THE SOLUTION OF CAUCHY'S PROBLEM FOR A THIRD-ORDER LINEAR HYPERBOLIC DIFFERENTIAL EQUATION BY MEANS OF RIESZ INTEGRALS

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1. Introduction. M. Riesz [3] solved Cauchy's problem for the wave equation by means of a generalization of the Riemann-Liouville integral and a consideration of Lorentz space. L. Gårding [1] solved Cauchy's problem for two linear hyperbolic differential equations arising from a consideration of spaces of symmetric and Hermitian matrices by means of similar generalizations of the Riemann-Liouville integral. Gårding [2] also proved some general results for the solution of Cauchy's problem for general linear hyperbolic partial differential equations with constant coefficients again using Riesz-type integrals.

In the present paper the explicit solution of Cauchy's problem for the third-order partial differential equation

$$\Delta u = h(x_1, x_2, x_3) \ , \tag{1.1}$$

where Δ denotes the operator $\partial^3/(\partial x_1 \partial x_2 \partial x_3)$, is given by means of a similar generalization of the Riemann-Liouville integral. We restrict our attention to the case in which u and its first and second derivatives are given on the plane S whose equation is $x_1+x_2+x_3=0$. We verify in detail that the solution given actually satisfies the differential equation (1.1), and also that it and its derivatives assume the proper values on S.

Before proceeding to a study of (1.1), we give a brief discussion of the Riemann-Liouville integral and Riesz's generalization of it. (We use mainly the notation of Gårding [1].) Let p be a complex variable, and consider the Riemann-Liouville integral

$$I^p f(x) = \frac{1}{\Gamma(p)} \int_a^x f(t)(x-t)^{p-1} dt \qquad (a \leq x < b \leq \infty) \ , \tag{1.2}$$

where $\mathscr{R}(p)>0$,[1] and $f(x)$ is a continuous function when $a \leq x < b \leq \infty$. This integral diverges if $\mathscr{R}(p) \leq 0$. If p and q are such that $\mathscr{R}(p)>0$, $\mathscr{R}(q)>0$ we have

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[1] $\mathscr{R}(p)$ denotes the real part of p.

$$I^p I^q f(x) = I^{p+q} f(x) \tag{1.3}$$

and

$$\frac{d}{dx} I^{p+1} f(x) = I^p f(x) . \tag{1.4}$$

Clearly $I^p f(x)$ is an analytic function of p, regular for $\mathscr{R}(p) > 0$, and depending on the parameter x. It can, however, be continued analytically beyond this region provided that $f(x)$ has a sufficient number of continuous derivatives. Let us write

$$f(t) = \sum_{j=0}^{k-1} \frac{f^{(j)}(x)(t-x)^j}{j\,!} + r(x, t, k) , \tag{1.5}$$

so that $r(x, t, k)/(t-x)^k$ is bounded when $a < t < x$. Then on substituting in equation (1.2) we find that

$$I^p f(x) = \frac{1}{\Gamma(p)} \int_a^x r(x, t, k)(x-t)^{p-1} dt + \sum_{j=0}^{k-1} \frac{(-1)^j f^{(j)}(x)(x-a)^{p+j} p(p+1)\cdots(p+j-1)}{j\,!\,\Gamma(p+j+1)} . \tag{1.6}$$

Here the integral converges for $\mathscr{R}(p) > -k$, and (1.6) provides an analytic continuation of $I^p f(x)$ for such values of p. In particular,

$$I^{-j} f(x) = f^{(j)}(x) \qquad (j = 0, 1, 2, \cdots) . \tag{1.7}$$

By successive integrations by parts we can find another formula which is also useful for the analytic continuation of $I^p f(x)$. We have

$$I^p f(x) = I^{p+m} f^{(m)}(x) + \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^j}{\Gamma(p+j+1)} \tag{1.8}$$

If we let $p \to 0$ we find that

$$f(x) = I^m f^{(m)}(x) + \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^j}{j\,!} \tag{1.9}$$

The right member of (1.9) gives the solution of the differential equation

$$\frac{d^m u(x)}{dx^m} = f^{(m)}(x) \tag{1.10}$$

whose derivatives of order less than m assume the values $f(a), \cdots, f^{(m-1)}(a)$ when $x = a$.

When generalizing (1.2), Riesz considers Lorentz space L with points $x = (x_1, x_2, \cdots, x_n)$. The square of the distance of $x = (x_1, x_2, \cdots, x_n)$ from

$\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is

$$r(x-\xi)=(x_1-\xi_1)^2-(x_2-\xi_2)^2-\cdots-(x_n-\xi_n)^2 .$$

The interior of the light cone with its vertex at a fixed point x is characterized by $r(x-\xi)>0$ where ξ is variable. It consists of two parts, the direct and the retrograde cone, characterized by

$$r(x-\xi)>0 , \quad \xi_1-x_1>0 \quad \text{and} \quad r(x-\xi)>0 , \quad \xi_1-x_1<0 ,$$

respectively. It is the retrograde cone denoted by $D(x)$ which is mainly considered by Riesz. The domain of integration used is the bounded domain $D_S(x)$ limited by the nappe $C(x)$ of the retrograde cone $D(x)$ and a certain sufficiently regular surface S having the property that every straight line in L with a direction of nonnegative square length meets S in at most one point. The volume element in L is $d\xi=d\xi_1 d\xi_2\cdots d\xi_n$. Let $f(x)=f(x_1, x_2, \cdots, x_n)$ de a real function defined in the region consisting of all points x whose retrograde cones $D(x)$ intersect S. Then Riesz's generalization of (1.2) is[2]

$$I^p f(x)=\frac{1}{H_n(p)}\int_{D_S(x)} f(\xi)[r(x-\xi)]^{p-(1/2)n}d\xi , \tag{1.11}$$

with

$$H_n(p)=2^{2p-1}\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-2}\Gamma(p)\Gamma\left(p-\frac{n-2}{2}\right) .$$

If $f(x)$ is bounded, the integral is a regular analytic function of p for $\mathscr{R}(p)>(n-2)/2$. It can be shown that (1.3) is valid and, corresponding to (1.4),

$$\Delta_W I^{p+1} f(x)=I^p f(x) , \tag{1.12}$$

where Δ_W is the wave operator

$$(\partial/\partial x_1)^2-(\partial/\partial x_2)^2-\cdots-(\partial/\partial x_n)^2 .$$

If $f(x)$ has derivatives of sufficiently high order it is possible to continue $I^p f(x)$ beyond the region in which the integral converges. The generalizations of (1.7) are found to be

$$I^0 f(x)=f(x) , \quad I^{-j} f(x)=\Delta^j f(x) \qquad (j=1, 2, 3, \cdots) . \tag{1.13}$$

By means of Green's formula it is found that

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[2] To get uniform notations in this paper, as in Gårding [**1**], Riesz's variable α is replaced by $2p$ here.

$$(1.14)\quad I^p f(x) = I^{p+1}\Delta f(x)$$
$$+\frac{1}{H_n(p+1)}\int_{S(x)}\left\{\frac{df(\xi)}{d\nu}[r(x-\xi)]^{p+1-\frac{1}{2}n} - f(\xi)\frac{d[r(x-\xi)]^{p+1-\frac{1}{2}n}}{d\nu}\right\}dS,$$

where $S(x)$ is the portion of S interior to the cone $D(x)$, $d/d\nu$ is taken in the direction of the Lorentzian normal to the surface S, and dS is the Lorentzian element of surface area.

