Y_n$$

we have: $s\in S_I$ if and only if $s(i)\in Y$ for infinitely many i.

3. Lemmas.

LEMMA 1. *If Γ is a* GS *game with*

$$\textstyle\sum_{II}^{W}(\Gamma)=\Delta$$

and

$$T=S-\bigcup\{\mathfrak{U}(x):\textstyle\sum_{II}^{W}(\Gamma_x)\neq\Delta\}\ ,$$

then

$$\Gamma_T=(x_0,X_I^T,X_{II}^T,X^T,f^T,T,S_I^T,S_{II}^T)$$

is a subgame of Γ,

$$\textstyle\sum_{I}^{W}(\Gamma_T)\neq\Delta$$

implies

$$\textstyle\sum_{I}^{W}(\Gamma)\neq\Delta\ ,$$

and
$$\textstyle\sum_{II}^{W}((\Gamma_T)_x)=\Delta$$

for all $x\in X^T$.

Proof. Since T is a closed nonempty subset of S, Γ_T is a subgame of Γ by Theorem 5 of [**1**]. The second statement follows from assertion B [**1**, p. 260]. Finally suppose that

$$\textstyle\sum_{II}^{W}((\Gamma_T)_x)\neq\Delta$$

for some $x\in X^T$. Letting, in assertion A [**1**, p. 260],

$$F=\mathfrak{U}(x)\cap T\ ,$$

and noting that F is closed and nonempty and that

$$(\Gamma'_T)_x = (\Gamma'_x)_F ,$$

we have

$$\sum_{II}^{W}(\Gamma'_x) \neq \Lambda ,$$

which is impossible in view of the construction of T.

We assume hereafter that Γ is a GS game with S_I described in terms of $Y \subseteq X$ as in § 2, and that

$$\sum_{II}^{W}(\Gamma) = \Lambda ,$$

whence

$$\sum_{II}^{W}(\Gamma_T) = \Lambda$$

by Lemma 1. The strict determinateness of Γ will follow from Lemma 1 and the fact that

$$\sum_{I}^{W}(\Gamma_T) \neq \Lambda ,$$

proved in § 4.

LEMMA 2. *For $x \in X^T$, we have*

$$s \in S_I^{Tx}$$

if and only if

$$s \in S^{Tx} \quad \textit{and} \quad s(i) \in Y$$

for infinitely many i.

LEMMA 3. *For $x \in X^T$ there exists*

$$\sigma_x \in \sum_I((\Gamma_T)_x)$$

such that for any

$$\tau \in \sum_{II}((\Gamma_T)_x)$$

we have

$$\langle \sigma_x, \tau \rangle(i) \in Y$$

for some $i > rk(x)$.

Proof. Let Y_x be the set of all

$$y \in Y \cap X^T$$

such that $y > x$ and no members of Y fall between x and y. Let Γ' be the game

$$(x_0, X_I^{Tx}, X_{II}^{Tx}, X^{Tx}, f^{Tx}, S^{Tx}, S'_I, S'_{II}) ,$$

where

$$S'_I = S^{Tx} \cap \bigcup \{\mathfrak{U}(y) : y \in Y_x\}$$

and

$$S'_{II} = S^{Tx} - S'_I$$

(that is, the game in which I wins if the play passes through any member of Y following x). Noting that

$$S_I^{Tx} \subseteq S'_I ,$$

we have

$$S'_{II} \subseteq S_{II}^{Tx}$$

and hence

$$\sum_{II}^W(\Gamma'') = \Lambda .$$

But S'_I is open in S^{Tx} and so Γ'' is strictly determined by Corollary 10 of [1], whence there exists

$$\sigma_x \in \sum_I^W(\Gamma'') ,$$

which satisfies the conclusion of the lemma.

