NOTE ON NONCOOPERATIVE CONVEX GAMES

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1. Introduction. Nash's equilibrium-point theorem for many-person games can be approached by two methods: first, the Kakutani-type fixed-point theorem[1] is very useful for this game problem; second, in case of finite-dimensional multilinear payoffs, J. Nash himself has given an elegant procedure [7] which is directly based on Brouwer's fixed-point theorem. In a previous paper [10] one of us proved a general minimax theorem in making use of a procedure analogous to that of Nash. The present note is a continuation of this paper, and its main purpose is to offer further improvements of Nash's method so as to treat noncooperative many-person games played over infinite-dimensional convex sets, based on a generalization of von Neumann's symmetrization method[2] of game matrices. The results thus obtained contain further weakening of (especially topological) assumptions of the equilibrium-point theorem.

Next we shall discuss the equilibrium-point problem of some general noncooperative games by reducing them to suitable convex games. This will clarify the relevance of convex games to general games.

2. Definitions and notations. We mean by a *convex game* [3] a noncooperative n-person game with the following conditions:

a) The ith player's strategy space is a compact convex set X_i of a topological linear space E_i.

b) The ith player's payoff $K_i(x_1, \cdots, x_i, \cdots, x_n)$ is concave with respect to his own strategy variable $x_i \in X_i$.

c) The sum of payoffs $\sum_{i=1}^{n} K_i(x_1, \cdots, x_i, \cdots, x_n)$ is continuous over the cartesian product space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$.

d) For each fixed x_i, $K_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a continuous

Received October 27, 1953. This work was partly sponsored by the Ministry of Education of Japan. The writers wish to express their thanks to Professor S. Iyanaga, Tokyo University, for his comments.

[1] See [6], [4], [5], or [9]. A supplementary note to [9] will be published shortly.

[2] See G. W. Brown and J. von Neumann, *Solutions of games by differential equations* in [1], and D. Gale, H. W. Kuhn and A. W. Tucker, *On symmetric games* in [1].

function of the $(n-1)$-tuple $[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n] \in X_1 \otimes \cdots \otimes X_{i-1} \otimes X_{i+1} \otimes \cdots \otimes X_n$ respectively.

REMARK In view of the usual classification of games in terms of total gains, c) may be of interest. Indeed, in case of constant-sum games, c) is automatically fulfilled. If all the payoffs are continuous over $X_1 \otimes \cdots \otimes X_n$, c) and d) are also fulfilled.

A point $[\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n] \in X_1 \otimes X_2 \otimes \cdots \otimes X_n$ is said to be an *equilibrium* point if the x_i-function $K_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \cdots, \hat{x}_n)$ assumes its maximum at $x_i = \hat{x}_i$ $(i=1, 2, \cdots, n)$.

REMARK The notion of equilibrium points first appeared in the celebrated work of Augustin Cournot (see [2]) and was investigated by him by means of differential calculus. But the contemporary concern about it is to see the existence of these points in the global sense by topological methods. The equilibrium-point problem under conditions a)–d) cannot, however, be treated by the Kakutani fixed-point theorem, since the required upper semi-continuity is not always assured in these cases. Thus, the proof in the following section may deserve some general attention.

3. Generalization of von Neumann's symmetrization and proof of the equilibrium-point theorem. To see the existence of equilibrium points for a convex game, we introduce an auxiliary function. To begin with, denote by

$$x = [x_1, x_2, \cdots, x_n], \qquad y = [y_1, y_2, \cdots, y_n]$$

two mutually independent variables with the same domain

$$X = X_1 \otimes X_2 \otimes \cdots \otimes X_n,$$

which is again compact and convex.

Next put

$$\Phi(x, y) = \sum_{i=1}^{n} K_i(y_1, y_2, \cdots, y_{i-1}, x_i, y_{i+1}, \cdots, y_n). \tag{1}$$

It is noted that $\Phi(x, y)$ is also concave with respect to $x \in X$. The importance of this function is clarified by:

LEMMA 3. 1. *A point*

$$\hat{y} = [\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n] \in X$$

is an equilibrium point for the given game, if and only if $\Phi(x, \hat{y})$ *assumes* *its maximum at* $x=\hat{y}$.

Proof. The necessity is obvious. If, conversely,

$$\Phi(\hat{y}, \hat{y}) \geqq \Phi(x, \hat{y})$$

for any $x \in X$, setting

$$x=[\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_{i-1}, x_i, \hat{y}_{i+1}, \cdots, \hat{y}_n]$$

gives

$$K_i(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_{i-1}, \hat{y}_i, \hat{y}_{i+1}, \cdots, \hat{y}_n) \geqq K_i(\hat{y}_1, \cdots, \hat{y}_{i-1}, x_i, \hat{y}_{i+1}, \cdots, \hat{y}_n)$$

for any $x_i \in X_i$.