If we let $p\to 0$ in (1.14), the right side gives the solution of the differential equation

$$(1.15)\qquad \Delta_W u(x) = h(x)\ ,$$

$u(x)$ and its (Lorentzian) normal derivative being given on S.

In the present paper we consider three-dimensional Euclidean space with points $x=(x_1, x_2, x_3)$. In this case the retrograde light cone $D(x)$ with its vertex at a fixed point x is characterized by $x_1-\xi_1>0$, $x_2-\xi_2>0$, $x_3-\xi_3>0$, where $\xi=(\xi_1, \xi_2, \xi_3)$ is variable. We denote by S the plane $\xi_1+\xi_2+\xi_3=0$. The domain of integration used is the bounded domain $D_S(x)$ limited by the boundary of $D(x)$ and the plane S. Then our generalization of (1.2) is

$$(1.16)\qquad I^p f(x) = \frac{1}{[\Gamma(p)]^3}\iiint_{D_S(x)} f(\xi)[r(x-\xi)]^{p-1}d\xi\ ,$$

where $r(x-\xi)=(x_1-\xi_1)(x_2-\xi_2)(x_3-\xi_3)$ and $d\xi = d\xi_1 d\xi_2 d\xi_3$. If $f(x)$ is bounded, the integral is a regular analytic function of p for $\mathscr{R}(p)>0$. We show that (1.3) is valid and, corresponding to (1.4),

$$(1.17)\qquad \Delta I^{p+1}f(x) = I^p f(x)\ .$$

As before, $I^p f(x)$ can be continued analytically if $f(x)$ is sufficiently differentiable. The generalizations of (1.7) which we prove are

$$(1.18)\qquad I^0 f(x) = f(x)\ ,\quad I^{-1}f(x) = \Delta f(x)\ .$$

In § 3 we apply Green's formula to discover a formula similar to (1.14), namely,

$$(1.19)\qquad I^p f(x) = I^{p+1}\Delta f(x) + I_*^{p+1} f(x)\ ,$$

where $I_*^{p+1}f(x)$ is an integral over $S(x)$, the portion of S interior to $D(x)$, involving f and its first and second derivatives. If we let $p\to 0$ in (1.19), we obtain the solution of Cauchy's problem for the equation (1.1). The verification of the solution is carried out in § 5 making use of a series of lemmas developed in § 4.

The methods of this paper can be applied to the solution of the nth order partial differential equation

$$\frac{\partial^n u}{\partial x_1 \partial x_2 \cdots \partial x_n} = h(x_1, x_2, \cdots, x_n) .$$

However, the formulas required are very cumbersome to write and for this reason the present discussion has been limited to equations of third order.

2. Generalization of the Riemann-Liouville integral. Since we wish to consider the differential equation

$$\Delta u \equiv \partial^3 u / (\partial x_1 \partial x_2 \partial x_3) = h(x_1, x_2, x_3) , \tag{2.1}$$

the appropriate formula for the cube of the distance between points $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ is

$$r(x - \xi) = (x_1 - \xi_1)(x_2 - \xi_2)(x_3 - \xi_3) . \tag{2.2}$$

The retrograde light cone $D(x)$ with vertex at a fixed point x is characterized by $x_1 - \xi_1 > 0$, $x_2 - \xi_2 > 0$, $x_3 - \xi_3 > 0$, where ξ is variable. We do not make any use of the geometry of the space based on this distance formula but in finding volume elements and surface elements we regard the space as ordinary three-dimensional Euclidean space. It is only in determining the proper generalizations of the Riemann-Liouville integral that (2.2) plays a role. We first consider an integral extended over the whole of $D(x)$. We suppose $f(x)$ defined in a region such that if this region contains a certain point x it contains also the retrograde cone $D(x)$. In order to assure the absolute convergence of the integral considered we suppose among other things that $f(x)$ tends toward zero sufficiently rapidly when $x_1, x_2, x_3 \to -\infty$. We then define, for complex values of p such that $\mathscr{R}(p) > 0$,

$$I^p f(x) = \frac{1}{H_3(p)} \iiint_{D(x)} f(\xi) [r(x - \xi)]^{p-1} d\xi . \tag{2.3}$$

We should like to have

$$\Delta I^{p+1} f(x) = I^p f(x) \tag{2.4}$$

and

$$I^p I^q f(x) = I^{p+q} f(x) . \tag{2.5}$$

In order to find the correct form of $H_3(p)$ to accomplish this we consider the particular function

$$f_1(x) = \exp (x_1 + x_2 + x_3) .$$

Clearly $\Delta f_1(x) = f_1(x)$, so we should have $I^p f_1(x) = f_1(x)$. Introducing this function into (2.3) we easily find that we should choose $H_3(p) = [\Gamma(p)]^3$.

With this choice of $H_3(p)$, it is easy to verify that (2.4) holds by merely carrying out the necessary differentiations. We proceed to verify that also (2.5) holds with this choice of $H_3(p)$. After interchanging the order of integration we find that

(2.6)

$$I^p I^q f(x) = \frac{1}{[\Gamma(p)\Gamma(q)]^3}\iiint_{D(x)} f(\eta)\,d\eta \int_{\eta_1}^{x_1}\int_{\eta_2}^{x_2}\int_{\eta_3}^{x_3} [r(\xi-\eta)]^{q-1}[r(x-\xi)]^{p-1}d\xi\ .$$

If we make use of the well-known formulas

$$\int_a^b (\xi-a)^{\alpha-1}(b-\xi)^{\beta-1}d\xi = (b-a)^{\alpha+\beta-1}B(\alpha,\beta) \tag{2.7}$$

and

$$B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)\ , \tag{2.8}$$

we find that the right member of (2.6) reduces to $I^{p+q}f(x)$. Thus (2.5) is established.

