ON UNIFORM DISTRIBUTION MODULO A SUBDIVISION

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1. **Let** Δ be a subdivision of the interval $(0, \infty)$: $\Delta = (z_0, z_1, \cdots)$, where

$$0 = z_0 < z_1 < \cdots \quad \text{and} \quad \lim_{n \to \infty} z_n = \infty.$$

For $z_{n-1} \le x < z_n$, put

$$[x]_\Delta = z_{n-1}, \quad \delta(x) = z_n - z_{n-1}, \quad \langle x \rangle_\Delta = \frac{x - [x]_\Delta}{\delta(x)}, \quad \phi(x) = n + \langle x \rangle_\Delta,$$

so that $0 \le \langle x \rangle_\Delta < 1$. Let $\{x_k\}$ be an increasing sequence of positive numbers. If the sequence $\{\langle x_k \rangle_\Delta\}$ is uniformly distributed over $[0, 1]$, in the sense that the proportion of the numbers $\langle x_1 \rangle_\Delta, \cdots, \langle x_k \rangle_\Delta$ which lie in $[0, \alpha)$ approaches α as $k \longrightarrow \infty$, for each $\alpha \in [0, 1)$, then we shall say that the sequence $\{x_k\}$ is *uniformly distributed modulo* Δ. If Δ is the subdivision Δ_0 for which $z_n = n$, this reduces to the ordinary concept of uniform distribution (mod 1), since then $[x]_\Delta = [x]$, $\delta(x) = 1$ for all x, and $\langle x \rangle_\Delta = x - [x]$ is the fractional part of x. Even in other cases, the generalization is more apparent than real, since the uniform distribution of one sequence (mod Δ) is equivalent to the uniform distribution of another sequence (mod 1). But most of the known theorems concerning uniform distribution (mod 1) are not applicable to the sequences $\{\langle x_k \rangle_\Delta\}$, if Δ is not Δ_0, for in such theorems x_k is ordinarily taken to be the value $f(k)$ of a function whose derivative exists and is monotonic for positive x. Here, on the other hand, $\langle x_k \rangle_\Delta \equiv \phi(x_k)$ (mod 1), and ϕ, although a continuous polygonal function, is not necessarily everywhere differentiable; and unless $\delta(x)$ is assumed monotonic, ϕ' is not monotonic even over the set on which it exists. This lack of monotonicity introduces serious difficulties; it is the object of the present work to show how they can be dealt with in certain cases.

For brevity, "uniformly distributed" will be abbreviated to "u.d.". The symbols "$\uparrow$", "$\nearrow$", "$\downarrow$" and "$\searrow$" indicate monotonic approach: increasing, non-decreasing, decreasing, and non-increasing, respectively.

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2. Put

$$N(\alpha, x) = \sum_{\substack{x_k \leq x \\ \langle x_k \rangle_\Delta < \alpha}} 1, \quad N(x) = N(1, x);$$

then $\{x_k\}$ is u.d. (mod Δ) if and only if, for each $\alpha \in [0, 1)$,

$$\lim_{x \to \infty} \frac{N(\alpha, x)}{N(x)} = \alpha.$$

THEOREM 1. *A necessary condition that* $\{x_k\}$ *be u.d.* (mod Δ) *is that*

$$N(z_{n+1}) \sim N(z_n)$$

as $n \longrightarrow \infty$.

For suppose that $\{x_k\}$ is u.d. (mod Δ). Then since

$$N\left(\frac{1}{2}, \frac{z_n + z_{n+1}}{2}\right) - N(1/2, z_n) = N\left(\frac{z_n + z_{n+1}}{2}\right) - N(z_n),$$

we have

$$\frac{1}{2} \sim \frac{N(1/2, (z_n + z_{n+1})/2)}{N((z_n + z_{n+1})/2)} = \frac{N(1/2, z_n)}{N((z_n + z_{n+1})/2)} + \frac{N((z_n + z_{n+1})/2) - N(z_n)}{N((z_n + z_{n+1})/2)}$$

$$= \frac{N(1/2, z_n)}{N(z_n)} \cdot \frac{N(z_n)}{N((z_n + z_{n+1})/2)} + 1 - \frac{N(z_n)}{N((z_n + z_{n+1})/2)}$$