4. Winning Γ_T. Let

$$Y' = (Y \cap X^T) \cup \{x_0\} .$$

For each $x \in Y'$ let σ_x be as given by Lemma 3, and let σ'_x be the restriction of σ_x to the set of all z in X^T such that $x \leqq z$ and that there exists no y in Y' with $x < y \leqq z$. We show that the domains of the σ'_x cover X^T and are disjoint: First, if $x_0 \in X_I^T$, then x_0 belongs to the domain of σ_{x_0}. For

$$z \in X_I^T - \{x_0\} ,$$

let

$$x = \max\{z' : z' \in Y' \ \& \ z' < z\} .$$

Then $x \in Y'$ and z belongs to the domain of σ'_x; thus the domains of the σ'_x cover X_I^T. Now suppose that $x_1, x_2 \in Y'$, $x_1 \neq x_2$, and that there exists x_3 common to the domains of σ'_{x_1} and σ'_{x_2}; then $x_1 \leqq x_3$ and $x_2 \leqq x_3$, so that either $x_1 < x_2 \leqq x_3$ or $x_2 < x_1 \leqq x_3$, which is impossible in view of the restriction imposed upon σ_x in obtaining σ'_x.

Since the domains of the σ'_x cover X_I^T and are disjoint, they have a common extension σ^*, which necessarily maps the elements of X_I^T on their immediate successors, and thus belongs to $\sum_I(\Gamma_T)$.

We show that σ^* wins Γ_T. Let

$$\tau\in\sum_{II}(\Gamma_T) .$$

For this τ and any x in Y', let $i(x)$ be the least i such that $\langle\sigma_x, \tau\rangle(i)\in Y'$, whose existence is given by Lemma 3. Define $\{x_n\}$ inductively by

$$x_{n+1}=\langle\sigma^*, \tau\rangle(i(x_n)) \qquad n=0, 1, \cdots$$

(x_0 is the distinguished vertex). Since

$$rk(x_{n+1})=i(x_n)>rk(x_n) ,$$

and x_n, x_{n+1} are on a common path, we have $x_{n+1}>x_n$ for all n, and so if $x_n\in Y'$ then

$$x_{n+1}=\langle\sigma^*, \tau\rangle(i(x_n))=\langle\sigma_{x_n}, \tau_{x_n}\rangle(i(x_n))\in Y' ,$$

where

$$\tau_{x_n}\in\sum_{II}((\Gamma_T)_{x_n})$$

is the restriction of τ to $X_{II}^{Tx_n}$. Thus by induction $x_n\in Y'$ for all n, and hence

$$\langle\sigma^*, \tau\rangle(i)\in Y$$

for infinitely many values of i, so that

$$\langle\sigma^*, \tau\rangle\in S_I^T .$$

Since τ is arbitrary,

$$\sigma^*\in\sum_I^W(\Gamma_T) ,$$

so that by Lemma 1, we have

$$\sum_I^W(\Gamma)\neq\Lambda .$$

As this is the consequence of the sole fact that

$$\sum_{II}^W(\Gamma)=\Lambda ,$$

Γ is strictly determined.

Reversing the roles of the players in the above gives the result that a GS game is strictly determined if S_I is an F_σ.

The strict determinateness of a two-person zero-sum game with G payoff having *chance moves* can be shown. The proof is more complicated, but uses the same ideas [4].

5. An application. Let

$$\Gamma=(x_0, X_I, X_{II}, X, f, S, \Phi)$$

be a zero-sum two-person infinite game of perfect information with no chance moves having payoff Φ such that there exists a real function h on X ($|h(x)|<K<\infty$) with

$$\Phi(s)=\limsup_{i\to\infty} h(s(i)) \quad \text{for all} \quad s\in S .$$

Γ is the result of an attempt to reduce the following situation to a game: The tree K of a GS game and a function h as above are given; the two players make choices in K in the belief that every play will terminate in some unknown, but distant, vertex x, at which time player I will receive the amount $h(x)$ from player II. A payoff function Φ is sought such that $\Phi(s)$ ($-\Phi(s)$) expresses the utility to player I (II) of a play s in K.