In the applications to follow, the domain $D(x)$ will be replaced by a bounded domain $D_S(x)$ which is limited by the boundary of the retrograde cone $D(x)$ and by the plane S whose equation is $\xi_1+\xi_2+\xi_3=0$. We shall therefore in all that follows use the following definition of $I^pf(x)$:

$$I^p f(x) = \frac{1}{[\Gamma(p)]^3}\iiint_{D_S(x)} f(\xi)[r(x-\xi)]^{p-1}d\xi\ . \tag{2.9}$$

Since this is the same as (2.3) if only we assume that $f(\xi)=0$ when $\xi_1+\xi_2+\xi_3<0$, it is clear that the relations (2.4) and (2.5) hold also when $I^pf(x)$ is defined by (2.9).

In the application of (2.9) to the solution of Cauchy's problem we shall be concerned with the limit of $I^pf(x)$ as $p\to 0$. We therefore prove:

THEOREM 2.1. *If $f(x)$ is continuous in the region $x_1+x_2+x_3\geq 0$ then $I^pf(x)$ defined by* (2.9) *is a regular analytic function of p for $\mathscr{R}(p)>0$, and*

$$\lim_{p\to 0} I^p f(x) = f(x) \tag{2.10}$$

in the region $x_1+x_2+x_3>0$.

Proof. That $I^pf(x)$ is analytic when $\mathscr{R}(p)>0$ follows at once from its definition by equation (2.9).

In order to prove (2.10) we make a change of variables by writing, in (2.9),

$$x_1-\xi_1=d\sigma\cos^2\theta_1\ ,\quad x_2-\xi_2=d\sigma\sin^2\theta_1\cos^2\theta_2\ ,\quad x_3-\xi_3=d\sigma\sin^2\theta_1\sin^2\theta_2\ ,$$

where $d=x_1+x_2+x_3>0$. If we also make use of (2.8) and the well-known formula

$$B(\alpha,\beta)=2\int_0^{\pi/2}\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\,d\theta\ , \tag{2.11}$$

we find that

$$I^pf(x)-\frac{d^{3p}}{\Gamma(3p+1)}f(x)=\frac{d^{3p}}{[\Gamma(p)]^3}\int_0^1\int_0^{\pi/2}\int_0^{\pi/2}[F(\sigma,\theta_1,\theta_2)-F(0,\theta_1,\theta_2)] \tag{2.12}$$
$$\cdot 4\sigma^{3p-1}\sin^{4p-1}\theta_1\cos^{2p-1}\theta_1\sin^{2p-1}\theta_2\cos^{2p-1}\theta_2 d\theta_2 d\theta_1 d\sigma\ ,$$

where

$$F(\sigma,\theta_1,\theta_2)=f(x_1-d\sigma\cos^2\theta_1,\,x_2-d\sigma\sin^2\theta_1\cos^2\theta_2,\,x_3-d\sigma\sin^2\theta_1\sin^2\theta_2)\ .$$

But since $f(x)$ is continuous, if $\varepsilon>0$ is assigned we can find a δ such that $0<\delta<1$ and such that $|F(\sigma,\theta_1,\theta_2)-F(0,\theta_1,\theta_2)|<\varepsilon$ when $0<\sigma<\delta$, uniformly in θ_1 and θ_2. We now break the integral in (2.12) into two parts J_1 and J_2; in J_1, σ goes from 0 to δ, and in J_2 from δ to 1, while θ_1 and θ_2 assume all values between 0 and $\pi/2$ in both J_1 and J_2. We see at once that

$$|J_1|<\varepsilon d^{3p}/\Gamma(3p+1)\ .$$

If M is the maximum of $F(\sigma,\theta_1,\theta_2)$ in the region of integration, an easy calculation shows that

$$|J_2|\le 2d^{3p}M\delta^{3p-1}/\Gamma(3p)$$

if $0<p<1/3$. By choosing p sufficiently close to zero, we can make J_2 arbitrarily small, and it follows that

$$\lim_{p\to 0}\,[I^pf(x)-\frac{d^{3p}}{\Gamma(3p+1)}f(x)]=0\ .$$

Equation (2.10) follows at once from this since $d^{3p}/\Gamma(3p+1)\to 1$ as $p\to 0$.

3. Green's formula for $I^pf(x)$. We shall find it convenient to make use of the function

$$v=v(x,\xi)=\frac{[r(x-\xi)]^p}{[\Gamma(p+1)]^3}=\frac{[(x_1-\xi_1)(x_2-\xi_2)(x_3-\xi_3)]^p}{[\Gamma(p+1)]^3}\ . \tag{3.1}$$

We wish to transform the volume integral

$$(3.2) \qquad \iiint_{D_S(x)} (f\Delta_\xi v + v\Delta_\xi f)d\xi$$

into a surface integral. Here Δ_ξ denotes the operator Δ with respect to the variable ξ.

The function to be integrated must first be transformed into the form of a divergence. We easily find that

$$f\Delta_\xi v = (fv_{\xi_1\xi_2})_{\xi_3} - (f_{\xi_3}v_{\xi_1})_{\xi_2} + (f_{\xi_3\xi_2}v)_{\xi_1} - v\Delta_\xi f \ .$$

By permutation of ξ_1, ξ_2, ξ_3 we obtain altogether a total of 3! such equations. The left member and the last term of the right member are unaltered by such permutations. Adding these 3! equations and dividing by 3! we obtain

$$(3.3) \qquad \begin{aligned} f\Delta_\xi v + v\Delta_\xi f = & \left[\frac{1}{3}(fv_{\xi_2\xi_3} + vf_{\xi_2\xi_3}) - \frac{1}{6}(f_{\xi_2}v_{\xi_3} + v_{\xi_2}f_{\xi_3})\right]_{\xi_1} \\ & + \left[\frac{1}{3}(fv_{\xi_3\xi_1} + vf_{\xi_3\xi_1}) - \frac{1}{6}(f_{\xi_3}v_{\xi_1} + v_{\xi_3}f_{\xi_1})\right]_{\xi_2} \\ & + \left[\frac{1}{3}(fv_{\xi_1\xi_2} + vf_{\xi_1\xi_2}) - \frac{1}{6}(f_{\xi_1}v_{\xi_2} + v_{\xi_1}f_{\xi_2})\right]_{\xi_3} . \end{aligned}$$

We note that if $\mathscr{R}(p)>0$, v vanishes on the boundary of the retrograde cone $D(x)$, v_{ξ_i} vanishes for $\xi_j = x_j$, $(j \neq i)$, and $v_{\xi_i\xi_j}$ vanishes for $\xi_k = x_k$, $(k \neq i,\ k \neq j,\ i \neq j)$.

