BOOLEAN ALGEBRAS WITH PATHOLOGICAL ORDER TOPOLOGIES

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If L is a partially ordered set, there are a variety of known ways in which L may be given a topology compatible, in some sense, with its partial ordering (see [1, 6]). Examples, by Northam [3] and Floyd and Klee [2], have very recently appeared of complete lattices which are not Hausdorff in their order topologies. It appears, then, that the various topologies will not be central in the study of *all* complete lattices. The question remains as to whether or not there is some wide and natural class of lattices in which some compatible topology has nice properties. We give a very simple example of a complete Boolean algebra which is not Hausdorff in any topology compatible with the order. We also give an example of a conditionally complete vector lattice in which addition is not continuous in any compatible topology. This is a counterexample to a result of Birkhoff [1, p. 242], who overlooked the possibility that convergence in the order topology differs from order convergence.

DEFINITION. Suppose that $(P, \geq)$ is a partially ordered set, and suppose that T is a topology for the set P (that is, T is a collection of subsets of P closed under arbitrary unions and finite intersections, and with $\phi \in T$, $P \in T$). We say that T is *σ-compatible with* $\geq$ if and only if whenever (x_i) is a sequence in P with

$$x_1 \geq x_2 \geq \cdots \text{ and } \bigwedge_i x_i = x$$

or

$$x_1 \leq x_2 \leq \cdots \text{ and } \bigvee_i x_i = x\,,$$

then the sequence (x_i) T-converges to x.

THEOREM 1. *Let L denote the complete Boolean algebra of all regular open subsets of the unit interval I, partially ordered by inclusion $>$. Suppose that T is a topology for L which is σ-compatible with $>$. Then the topology T is not Hausdorff.*

Proof. Recall that a subset b of I is a regular open set if and only if b is the interior of its closure. L is known to be a complete

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Boolean algebra [1]. Let $\mathscr{U}$ be a T-neighbourhood of the empty set $\phi \in L$. We show that $I \in \overline{\mathscr{U}}$. Suppose that $U_1, U_2, \cdots$, is a basis for the open sets of I, with each U_i nonempty. There exists for each i a sequence $(A^i_j | j=1, 2, \cdots)$ in L with

$$A^i_j \subset U_i, \quad A^i_j \neq \phi,$$

so that $A^i_1 > A^i_2 > \cdots$ and

$$\bigwedge (A^i_j | j=1, 2, \cdots) = \phi.$$

Since (A^1_i) converges to ϕ, there exists $A^1_i \in \mathscr{U}$. Define $B_1 = A^1_i$. Since the sequence $(B_1 \vee A^2_j)$ converges to B_1, there exists j with $B_1 \vee A^2_j \in \mathscr{U}$. Define $B_2 = B_1 \vee A^2_j$. Similarly there exists $B_3 = B_2 \vee A^3_k \in \mathscr{U}, \cdots$. Now (B_i) is a sequence in $\mathscr{U}$ with $B_1 < B_2 < \cdots$. Moreover, since the only regular open set containing $\bigcup B_i$ is I, we have $\bigvee_i B_i = I$. Hence $I \in \overline{\mathscr{U}}$ and the theorem follows.

The following remark answers Problem 77 of Birkhoff [1, p. 167].

THEOREM 2. *If L is the complete Boolean algebra of Theorem 1, then there exist, for $i=1, 2 \cdots$, sequences $(X_{i,j} | j=1, 2, \cdots)$ with $(X_{i,j})$ order-converging to ϕ for each i but such that for no function $j(i)$ is it true that $(X_{i, j(i)})$ order-converges to ϕ.*

Proof. Let $(X_{i,j})$ denote the sequence (A^i_j) of the proof of Theorem 1. Consider any function $j(i)$, then

$$\bigvee_{i \geq k} A^i_{j(i)} = I\,.$$

Hence

$$\bigwedge_k \bigvee_{i \geq k} A^i_{j(i)} = I\,.$$

Hence the sequence $(X_{i, j(i)})$ does not order-converge to ϕ.

THEOREM 3. *Let L be the complete Boolean algebra of Theorem 1, and let M be a Stone representation space for L. Let N denote the lattice of all continuous real-valued functions on M. Then N is a conditionally complete vector lattice in which the function $x-y$ is not T-continuous simultaneously in x and y for any T_1-topology T for N which is σ-compatible with $>$.*

Proof. It is known [4, 7] that N is conditionally complete. We may consider L as identical with the algebra of all open and closed subsets of M. There is a function $t: L \to N$ which assigns to $u \in L$ the characteristic function $t(u)$ of the open and closed set u. We show that t is an embedding of L in N. It is seen that t is an isotone one-to-one map of L onto $t(L)$, and t^{-1} is an isotone map of $t(L)$ on L. We prove that if $K \subset L$ then

$$\bigvee t(K) = t(\bigvee K) ,$$

where $\bigvee t(K)$ denotes the least upper bound in N. Clearly

$$t(\bigvee K) \geq \bigvee t(K) .$$

Now $\bigvee t(K)$ is a nonnegative continuous function whose value is ≥ 1 on the set $\bigcup K$, and hence ≥ 1 also on its closure. But the closure of $\bigcup K$ is $\bigvee K$ [7]. Hence

$$t(\bigvee K) \leq \bigvee t(K)$$

and equality holds. The dual also follows. So t embeds L in N. It follows that $t(L)$ is not Hausdorff in the topology T restricted to $t(L)$. Hence N is not Hausdorff in the topology T. But if $x-y$ is T-continuous in x and y, it is known that N is then regular [5, p. 54] and hence Hausdorff.

COROLLARY. *Suppose, in addition to the hypotheses of Theorem* 3, *that the function $y \to -y$ on N is T-continuous. Then $x+y$ is not T-continuous in x and y simultaneously.*

This answers, in the negative, a part of Problem 4 of Rennie [6, p. 51].

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