The payoff Φ defined above arises from ascription to players I and II respectively of "optimistic" and "pessimistic" behaviors in this way: Player I assumes that the play s will terminate in some "distant" vertex $s(i)$ at which h assumes nearly its supremum on all "distant" vertices of s; he thus makes his choices so as to maximize the expression

$$\limsup_{i\to\infty} h(s(i))=\Phi(s) ;$$

and player II supposes that s will terminate in some "distant" vertex at which his gain $-h(s(i))$ assumes nearly its infimum for all such vertices, and thus seeks to maximize

$$\liminf_{i\to\infty} -h(s(i))=-\Phi(s) ,$$

that is, to minimize Φ. The derived game is thus zero-sum. Ascription, however, of such "optimistic" or "pessimistic" payoffs to both players yields, in general, a non-zero sum game.

We show now that the game Γ of this section is strictly determined, using the method of Theorem 15 of [1] which asserts the strict determinateness of Γ for the more special case of continuous Φ. (Gillette [2] has shown the strict determinateness of an infinite game of perfect information with chance moves which consists in repeated play from a finite set of finite games and has payoff

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g_n(s) ,$$

where $g_n(s)$ is the gain from the nth game played.)

First, as a converse to the equivalence of § 2, let $Y\subseteq X$, and denote by Y_n the set of all members of Y having rank greater than n. Then

$$\begin{aligned}\{s : s(i)\in Y \text{ for infinitely many } i\} &= \bigcap_n \{s : s(i)\in Y_n \text{ for some } i\}\\ &= \bigcap_n \bigcup \{\mathfrak{U}(y) ; y\in Y_n\} ,\end{aligned}$$

which is a G_δ.

Now in Γ, for t real, let

$$S_I^t = \{s : h(s(i)) > t \text{ for infinitely many } i\} ,$$

and $S_{II}^t = S - S_I^t$. Then S_I^t is a G_δ, and thus the GS game

$$\Gamma_t = (x_0, X_I, X_{II}, X, f, S, S_I^t, S_{II}^t)$$

is strictly determined. Let

$$v = \sup \{t : \textstyle\sum_I^W(\Gamma_t) \neq \Lambda\} .$$

Since $S_I^K = \Lambda$, $S_I^{-K} = S$, and S_I^t is a decreasing function of t, we have

$$-K \leqslant v \leqslant K , \quad \textstyle\sum_I^W(\Gamma_t) \neq \Lambda \quad \text{if} \quad t < v ,$$

and

$$\textstyle\sum_{II}^W(\Gamma_t) \neq \Lambda \quad \text{if} \quad t > v .$$

Given $\varepsilon > 0$, choose

$$\sigma_0 \in \textstyle\sum_I^W(\Gamma_{v-\varepsilon}) \quad \text{and} \quad \tau_0 \in \textstyle\sum_{II}^W(\Gamma_{v+\varepsilon}) .$$

Then for any

$$\sigma \in \textstyle\sum_I(\Gamma) , \quad \tau \in \textstyle\sum_{II}(\Gamma) ,$$

we have

$$h(\langle \sigma_0, \tau \rangle (i)) > v - \varepsilon \quad \text{for infinitely many } i$$

and do not have

$$h(\langle \sigma, \tau_0 \rangle (i)) > v + \varepsilon \quad \text{for infinitely many } i ;$$

so that

$$\Phi(\langle \sigma_0, \tau \rangle) \geqslant v - \varepsilon \quad \text{and} \quad \Phi(\langle \sigma, \tau_0 \rangle) < v + 2\varepsilon .$$

Hence

$$v - \varepsilon \leqslant \sup_\sigma \inf_\tau \Phi(\langle \sigma, \tau \rangle) \leqslant \inf_\tau \sup_\sigma \Phi(\langle \sigma, \tau \rangle) \leqslant v + 2\varepsilon ;$$

thus Γ is strictly determined, and has value v.

References

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