Applying the divergence theorem and noting that

$$\Delta_\xi v = -[r(x-\xi)]^{p-1}/[\Gamma(p)]^3 \ ,$$

we obtain

$$(3.4) \quad \begin{aligned} I^p f(x) = & I^{p+1}\Delta f(x) \\ & + \frac{1}{\sqrt{3}} \iint_{S(x)} \left\{ \frac{1}{3}\left[f(v_{\xi_2\xi_3} + v_{\xi_3\xi_1} + v_{\xi_1\xi_2}) + v(f_{\xi_2\xi_3} + f_{\xi_3\xi_1} + f_{\xi_1\xi_2}) \right] \right. \\ & \left. - \frac{1}{6}\left[(f_{\xi_2} + f_{\xi_3})v_{\xi_1} + (f_{\xi_3} + f_{\xi_1})v_{\xi_2} + (f_{\xi_1} + f_{\xi_2})v_{\xi_3} \right] \right\} dS \ , \end{aligned}$$

where $S(x)$ is the portion of S included in the retrograde cone $D(x)$, and dS is the surface area element on $S(x)$. If $f(x)$ is continuous, then by Theorem 2.1 the left member of (3.4) becomes $f(x)$ when we let $p \to 0$. If $\Delta f(x)$ is given in $D_S(x)$, and f together with its first and second derivatives are given on S, then the right member of (3.4) can be calculated. We are going to show that it yields the solution of Cauchy's problem for the differential equation $\Delta u = h(x)$.

It is clear that if u and its first and second derivatives are prescribed on S, then these derivatives cannot be prescribed arbitrarily but certain relations exist between u and its derivatives. Only a complete independent set can be prescribed arbitrarily on S. For example, one may prescribe u and its first and second normal derivatives on S, or one may prescribe u, u_{ξ_1}, and $u_{\xi_1\xi_1}$ on S. It is easily shown that it is always possible to determine a function $g(\xi_1, \xi_2, \xi_3)$ which agrees with u on S and whose derivatives agree with the corresponding derivatives of u on S. This being the case, it is reasonable to introduce the following definition:

$$(3.5)\quad I_*^{p+1}f(x)=\frac{1}{\sqrt{3}}\iint_{S(x)}\Big\{\frac{1}{3}[f(v_{\xi_2\xi_3}+v_{\xi_3\xi_1}+v_{\xi_1\xi_2})+v(f_{\xi_2\xi_3}+f_{\xi_3\xi_1}+f_{\xi_1\xi_2})]$$
$$-\frac{1}{6}[(f_{\xi_2}+f_{\xi_3})v_{\xi_1}+(f_{\xi_3}+f_{\xi_1})v_{\xi_2}+(f_{\xi_1}+f_{\xi_2})v_{\xi_3}]\Big\}dS\,,$$

where v is defined by (3.1). We can then write (3.4) in the form

$$(3.6)\qquad I^pf(x)=I^{p+1}\Delta f(x)+I_*^{p+1}f(x)\,.$$

If we are to solve the differential equation $\Delta u=h(x)$ subject to the conditions that u and its first and second derivatives agree with g and its corresponding derivatives on S, then according to (3.6) and Theorem 2.1 we must have

$$(3.7)\qquad u(x)=I^1h(x)+\lim_{p\to 0} I_*^{p+1}g(x)$$

as the solution. We write the limit as $p\to 0$ in the second term on the right because some of the integrals fail to exist if $p=0$.

4. Lemmas for the evaluation of the surface integrals. The surface integral in (3.5) which is required for the solution of Cauchy's problem converges for $\mathscr{R}(p)>0$. In order to find the solution of Cauchy's problem according to equation (3.7) we need to show that the limit of $I_*^{p+1}g(x)$ exists when $p\to 0$. To verify that u and its derivatives assume the prescribed values on S it is necessary to differentiate (3.7). This is trivial for the first term on the right but not so simple for the second term. But if $\mathscr{R}(p)$ is sufficiently large the differentiation of $I_*^{p+1}g(x)$ is very easy. The resulting integrals fail to exist near $p=0$, and an analytic continuation is required. We wish to show how this analytic continuation can be accomplished and that instead of differentiating the second term on the right of (3.7) after letting $p\to 0$ we can differentiate $I_*^{p+1}(g)$ first and then let $p\to 0$. We, of course, make suitable assumptions concerning the differentiability of g.

We note that all of the integrals occurring in (3.5) are of the form

$$J^{\alpha,\beta,\gamma}f(x) = \frac{1}{\sqrt{3}\,\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\iint_{S(x)} f(\xi_1,\xi_2,\xi_3)(x_1-\xi_1)^{\alpha-1}(x_2-\xi_2)^{\beta-1}(x_3-\xi_3)^{\gamma-1}dS\ , \tag{4.1}$$

where we assume that $f(\xi_1, \xi_2, \xi_3)$ possesses continuous derivatives up to the first, second, or third order. We note that the integral in (4.1) converges when the real parts of α, β, and γ are greater than zero. We proceed to a study of this integral, proving a number of lemmas some of which are of interest in themselves.

The first lemma which we need is similar to one given by Riesz [3, p. 60].

LEMMA 4.1. *Let $G(u,v)$ be a function defined for $0\leq u\leq a<\infty$, $0\leq v<b\leq\infty$, and let it have continuous derivatives to the qth order. Then it may be written in the form*

$$G(u,v)=\pi(u,v)+\sum_{r=0}^{q-2}h_r(v)\frac{u^r}{r!}+k_0(u)+m(u,v)\ , \tag{4.2}$$

where

$$\pi(u,v)=\sum_{r=0}^{q-1}\sum_{s=0}^{q-r-1}\frac{G^{(r,s)}(0,0)}{r!\,s!}u^r v^s \tag{4.3}$$

and

$$h_r(v)=O(v^{q-r})\ ,\quad k_0(u)=O(u^q)\ ,\quad m(u,v)=O(u^{q-1}v)\ . \tag{4.4}$$

Here $G^{(r,s)}(u,v)=\partial^{r+s}G(u,v)/(\partial u^r\partial v^s)$.