$$= 1 + \frac{N(z_n)}{N((z_n + z_{n+1})/2)} \left(\frac{N(1/2, z_n)}{N(z_n)} - 1\right) \sim 1 - \frac{1}{2} \frac{N(z_n)}{N((z_n + z_{n+1})/2)}$$

as $n \longrightarrow \infty$, and so

$$N(z_n) \sim N\left(\frac{z_n + z_{n+1}}{2}\right).$$

In the same way it can be shown that

$$N\left(\frac{z_n + z_{n+1}}{2}\right) \sim N(z_{n+1}),$$

and consequently $N(z_n) \sim N(z_{n+1})$.

3. The following theorem, due in a slightly different form to Fejér (see [1, p. 88-89]), expresses the fact that if f is sufficiently smooth and $[f(x)]$ is constant over increasingly long intervals as x increases, such that the length of the n-th interval is of smaller order of magnitude than the total length of all preceding intervals, then $f(k)$ is u.d. (mod 1):

Suppose that $f(x)$ has the following properties:

(i) *f is continuously differentiable for $x > x_0$,*

(ii) *$f(x) \uparrow \infty$ as $x \uparrow \infty$,*

(iii) *$f'(x) \searrow 0$ as $x \uparrow \infty$,*

(iv) *$xf'(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.*

Then $f(k)$ is u.d. (mod 1).

The following theorem uses the same general idea:

THEOREM 2. *Suppose that, for a given subdivision Δ and a sequence $\{x_k\}$, $N(z_n) - N(z_{n-1}) \longrightarrow \infty$ as $n \longrightarrow \infty$. Then $\{x_k\}$ is u.d.* (mod Δ) *if the following conditions are satisfied:*

(i) *$N(z_{n-1}) \sim N(z_n)$ as $n \longrightarrow \infty$,*

(ii) *except possibly on a sequence of intervals $[z_{n_t-1}, z_{n_t})$ such that*

$$\sum_{t=1}^{m} (N(z_{n_t}) - N(z_{n_t-1})) = o(N(z_{n_m})), \tag{1}$$

the relation

$$\max(x_k - x_{k-1}) \sim \min(x_k - x_{k-1})$$

holds as $n \longrightarrow \infty$, the maximum and minimum being taken independently, for given $n \neq n_1, n_2, \cdots$, over all k for which at least one of x_{k-1} and x_k is in $[z_{n-1}, z_n]$.

Give the name δ_n to the interval $[z_{n-1}, z_n]$, and put

$$N(\alpha, \delta_n) = N(z_{n-1} + \alpha(z_n - z_{n-1})) - N(z_{n-1}),$$

$$N(\delta_n) = N(1, \delta_n) = N(z_n) - N(z_{n-1}).$$

It will be shown that

$$\lim_{\substack{n \to \infty \\ n \neq n_1, n_2, \cdots}} \frac{N(\alpha, \delta_n)}{N(\delta_n)} = \alpha;$$

in other words, that in the limit the x_k's which lie in $\delta_n \neq \delta_{n_t}$ are u.d. there. This implies the theorem, for using it, (1), and (i) we have, for $x \in \delta_n$,

$$\frac{N(\alpha, x)}{N(x)} = \frac{1}{N(x)} \left\{ \sum_{\nu=1}^{n-1} N(\alpha, \delta_n) + N(\min(x, z_{n-1} + \alpha(z_n - z_{n-1}))) - N(z_{n-1}) \right\}$$

$$= \frac{1}{N(x)} \sum{}^{\circ} N(\alpha, \delta_\nu) + o(1)$$

$$= \frac{\sum^{\circ} (\alpha + o(1))\, N(\delta_\nu)}{\sum^{\circ} N(\delta_\nu) + o\left(\sum^{\circ} N(\delta_\nu)\right) + N(x) - N(z_{n-1})} + o(1)$$

$$= \frac{\alpha}{1 + o(1)} + o(1) = \alpha + o(1),$$

where $\sum^{\circ}$ denotes summation from $\nu = 1$ to $\nu = n - 1$, $\nu \neq n_1, n_2, \cdots$.