REMARK For a zero-sum two-person game, we have

$$\Phi(x, y)=K(x_1, y_2)-K(y_1, x_2)\ , \qquad \Phi(y, y)=0\ ,$$

where $K(x_1, x_2)$ is the payoff from player 2 to player 1. This implies the functional form of von Neumann's symmetrization procedure[3]. We shall later present an interpretation of this function with regard to player's behavior.

With this setup, we next prove:

THEOREM 3. 1. *A convex game always has at least one equilibrium point.*

Proof. By Lemma 3. 1., we have only to see the existence of a point $\hat{y} \in X$ such that $\Phi(\hat{y}, \hat{y}) \geqq \Phi(x, \hat{y})$ for any $x \in X$. Suppose the contrary were valid. Then, to each $y \in X$, there exists some $x \in X$ such that

$$\Phi(y, y) < \Phi(x, y)\ . \tag{2}$$

Put $G_x=\{y; \Phi(y, y)<\Phi(x, y)\}$ then G_x is open by conditions c) and d), and

$$X \subset \bigcup_{x \in X} G_x$$

by (2). Hence, in view of the compactness of X, we can find a finite

[3] It is noted that $\Phi(x, y)$ does not provide a *real* generalization of von Neumann's symmetrization, since x_i's refer, in special cases, to mixed strategies. We can also construct, however, the function Φ in terms of pure strategies, and this will give a real generalization of von Neumann's method symmetrizing game matrices; instead of the cartesian product of mixed strategy spaces we must, then, consider the mixed strategies over the cartesian product of pure strategy spaces. But in either cases the *formal* procedures in constructing Φ are exactly the same.

set $A=\{a_1, a_2, \cdots, a_s\}\subset X$ such that

$$X\subset \bigcup_{j=1}^{s} G_{a_j} \ .$$

This implies $\Phi(y, y)<\max_j \Phi(a_j, y)$ for any $y\in X$. Now, put

$$f_j(y)=\max\,[\Phi(a_j, y)-\Phi(y, y), 0] \qquad (j=1, 2, \cdots, s) \ .$$

These s functions are all continuous by conditions c) and d), and satisfy $f_j(y)\geqq 0$, $\sum_{j=1}^{s} f_j(y)>0$ for any $y\in X$.

The continuous mapping

$$(3) \qquad y\rightarrow \sum_{j=1}^{s} f_j(y)a_j\Big/\sum_{j=1}^{s} f_j(y)$$

maps X into the convex hull $C(A)$ of A and therefore in particular $C(A)$ into $C(A)$. Since $C(A)$ is homeomorphic to a compact convex set in a Euclidean space, there exists a fixed point by Brouwer's fixed-point theorem.

Denote by $\hat{y}$ one such point. We have then

$$\hat{y}=\sum_{j=1}^{s} f_j(\hat{y})a_j\Big/\sum_{j=1}^{s} f_j(\hat{y})\in C(A)\subset X \ .$$

But for such a j that $f_j(y)>0$, we have, by definition, $\Phi(a_j, \hat{y})>\Phi(\hat{y}, \hat{y})$. Since $\Phi(x, y)$ is x-concave, this implies $\Phi(\hat{y}, \hat{y})>\Phi(\hat{y}, \hat{y})$, which is a contradiction.

REMARK The foregoing proof is essentially a repetition of the argument in [10]; the application of this argument to many-person cases is made possible by the use of $\Phi(x, y)$. It should be noticed, however, that despite the generality of Theorem 3. 1, it does not contain the result of [10]. The main reason for this fact is: the quasi-concavity (see [10]) of the original payoff may be lost in constructing $\Phi(x, y)$. So the theorem in [10] needs separate discussion.

4. An interpretation of $\Phi(x, y)$. Lemma 3. 1 can be rewritten as follows: *An n-person game has an equilibrium point if and only if*

$$(4) \qquad \min_{y\in X}\max_{x\in X}\,[\Phi(x, y)-\Phi(y, y)]=0 \ .$$

Now (4) may be interpreted in the following way: Suppose there are n persons $P_1, P_2, \cdots, P_n$. We consider the cases where all the persons $P_2, \cdots, P_n$ except P_1 cooperate. Denote the coalition consisting of only P_1 by Q_1 and that consisting of $P_2, P_3, \cdots, P_n$ by Q_2. Q_1 and Q_2 play n original games simultaneously, conforming to the following new rules: We denote these n games by $G_1, G_2, \cdots, G_n$, respectively. In G_i $(i=1,$