Proof. If $G(u,v)$ could be expanded in a Maclaurin's series for sufficiently small u and v, the result would be obvious. Since we do not assume this we proceed as Riesz does. We write

$$\begin{aligned} h_r(v) &= G^{(r,0)}(0,v)-\sum_{s=0}^{q-r-1}G^{(r,s)}(0,0)\frac{v^s}{s!} \\ &= \frac{1}{(q-r-1)!}\int_0^v G^{(r,q-r)}(0,\eta)(v-\eta)^{q-r-1}d\eta\ . \end{aligned}$$

and

$$k_0(u)=G(u,0)-\sum_{r=0}^{q-1}G^{(r,0)}(0,0)\frac{u^r}{r!}=\frac{1}{(q-1)!}\int_0^u G^{(q,0)}(\xi,0)(u-\xi)^{q-1}d\xi\ .$$

Then

$$m(u, v)=G(u, v)-\pi(u, v)-\sum_{r=0}^{q-2} h_r(v)\frac{u^r}{r!}-k_0(u)$$

$$=G(u, v)-\sum_{r=0}^{q-2} G^{(r,0)}(0, v)\frac{u^r}{r!}-G(u, 0)+\sum_{r=0}^{q-2} G^{(r,0)}(0, 0)\frac{u^r}{r!}$$

$$=\frac{1}{(q-2)!}\int_0^u\int_0^v G^{(q-1,1)}(\xi, \eta)(u-\xi)^{q-2}d\eta d\xi .$$

The equalities are verified by integrations by parts, and the order relations are now evident.

Clearly the roles of u and v may be interchanged in equations (4.2) and (4.4). Moreover, other similar lemmas may be found giving different powers of u and v in the estimate of $m(u,v)$.

The second lemma is an immediate consequence of equations (2.7) and (2.8).

LEMMA 4.2. *If $d=x_1+x_2+x_3>0$, we have*

$$J^{\alpha,\beta,\gamma}1=d^{\alpha+\beta+\gamma-1}/\Gamma(\alpha+\beta+\gamma) . \tag{4.5}$$

If the real parts of α, β, or γ are less than or equal to zero, this formula provides an analytic continuation of the left member.

The next three lemmas provide the principal tools for use in § 5.

LEMMA 4.3. *Suppose that $f(\xi_1, \xi_2, \xi_3)$ has continuous derivatives up to the third order. Let $d=x_1+x_2+x_3>0$. Then $J^{\alpha,\beta,\gamma}f(x)$, defined by (4.1), can be continued analytically throughout the region R in α, β, γ space, where R is defined by the fact that one of the following three conditions holds:*

(a) $\mathscr{R}(\alpha)>-2$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$,

or

(b) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-2$, $\mathscr{R}(\gamma)>-1$,

or

(c) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$.

Moreover, $J^{\alpha,\beta,\gamma}f(x)$ assumes the following special values. (In all cases, if α_0, β_0, γ_0 is on the boundary of R, the formula is to be interpreted as meaning the limit as $\alpha\to\alpha_0$, $\beta\to\beta_0$, $\gamma\to\gamma_0$.)

$$J^{1,1,1}f(x)=3^{-1/2}\iint_{S(x)} f(\xi_1, \xi_2, \xi_3)dS , \tag{4.6}$$

$$(4.7)\qquad J^{1,1,0}f(x)=\int_{x_1-d}^{x_1} f(\xi_1,\ -\xi_1-x_3,\ x_3)d\xi_1\ ,$$

$$(4.8)\qquad J^{1,1,-1}f(x)=\int_{x_1-d}^{x_1}[f_{\xi_3}(\xi_1,\ -\xi_1-x_3,\ x_3)-f_{\xi_2}(\xi_1,\ -\xi_1-x_3,\ x_3)]d\xi_1 \\ +f(x_1-d,\ x_2,\ x_3)\ ,$$

$$(4.9)\qquad J^{1,1,-2}f(x)=\int_{x_1-d}^{x_1}[f_{\xi_3\xi_3}(\xi_1,\ -\xi_1-x_3,\ x_3)-2f_{\xi_3\xi_2}(\xi_1,\ -\xi_1-x_3,\ x_3) \\ +f_{\xi_2\xi_2}(\xi_1,\ -\xi_1-x_3,\ x_3)]d\xi_1 \\ +2f_{\xi_3}(x_1-d,\ x_2,\ x_3)-f_{\xi_1}(x_1-d,\ x_2,\ x_3)-f_{\xi_2}(x_1-d,\ x_2,\ x_3)\ .$$

$$(4.10)\qquad J^{1,0,0}f(x)=f(x_1-d,\ x_2,\ x_3)=f(-x_2-x_3,\ x_2,\ x_3)\ ,$$

$$(4.11)\qquad J^{1,0,-1}f(x)=f_{\xi_3}(x_1-d,\ x_2,\ x_3)-f_{\xi}\ (x_1-d,\ x_2,\ x_3)\ ,$$

$$(4.12)\qquad J^{1,0,-2}f(x)=f_{\xi_3\xi_3}(x_1-d,\ x_2,\ x_3)-2f_{\xi_1\xi_3}(x_1-d,\ x_2,x_3)+f_{\xi_1\xi_1}(x_1-d,x_2,x_3),$$

$$(4.13)\qquad J^{1,-1,-1}f(x)=f_{\xi_3\xi_2}(x_1-d,\ x_2,\ x_3)-f_{\xi_1\xi_3}(x_1-d,\ x_2,\ x_3) \\ -f_{\xi_1\xi_2}(x_1-d,\ x_2,\ x_3)+f_{\xi_1\xi_1}(x_1-d,\ x_2,\ x_3)\ ,$$

$$(4.14)\qquad J^{0,0,0}f(x)=0\ ,$$

$$(4.15)\qquad J^{0,0,-1}f(x)=0\ ,$$

$$(4.16)\qquad J^{0,-1,-1}f(x)=0\ .$$

Formulas analogous to these can be obtained by permuting the superscripts.

Proof. Since $J^{\alpha,\beta,\gamma}f(x)$ is defined by (4.1) and is analytic for $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$, equation (4.6) is immediate. To obtain equations (4.7), (4.8), and (4.9) we have

$$(4.17)\qquad J^{1,1,\gamma}f(x)=\frac{1}{\Gamma(\gamma)}\int_{-x_1-x_2}^{x_3} F(\xi_3)(x_3-\xi_3)^{\gamma-1}d\xi_3\ ,$$

where

$$(4.18)\qquad F(\xi_3)=\int_{-\xi_3-x_2}^{x_1} f(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)d\xi_1\ .$$

Then (4.17) is an ordinary Riemann-Liouville integral which can be continued analytically for $\mathscr{R}(\gamma)>-3$ since $F(\xi_3)$ has a continuous third derivative. Also, by (1.7), we have

$$(4.19)\qquad J^{1,1,\gamma}f(x)=F^{(-\gamma)}(x_3)\ ,\qquad (\gamma=0,\ -1,\ -2)\ .$$

Equation (4.7) follows by setting $\xi_3=x_3$ in (4.18). Also from (4.18) we have

$$\frac{dF}{d\xi_3}=\int_{-\xi_3-x_2}^{x_1}[f_{\xi_3}(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)-f_{\xi_2}(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)]d\xi_1 + f(-\xi_3-x_2,\ x_2,\ \xi_3)\ , \tag{4.20}$$

and

$$\begin{aligned}\frac{d^2F}{d\xi_3^2} &= \int_{-\xi_3-x_2}^{x_1}[f_{\xi_3\xi_3}(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)-2f_{\xi_3\xi_2}(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)+f_{\xi_2\xi_2}(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)]d\xi_1 \\ &\quad +2f_{\xi_3}(-\xi_3-x_2,\ x_2,\ \xi_3)-f_{\xi_2}(-\xi_3-x_2,\ x_2,\ \xi_3)-f_{\xi_1}(-\xi_3-x_2,\ x_2,\ \xi_3)\ .\end{aligned} \tag{4.21}$$

Equations (4.8) and (4.9) follow by setting $\xi_3=x_3$ in (4.20) and (4.21).