To prove (2), suppose that $n \neq n_1, n_2, \cdots$, that $z_{n-1} \in (x_{k_n}, x_{k_n+1}]$, and that

$$\min_{k_n \leq k \leq k_{n+1}} (x_k - x_{k-1}) = X_n.$$

Then for $k_n \leq k \leq k_{n+1}$, we have $x_k - x_{k-1} = (1 + \epsilon_{kn}) X_n$, where ϵ_{kn} is a positive quantity tending to zero as $n \longrightarrow \infty$. Put

$$\epsilon_n = \max_{k_n \leq k \leq k_{n+1}} \epsilon_{kn},$$

and put $\Delta x_k = x_k - x_{k-1}$. Now if

$$x_{k_n+t} \leq z_{n-1} + \alpha(z_n - z_{n-1}) < x_{k_n+t+1},$$

then

$$\alpha(z_n - z_{n-1}) = (x_{k_n+1} - z_{n-1}) + \sum_{k=k_n+2}^{k_n+t} \Delta x_k + (z_{n-1} + \alpha(z_n - z_{n-1}) - x_{k_n+t})$$

$$= \sum_{s=1}^{t} \Delta x_{k_n+s} + \epsilon_n' X_n,$$

where $\epsilon_n' = O(1)$ as $n \longrightarrow \infty$. But

$$tX_n \leq \sum_{s=1}^{t} \Delta x_{k_n+s} \leq tX_n + t\epsilon_n X_n \leq tX_n + u\epsilon_n X_n,$$

where $u = N(z_n) - N(z_{n-1})$. Hence

$$\alpha \frac{z_n - z_{n-1}}{X_n} - \epsilon_n' - u\epsilon_n \leq t \leq \alpha \frac{z_n - z_{n-1}}{X_n} - \epsilon_n'.$$

Similarly,

$$\frac{z_n - z_{n-1}}{X_n} - \epsilon_n' - u\epsilon_n \leq u \leq \frac{z_n - z_{n-1}}{X_n} - \epsilon_n',$$

so that

$$\frac{\alpha(z_n - z_{n-1})/X_n - \epsilon_n' - u\epsilon_n}{(z_n - z_{n-1})/X_n - \epsilon_n'} \leq \frac{t}{u} \leq \frac{(z_n - z_{n-1})/X_n - \epsilon_n'}{(z_n - z_{n-1})/X_n - \epsilon_n' - u\epsilon_n}.$$

Since $N(z_n) - N(z_{n-1}) \longrightarrow \infty$ as $n \longrightarrow \infty$, also $(z_n - z_{n-1})/X_n \longrightarrow \infty$, and so

$$\frac{\alpha + o(1) - u\epsilon_n X_n/(z_n - z_{n-1})}{1 + o(1)} \leq \frac{t}{u} \leq \frac{\alpha + o(1)}{1 + o(1) - u\epsilon_n X_n/(z_n - z_{n-1})}.$$

But since

$$uX_n \leq \sum_{k=k_n+1}^{k_{n+1}} \Delta x_k \leq z_n - z_{n-1},$$

$uX_n = O(z_n - z_{n-1})$; thus

$$\frac{\alpha + o(1)}{1 + o(1)} \leq \frac{t}{u} \leq \frac{\alpha + o(1)}{1 + o(1)},$$

and therefore

$$\frac{N(\alpha, \delta_n)}{N(\delta_n)} = \frac{t}{u} \sim \alpha.$$

This completes the proof.

In case $\Delta = \Delta_0$ and $x_k = f(k)$, it is easily seen that the hypotheses of Fejér's theorem imply two of the hypotheses of Theorem 2, namely that $N(z_n) -$

$N(z_{n-1}) \uparrow \infty$ and $N(z_{n-1}) \sim N(z_n)$ as $n \longrightarrow \infty$. But I do not know whether Theorem 2 includes Fejér's theorem; the most that I can show is that the exceptional sequence $\{z_{n_t}\} = \{n_t\}$ mentioned in (ii) of Theorem 2 is in this case of density zero, which does not imply (1) for all functions f satisfying the hypotheses of Fejér's theorem. Certainly, however, Theorem 2 deals with cases not covered by the following direct extension of Fejér's theorem, since it does not require the monotonicity of either $z_n - z_{n-1}$ or Δx_k.