$2, \cdots, n)$, Q_1 participates in the n simultaneous games as the ith player, while Q_2 occupies all the other positions. Then

$$x=[x_1, x_2, \cdots, x_n]\in X$$

indicates the strategies of Q_1, and

$$y=[y_1, y_2, \cdots, y_n]\in X$$

indicates those of Q_2. If Q_1 chooses x and Q_2 chooses y, Q_2 pays to Q_1 the amount

$$K_i(y_1, \cdots, y_{i-1}, x_i, y_{i+1}, \cdots, y_n)$$

as the outcome of G_i. On the other hand, Q_1 pays to Q_2 the amount

$$\sum_{i=1}^{n} K_i(y_1, y_2, \cdots, y_n)$$

as the rent for gambling, after the game is over. Thus $\Phi(x, y)-\Phi(y, y)$ indicates the total gain of Q_1, while $\Phi(y, y)-\Phi(x, y)$ indicates that of Q_2. With the notion of this new zero-sum two-person game, (4) gives a criterion for the existence of equilibrium points for the original n-person game. If the given n-person game is constant sum, (4) is reduced to the more natural formula:

$$\min_{y\in X} \max_{x\in X} \Phi(x, y)=\pi ,$$

where π denotes the corresponding constant sum.

5. Reduction to convex games. In this section we assume E_i is a normed linear space. We further assume regarding the payoffs $H_i(x_1, x_2, \cdots, x_n)$ the following conditions:

(i) The x_i-function $H_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is upper semi-continuous for each fixed $(n-1)$-tuple $[x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$.

(ii) The x_i-set

$$\{x_i\,;\ \max_{x_i\in X_i} H_i(x_1, \cdots, x_i, \cdots, x_n)=H_i(x_1, \cdots, x_i, \cdots, x_n)\}$$

is convex for each fixed $(n-1)$-tuple $[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$.

(iii) The family $\{H_i(x_1, \cdots, x_i, \cdots, x_n)\,;\ x_i\in X_i\}$ is a uniformly equi-continuous family of functions on $X_1\otimes\cdots\otimes X_{i-1}\otimes X_{i+1}\otimes\cdots\otimes X_n$.

These games are usually treated by means of Kakutani's fixed-point theorem. We shall next, however, prove the following:

THEOREM 5. 1. *To each game of foregoing type there exists a convex game with the same strategy spaces whose equilibrium points are exactly those of the original game.*

As a direct application of Theorems 3.1 and 5.1 we can see the existence of equilibrium points for games of the foregoing type without Kakutani's theorem.

We now proceed to prove some lemmas.

Let R and S be normed linear spaces. We denote by $||x||$ the norm of a point $x \in R$. A continuous function $f(x)$ over R will be called linear if

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

for $x_1, x_2 \in R$, $\alpha_1 + \alpha_2 = 1$. We define the norm of f as usual:

$$||f|| = \sup_{||x|| \leqq 1} |f(x) - f(0)| .$$

Now, let $H(x, y)$ be a function on $X \otimes Y$, where X and Y are compact convex sets in R and S, respectively, and suppose that the family of functions $\{H(x, y) ; x \in X\}$ is uniformly equi-continuous.

Let further F_y be the totality of linear functions f over R such that (I) $||f|| \leqq 1$ and (II) $f(x) \geqq H(x, y)$ for any $x \in X$.

Putting

$$K(x, y) = \inf_{f \in F_y} f(x) ,$$

we obtain an x-concave function on $X \otimes Y$. We call $K(x, y)$ the x-concave envelope of $H(x, y)$. We shall show the continuity of this function by proving the following lemmas.

LEMMA 5. 1. $\{K(x, y) ; x \in X\}$ is a uniformly equi-continuous family of functions on Y.

Proof. Since $\{H(x, y) ; x \in X\}$ is uniformly equi-continuous, we can find for $\varepsilon > 0$ a $\delta > 0$ such that $||y_1 - y_2|| \leqq \delta$ implies $|H(x, y_1) - H(x, y_2)| \leqq \varepsilon$ for any $x \in X$. We shall show that, for this same δ, $||y_1 - y_2|| \leqq \delta$ implies $|K(x, y_1) - K(x, y_2)| \leqq \varepsilon$ for any $x \in X$.

Indeed, if $f \in F_{y_1}$, then

$$f(x) \geqq H(x, y_1) \geqq H(x, y_2) - \varepsilon$$

for all $x \in X$, and $||f + \varepsilon|| = ||f||$; namely, we have $f + \varepsilon \in F_{y_2}$.

In the same way, we have $g + \varepsilon \in F_{y_1}$ for $g \in F_{y_2}$.