Turning our attention to equations (4.10)-(4.13), we shall express the integral (4.1) in terms of the variables ξ_2 and ξ_3 and use Lemma 4.1 with $q=3$ to expand $f(\xi_1, \xi_2, \xi_3)$ in the form

$$\begin{aligned} f(\xi_1,\ \xi_2,\ \xi_3) &= f(x_1-d,\ x_2,\ x_3)+(x_2-\xi_2)(f_{\xi_1}-f_{\xi_2})_0+(x_3-\xi_3)(f_{\xi_1}-f_{\xi_3})_0 \\ &\quad +\frac{(x_2-\xi_2)^2}{2}(f_{\xi_1\xi_1}-2f_{\xi_1\xi_2}+f_{\xi_2\xi_2})_0+\frac{(x_3-\xi_3)^2}{2}(f_{\xi_1\xi_1}-2f_{\xi_1\xi_3}+f_{\xi_3\xi_3})_0 \\ &\quad +(x_2-\xi_2)(x_3-\xi_3)(f_{\xi_1\xi_1}-f_{\xi_1\xi_3}-f_{\xi_1\xi_2}+f_{\xi_2\xi_3})_0+L(\xi_2,\ \xi_3)\end{aligned} \tag{4.22}$$

where

$$L(\xi_2,\ \xi_3)=L_1(\xi_2)+(x_3-\xi_3)L_2(\xi_2)+L_3(\xi_3)+L_4(\xi_2,\ \xi_3) \tag{4.23}$$

with

$$\begin{aligned} &L_1(\xi_2)=O((x_2-\xi_2)^3)\ , \qquad L_2(\xi_2)=O((x_2-\xi_2)^2)\ , \\ &L_3(\xi_3)=O((x_3-\xi_3)^3)\ , \qquad L_4(\xi_2,\ \xi_3)=O((x_2-\xi_2)(x_3-\xi_3)^2)\ .\end{aligned} \tag{4.24}$$

Here the subscript 0 indicates that the values of the derivatives are calculated at the point (x_1-d, x_2, x_3).

Considering the first six terms of (4.22), we deal with the term involving $(x_2-\xi_2)^\lambda(x_3-\xi_3)^\mu/(\lambda\,!\ \mu\,!)$ where $\lambda+\mu\leq 2$. The contribution to $J^{\alpha,\beta,\gamma}f(x)$ of this term is found to be

$$\frac{\Gamma(\beta+\lambda)\Gamma(\gamma+\mu)d^{\alpha+\beta+\gamma+\lambda+\mu-1}}{\lambda!\ \mu!\ \Gamma(\beta)\Gamma(\alpha)\Gamma(\alpha+\beta+\gamma+\lambda+\mu)}\ ,$$

by Lemma 4.2. We note that this function is analytic for all values of α, β, γ. When $\alpha=1$, $\beta=\gamma=0$, it reduces to 1 if $\lambda=\mu=0$ and to zero otherwise. When $\alpha=1$, $\beta=0$, $\gamma=-1$, it reduces to -1 if $\lambda=0$, $\mu=1$, and to zero otherwise. When $\alpha=1$, $\beta=0$, $\gamma=-2$, it reduces to 1 if $\lambda=0$, $\mu=2$, and to zero otherwise. When $\alpha=1$, $\beta=\gamma=-1$, it reduces to 1 if $\lambda=\mu=1$, and to zero otherwise. Thus these terms yield the values stated in equations (4.10)-(4.13). We have only to show that the contribution of $L(\xi_2, \xi_3)$ to $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R and reduces to zero when α, β, γ assume the values needed in (4.10)-(4.13).

We first show that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region R_1 where $\mathscr{R}(\alpha)\geq 2$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. We consider in turn the contributions arising from the four terms of $L(\xi_2, \xi_3)$ given in (4.23).

We have, for $L_1(\xi_2)$,

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_{-x_1-x_3}^{x_2} L_1(\xi_2)(x_2-\xi_2)^{\beta-1}d\xi_2\int_{-\xi_2-x_1}^{x_3}(x_3-\xi_3)^{\gamma-1}(x_1+\xi_2+\xi_3)^{\alpha-1}d\xi_3$$

$$=\frac{1}{\Gamma(\beta)\Gamma(\alpha+\gamma)}\int_{-x_1-x_3}^{x_2} L_1(\xi_2)(x_2-\xi_2)^{\beta-1}(x_1+x_3+\xi_2)^{\alpha+\gamma-1}d\xi_2$$

on using (2.7) and (2.8). On taking account of (4.24) we see that the integral is analytic in R_1. Moreover, the expression is zero if $\beta=0$ or -1 even when $\gamma\to-2$.

The contribution of $(x_3-\xi_3)L_2(\xi_2)$ is similarly

$$\frac{\gamma}{\Gamma(\beta)\Gamma(\alpha+\gamma+1)}\int_{-x_1-x_3}^{x_2} L_2(\xi_2)(x_2-\xi_2)^{\beta-1}(x_1+x_3+\xi_2)^{\alpha+\gamma}d\xi_2\ ,$$

which is also analytic in R_1. It is also zero if $\beta=0$ or -1 even when $\gamma=-2$.

The contribution of $L_3(\xi_3)$ is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_{-x_1-x_2}^{x_3} L_3(\xi_3)\,(x_3-\xi_3)^{\gamma-1}d\xi_3\int_{-\xi_3-x_1}^{x_2}(x_2-\xi_2)^{\beta-1}(x_1+\xi_2+\xi_3)^{\alpha-1}d\xi_2$$

$$=\frac{1}{\Gamma(\gamma)\Gamma(\alpha+\beta)}\int_{-x_1-x_2}^{x_3} L_3(\xi_3)(x_3-\xi_3)^{\gamma-1}(x_1+x_2+\xi_3)^{\alpha+\beta-1}d\xi_3\ .$$

The integral is again analytic in R_1. This contribution is zero if $\gamma=0$, -1, or -2 even when $\beta=-1$.