THEOREM 3. *The sequence* $\{x_k\}$ *is u.d.* (mod Δ) *if the following conditions are satisfied*:

(i) $z_n - z_{n-1} \geq z_{n-1} - z_{n-2}$ *for* $n = 2, 3, \cdots$,

(ii) $\Delta x_k \downarrow 0$ *as* $k \uparrow \infty$,

(iii) $N(z_{n-1}) \sim N(z_n)$ *as* $n \longrightarrow \infty$.

We sketch the proof. Let ψ be the continuous polygonal function such that $\psi(x_k) = k$; then $0 \leq \psi(x) - N(x) < 1$. Let $\{\epsilon_k\}$ be such that $\epsilon_k = o(\Delta x_k)$ and $0 < \epsilon_k < \Delta x_k/2$ for $k = 1, 2, \cdots$. Define ψ_1 as follows:

$$\psi_1(x) = \frac{1}{2\epsilon_k} \int_{x-\epsilon_k}^{x+\epsilon_k} \psi(t)\, dt \quad \text{for} \quad x \in \left[x - \frac{1}{2}\Delta x_k,\, x_k + \frac{1}{2}\Delta x_{k+1}\right]$$

$$(k = 2, 3, \cdots).$$

Then ψ_1 is continuously differentiable, and is identical with ψ except at the corners of ψ, where it is smooth. For $0 \leq \alpha \leq 1$, $n = 1, 2, 3, \cdots$, put

$$\rho(n + \alpha) = \psi_1(z_{n-1} + \alpha(z_n - z_{n-1}));$$

ρ is continuously differentiable except at $x = 1, 2, \cdots$. A function ρ_1 can now be defined in terms of ρ, just as ψ_1 was determined from ψ, so that ρ_1 is everywhere continuously differentiable, and ρ_1 differs from ρ only on an interval about $x = n (n = 1, 2, \cdots)$ whose length ϵ_n' is of lower order of magnitude than Δx_{k_n} if $z_n \in [x_{k_n - 1}, x_{k_n})$. If $x = n + \alpha$ is such that

$$\rho_1(x) = \rho(x), \quad \psi_1(z_{n-1} + \alpha(z_n - z_{n-1})) = \psi(z_{n-1} + \alpha(z_n - z_{n-1})),$$

and

$$z_{n-1} + \alpha(z_n - z_{n-1}) \in (x_{k-1}, x_k),$$

then

$$\rho_1'(x) = \frac{z_n - z_{n-1}}{\Delta x_k};$$

it follows that $\rho_1'(x) \nearrow \infty$. Moreover, since

$$\frac{\rho_1(n+1)}{\rho_1(n)} \sim \frac{\psi(z_n)}{\psi(z_{n-1})} \sim \frac{N(z_n)}{N(z_{n-1})} \longrightarrow 1,$$

it follows that $\rho_1'(x)/\rho_1(x) \longrightarrow 0$ as $x \longrightarrow \infty$. But if f is the function inverse to ρ_1, these facts imply that $f(x) \uparrow \infty$, $f'(x) \searrow 0$, and $xf'(x) \longrightarrow \infty$ as $x \uparrow \infty$. Since $f(k) \longrightarrow x_k$ as the arbitrary numbers ϵ_k and ϵ_n' approach zero, the conclusion follows from Fejér's theorem.

A trivial variation of Theorem 3 has, instead of (i) and (ii), the hypotheses

(i′) $z_n - z_{n-1} \uparrow \infty$,

(ii′) $\Delta x_{k-1} \geq \Delta x_k$ for $k = 2, 3, \cdots$.

For then it will still be true that $\rho_1'(x) \nearrow \infty$ as $x \uparrow \infty$.