Hence, if $||y_1 - y_2|| \leqq \delta$, we obtain

$$K(x, y_1) + \varepsilon = \inf_{f \in F_{y_1}} f(x) + \varepsilon = \inf_{f \in F_{y_1}} [f(x) + \varepsilon] \geqq \inf_{g \in F_{y_2}} g(x) = K(x, y_2) ,$$

and similarly $K(x, y_2)+\varepsilon \geqq K(x, y_1)$ for any $x \in X$. This means that $|K(x, y_1)-K(x, y_2)| \leqq \varepsilon$ for $y_1, y_2 \in Y$, $\|y_1-y_2\| \leqq \delta$, and all $x \in X$.

LEMMA 5. 2. *$K(x, y)$ is continuous on X for each fixed $y \in Y$.*

Proof. Let y be an arbitrary fixed point in Y. If $\|x-\hat{x}\| \leqq \varepsilon$, then $f \in F_y$ implies

$$|f(x)-f(\hat{x})| \leqq \|f\| \, \|x-\hat{x}\| \leqq \varepsilon .$$

It follows that

$$\left| \inf_{f \in F_y} f(x) - \inf_{f \in F_y} f(\hat{x}) \right| \leqq \varepsilon ,$$

proving the desired continuity.

LEMMA 5. 3. *$K(x, y)$ is continuous on $X \otimes Y$.*

Proof. We have this lemma immediately by taking Lemmas 5. 1 and 5. 2 together into consideration.

LEMMA 5. 4. *Suppose $H(x, y)$ is upper semi-continuous in x for each fixed $y \in Y$, and the x-set*

$$\Gamma_y = \{x \, ; \, \max_{x \in X} H(x, y) = H(x, y)\}$$

is a convex subset of X for each fixed $y \in Y$. Put

$$\Delta_y = \{x \, ; \, \max_{x \in X} K(x, y) = K(x, y)\} .$$

Then we have $\Gamma_y = \Delta_y$ for each fixed $y \in Y$.

Proof. Let y be any fixed point $\in Y$, and put

$$\omega_y = \max_{x \in X} H(x, y) .$$

Then the linear function $g(x) \equiv \omega_y$ belongs to F_y. Hence we have

$$H(x, y) \leqq K(x, y) = \inf_{f \in F_y} f(x) \leqq \omega_y$$

for all $x \in X$, which implies $\Gamma_y \subset \Delta_y$.

Conversely, by the above formula, it is obvious that if $\hat{x} \in \Delta_y$ then $K(\hat{x}, y) = \omega_y$. Thus, to see that $\Delta_y \subset \Gamma_y$, it suffices to show that $K(\hat{x}, y) < \omega_y$ for $\hat{x} \notin \Gamma_y$.

Let $\hat{x}$ be any point not belonging to Γ_y. Then

$$\operatorname{dist}(\hat{x}, \Gamma_y) = 2\alpha > 0 .$$

Putting

$$M = \{x ; \operatorname{dist}(x, \Gamma_y) < \alpha\} ,$$

we obtain an open convex set M and a closed convex set $\overline{M}$ (the closure of M) in R. Moreover, it is clear that $\hat{x} \notin \overline{M}$.

Let $e(x)$ be such a linear function that $e(x) \geqq 0$ on $\overline{M}$ and $e(\hat{x}) = -1$; its existence is a well-known fact (known as Mazur's theorem) in the theory of convex sets. Denote by N the complement of M within X; N is compact and, in view of the definition of M, we have

$$\omega_y - \max_{x \in N} H(x, y) = \gamma > 0 ; \qquad \min_{x \in N} e(x) = \eta < 0 .$$

Put

$$f(x) = \omega_y + \frac{\delta e(x)}{|\eta|} ,$$

where $\delta > 0$ is so small that $\delta \leqq \gamma$ and $\delta \|e\| \leqq |\eta|$. Then $\|f\| \leqq 1$, $f(x) \geqq \omega_y$ on M, and

$$f(x) = \omega_y + \frac{\delta e(x)}{|\eta|} \geqq \omega_y + \frac{\gamma\eta}{|\eta|} = \omega_y - \gamma \geqq H(x, y)$$

for any $x \in N$.

Hence $f \in F_y$. Moreover,

$$f(\hat{x}) = \omega_y + \frac{\delta e(\hat{x})}{|\eta|} = \omega_y - \frac{\delta}{|\eta|} < \omega_y ,$$

which means $K(\hat{x}, y) < \omega_y$, proving the lemma.

The proof of Theorem 5. 1. is now immediate. Indeed, let us construct the x_i-concave envelope $K_i(x_1, \cdots, x_i, \cdots, x_n)$ of

$$H_i(x_1, \cdots, x_i, \cdots, x_n) \qquad (i = 1, 2, \cdots, n) .$$

Then $K_i(x_1, x_2, \cdots, x_n)$ is clearly x_i-concave, and is continuous by Lemma 5. 3. Thus, we obtain a convex game. Moreover, the set of equilibrium points of this game coincides with that of the original game, by Lemma 5. 4.

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