On taking account of (4.24) it is at once evident that the contribution of $L_4(\xi_2, \xi_3)$ is analytic in R_1 and vanishes when $\beta=0$ or $\gamma=-1$ even on the boundary of R_1.

Thus we have shown that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R_1. Since the roles of α, β, γ may be interchanged it can also be continued analytically throughout five similar regions obtained by permuting α, β, γ in the definition of R_1.

We note that $d=(x_1-\xi_1)+(x_2-\xi_2)+(x_3-\xi_3)$ on $S(x)$ and we multiply equation (4.1) through by these expressions to obtain

$$dJ^{\alpha,\beta,\gamma}f(x)=\alpha J^{\alpha+1,\beta,\gamma}f(x)+\beta J^{\alpha,\beta+1,\gamma}f(x)+\gamma J^{\alpha,\beta,\gamma+1}f(x)\ . \tag{4.25}$$

We use (4.25) to show that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R.

We first suppose that $\mathscr{R}(\alpha)\geq 1$. If $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$, $J^{\alpha,\beta,\gamma}f(x)$ is clearly analytic on using the integral definition in (4.1). If $\mathscr{R}(\beta)>1$, $\mathscr{R}(\gamma)>-1$, then $J^{\alpha+1,\beta,\gamma}f(x)$ is analytic since $(\alpha+1, \beta, \gamma)$ belongs to R_1, $J^{\alpha,\beta+1,\gamma}f(x)$ is analytic since $(\alpha, \beta+1, \gamma)$ belongs to a region similar to R_1, and $J^{\alpha,\beta,\gamma+1}f(x)$ is analytic by (4.1). Thus $J^{\alpha,\beta,\gamma}f(x)$ is analytic if $\mathscr{R}(\beta)>1$, $\mathscr{R}(\gamma)>-1$. We proceed in this way using (4.25) to show the possibility of continuing analytically $J^{\alpha,\beta,\gamma}f(x)$ in turn into the regions $\mathscr{R}(\beta)>1$, $\mathscr{R}(\gamma)>-2$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-2$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. At any stage we remember that the roles of β and γ can be interchanged where necessary. We conclude that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\alpha)\geq 1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$ and throughout five similar regions obtained by permuting α, β, γ.

We next suppose $\mathscr{R}(\alpha)\geq 0$, We proceed as before using (4.25) to show the possibility of continuing $J^{\alpha,\beta,\gamma}f(x)$ analytically in turn throughout the regions $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-2$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. We conclude that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\alpha)\geq 0$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$, and throughout five similar regions obtained by permuting α, β, γ.

We next suppose $\mathscr{R}(\alpha)>-1$. We have already shown that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-2$. We then use (4.25) to show that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically in turn throughout the regions $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. We conclude that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$, and throughout the two similar regions obtained by permuting α, β, γ. Thus we have shown that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R.

We have yet to show that the contribution of $L(\xi_2, \xi_3)$ to $J^{\alpha,\beta,\gamma}f(x)$ reduces to zero when α, β, γ assume the values needed in (4.10)-(4.13). If α were 2 instead of 1, and β and γ were as in (4.10)-(4.13), our analyticity discussion would show that this contribution is zero. If we apply (4.25) using L instead of f we find that the desired result follows easily. This completes the proof of formulas (4.10)-(4.13).

The formulas (4.14)-(4.16) follow immediately from equation (4.25).

If $f(x)$ has continuous derivatives up to only the second or first order we can still get results similar to Lemma 4.3, but the region into which $J^{\alpha,\beta,\gamma}f(x)$ can be continued will be smaller; however, those of formulas (4.6)-(4.16) which are still valid are unchanged. The method of proof is the same as for Lemma 4.3 and the results can be expressed in the form of two lemmas:

LEMMA 4.3.1. *If $f(\xi_1, \xi_2, \xi_3)$ has continuous derivatives up to the second order, then Lemma* 4.3 *holds if the region R is replaced by the region R^* in which* (a) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$, *or* (b) $\alpha=\beta=1$, $\mathscr{R}(\gamma)>-2$, *or* (c) $\alpha=\gamma=1$, $\mathscr{R}(\beta)>-2$, *or* (d) $\beta=\gamma=1$, $\mathscr{R}(\alpha)>-2$, *and if formulas* (4.12), (4.13), *and* (4.16) *are deleted.*

LEMMA 4.3.2. *If $f(\xi_1, \xi_2, \xi_3)$ has continuous derivatives of first order, then Lemma* 4.3 *holds if the region R is replaced by the region R^{**} in which* (a) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$, *or* (b) $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>0$, *or* (c) $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-1$, *and if only formulas* (4.6), (4.7), (4.8), (4.10), and (4.14) *are retained.*

From equation (4.1) it follows immediately that

$$\frac{\partial}{\partial_{x_1}} J^{\alpha+1,\beta,\gamma} f(x) = J^{\alpha,\beta,\gamma}(x) \tag{4.26}$$

as long as $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$. Similar formulas hold, of course, for derivatives with respect to x_2 and x_3. By analytic continuation the validity of (4.26) follows as long as (α, β, γ) lies in the interior of a region into which $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically. But even if (α, β, γ) should lie on the boundary of such a region, if it assumes one of the sets of values occurring in equations (4.6)–(4.16) then (4.26) remains valid, as is easily verified by carrying out the appropriate differentiation of the right members of equations (4.6)–(4.16).

The importance of this lies in the fact that it shows that in finding the derivative of $u(x)$ as given by (3.7) we may interchange the order of the limiting procedure $p\to 0$ and the differentiation in the term $I_*^{p+1}g(x)$. This simplifies materially the task of verifying that (3.7) gives the solution of Cauchy's problem for the differential equation (1.1).

5. The solution of Cauchy's problem for the equation $\Delta u=h(x)$. It has already been pointed out in § 3 that if the Cauchy problem for the differential equation (1.1) is to have a solution, this solution must be given by (3.7). We are now able to prove the following theorem which gives the solution of Cauchy's problem.

THEOREM 5.1. *Let $h(x)$ be continuous and let $g(x)$ have continuous derivatives up to the third order in the region $x_1+x_2+x_3\geq 0$. Then, in the notation of equations* (2.9) *and* (3.5),

$$u(x)=I^1h(x)+\lim_{p\to 0} I_*^{p+1}g(x) \tag{5.1}$$

is, when $x_1+x_2+x_3\geq 0$, *a solution of the equation* $\Delta u=h(x)$; *moreover, when* $x_1+x_2+x_3=0$, *we have* $u(x)=g(x)$, *and all the derivatives of* $u(x)$ *of first and second order equal the corresponding derivatives of* $g(x)$.

Proof. We first note that

$$I^1h(x)=\iiint_{D_S(x)} h(\xi)d\xi=\int_{-x_1-x_2}^{x_3}\int_{-\xi_3-x_1}^{x_2}\int_{-\xi_2-\xi_3}^{x_1} h(\xi_1, \xi_2, \xi_3)d\xi_1 d\xi_2 d\xi_3 , \tag{5.2}$$

and

$$\begin{aligned} I_*^{p+1}g(x)= & \frac{1}{3}[J^{p+1,p,p}\, g(x)+J^{p,p+1,p}\, g(x)+J^{p,p,p+1}g(x) \\ & +J^{p+1,p+1,p+1}(g_{x_2x_3}+g_{x_3x_1}+g_{x_1x_2})] \\ & +\frac{1}{6}[J^{p,p+1,p+1}(g_{x_2}+g_{x_3})+J^{p+1,p,p+1}(g_{x_3}+g_{x_1})+J^{p+1,p+1,p}(g_{x_1}+g_{x_2})] , \end{aligned} \tag{5.3}$$

by (3.5), (3.1), and (4.1).

We now verify that (5.1) satisfies the differential equation $\Delta u=h(x)$. We have

$$\Delta u=\Delta I^1h(x)+\lim_{p\to 0}\Delta I_*^{p+1}g(x) \tag{5.4}$$

on account of the remark at the end of § 4. If (5.2) is used, an elementary calculation shows that $\Delta I^1h(x)=h(x)$. It follows directly from (3.5) and (3.1) that

$$\Delta I_*^{p+1}g(x)=I_*^p g(x) \tag{5.5}$$

if $\mathscr{R}(p)>1$, and a suitable analytic continuation as indicated in § 3 establishes the validity of (5.5) for $\mathscr{R}(p)>0$. If we now let $p\to 0$ and make use of (5.5), (5.3), and (4.14)-(4.16), we find that

$$\lim_{p\to 0}\Delta I_*^{p+1}g(x)=\lim_{p\to 0} I_*^p g(x)=\lim_{p\to -1} I_*^{p+1}g(x)=0 . \tag{5.6}$$

This completes the verification that (5.1) satisfies the differential equation $\Delta u=h(x)$.

Next we show that $u(x)$ assumes the correct value $g(x)$ on the plane S whose equation is $x_1+x_2+x_3=0$. We consider $u(x)$ at the point $x=(x_1, x_2, x_3)$, where $x_1+x_2+x_3=d>0$, and let $d\to 0$. From (5.1), (5.3), and (4.10), we find that

$$\begin{aligned} u(x)= & I^1h(x) \\ & +\frac{1}{3}[g(x_1-d, x_2, x_3)+g(x_1, x_2-d, x_3)+g(x_1, x_2, x_3-d)+J^{1,1,1}(g_{x_2x_3}+g_{x_3x_1}+g_{x_1x_2})] \\ & +\frac{1}{6}[J^{0,1,1}(g_{x_2}+g_{x_3})+J^{1,0,1}(g_{x_3}+g_{x_1})+J^{1,1,0}(g_{x_1}+g_{x_2})] . \end{aligned} \tag{5.7}$$

Since $h(x)$ is continuous, equation (5.2) shows that $I^1h(x)=O(d^3)$. On account of Lemma 4.2, we see that if $f(x)$ is continuous, and α, β, γ are real and nonnegative, then

$$J^{\alpha,\beta,\gamma}f(x)=O(d^{\alpha+\beta+\gamma-1}) . \tag{5.8}$$

Thus when x approaches S, that is, when $d\to 0$, (5.7) shows that $u(x)\to g(x)$.

If it is desired, $u(x)$ can be written explicitly in terms of $h(x)$ and $g(x)$ and its derivatives by using (4.6) and (4.7).

Next we consider $\partial u/\partial x_1$. On account of the remark at the end of § 4 we have, from (5.1),

$$\frac{\partial u}{\partial x_1}=\frac{\partial I^1h(x)}{\partial x_1}+\lim_{p\to 0}\frac{\partial I_*^{p+1}g(x)}{\partial x_1} . \tag{5.9}$$

We calculate $\partial I_*^{p+1}g(x)/\partial x_1$ by differentiating (5.3) and using (4.26). We then let $p\to 0$ and make use of equations (4.14), (4.11), (4.7), (4.8), and (4.10). On using equation (5.2) it is easily verified that $\partial I^1h(x)/\partial x_1=O(d^2)$, and hence tends to zero with d. We also note that the integrals in (4.7) and (4.8) tend to zero with d. We thus find that $\partial u/\partial x_1\to g_{x_1}(x_1, x_2, x_3)$ when x approaches S. In the same way we can consider $\partial u/\partial x_2$ and $\partial u/\partial x_3$.

In a similar manner we treat $\partial^2u/\partial x_i^2$ $(i=1, 2, 3)$. We have only to use equations (4.15), (4.12), (4.8), (4.9), and (4.11) and observe that $\partial^2I^1h(x)/\partial x_i^2=O(d)$.

The treatment of $\partial^2u/\partial x_i\partial x_j$ $(i, j=1, 2, 3\,;\ i\neq j)$ is also similar and makes use of equations (4.15), (4.13), (4.10), (4.11), and (4.14).

This completes the verification of the solution.

In Theorem 2.1 we showed that, if $f(x)$ is continuous, $I^pf(x)$ is analytic for $\mathscr{R}(p)>0$ and $I^pf(x)\to f(x)$ when $p\to 0$. The following theorem shows that $I^pf(x)$ can be continued analytically when $f(x)$ is sufficiently differentiable.

THEOREM 5.2. *If $f(x)$ has continuous derivatives up to the third order in the region $x_1+x_2+x_3\geq 0$ then $I^pf(x)$ can be continued analytically throughout the region $\mathscr{R}(p)>-1$, and*

$$\lim_{p\to -1}I^pf(x)=\Delta f(x) \tag{5.10}$$

if $x_1+x_2+x_3>0$.

Proof. We make use of equations (3.6) and (5.3) with $g(x)$ replaced by $f(x)$. Then if $\mathscr{R}(p)>-1$, Theorem 2.1 shows that $I^{p+1}\Delta f(x)$ is analytic, and Lemmas 4.3, 4.3.1, and 4.3.2 show that $I_*^{p+1}f(x)$ is analytic. If we let $p \to -1$, equation (5.10) is a consequence of Theorem 2.1 and the last equality in equations (5.6) with $f(x)$ in place of $g(x)$.

References

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