4. It follows from Theorem 2 (and also from the variation of Theorem 3 just mentioned) that *if $z_n - z_{n-1} \nearrow \infty$ in such a way that $z_{n-1} \sim z_n$, the sequence $\{k\theta\}$ is u.d.* (mod Δ) *for each $\theta > 0$.* In this section we examine the distribution of $\{k\theta\}$ (mod Δ) when $\delta(x) \searrow 0$. This is a problem of a very different kind from the earlier one; the result is expressed in the following metric theorem:

THEOREM 4. *If $\delta(x) \searrow 0$ and $\delta(x) = O(x^{-1})$ then $\{k\theta\}$ is u.d.* (mod Δ) *for almost all $\theta > 0$.*

The proof depends on a principle used in an earlier paper [2]:

If C and ϵ are positive constants and $\{f_k\}$ is a sequence of real-valued functions such that

$$(3) \quad \left|\int_a^b e^{i(f_j(x) - f_k(x))}\,dx\right| \leq \frac{C}{\max(1, |j-k|^{\epsilon})}, \qquad (j, k = 1, 2, \cdots),$$

then $\{f_k(x)\}$ is u.d. (mod 1) *for almost all $x \in (a, b)$.*

This will be applied with $f_k(x) = \phi(kx)$, where ϕ is the function defined in §1; it was noted there that the u.d. (mod Δ) of $\{x_k\}$ is equivalent to the u.d. (mod 1) of $\{\phi(x_k)\}$. Let a and b be arbitrary positive numbers with $a < b$, and put

$$J_{jk} = \int_a^b e^{i(f_j(x) - f_k(x))}\, dx\,;$$

since J_{kj} and J_{jk} are complex conjugates, it suffices to consider the case $j > k$. For fixed j and k, denote by $\xi_0, \cdots \xi_r$ all the numbers of the form z_m/j or z_m/k in the interval (a, b), so named that $\xi_0 < \cdots < \xi_r$. Then the function

$$f_j(x) - f_k(x) = \left(\frac{j}{\delta(jx)} - \frac{k}{\delta(kx)}\right) x - \left(\frac{[jx]_\Delta}{\delta(jx)} - \frac{[kx]_\Delta}{\delta(kx)}\right) = xA(x) + B(x)$$

is linear in each interval $[\xi_{l-1}, \xi_l)$, $A(x)$ and $B(x)$ being certain constants A_l and B_l there. Hence

$$J_{jk} = \sum_{l=1}^{r} \int_{\xi_{l-1}}^{\xi_l} e^{i(A_l x + B_l)}\, dx = \sum_{l=1}^{r} \frac{e^{i(A_l\xi_l + B_l)} - e^{i(A_l\xi_{l-1} + B_l)}}{iA_l}.$$

Since f is continuous,

$$A_l\, \xi_l + B_l = A_{l+1}\, \xi_l + B_{l+1},$$

and so for $1 \le t \le r$,

$$\sum_{l=1}^{t} [e^{i(A_l\xi_l + B_l)} - e^{i(A_l\xi_{l-1} + B_l)}] = e^{i(A_t\,\xi_t + B_t)} - e^{i(A_1\,\xi_0 + B_1)}$$

Thus, using the relation

$$\sum_{m=1}^{n} a_m b_m = \sum_{m=1}^{n-1} \left(\sum_{\mu=1}^{m} a_\mu\right)(b_m - b_{m+1}) + b_n \sum_{\mu=1}^{n} a_\mu,$$

we have

$$J_{jk} = \frac{1}{i} \sum_{t=1}^{r-1} (e^{i(A_t\,\xi_t + B_t)} - e^{i(A_1\,\xi_0 + B_1)})\left(\frac{1}{A_t} - \frac{1}{A_{t+1}}\right)$$

$$+ (e^{i(A_r\xi_r + B_r)} - e^{i(A_1\,\xi_0 + B_1)})\,\frac{1}{iA_r},$$

and so

$$|J_{jk}| \le 2 \sum_{t=1}^{r-1} \left| \frac{1}{A_t} - \frac{1}{A_{t+1}} \right| + \frac{2}{|A_r|}. \tag{4}$$

By the facts that $\xi_t \ge a > 0$, $\delta(x) \searrow 0$ as $x \longrightarrow \infty$, and

$$A_t = \frac{j}{\delta(j\xi_{t-1})} - \frac{k}{\delta(k\xi_{t-1})},$$

it is clear that

$$A_t > C(j-k) > 0$$

for $t = 1, 2, \cdots, r$, so that (3) will follow from (4) if it can be shown that for some c, $\epsilon > 0$, the inequality

$$\sum_{t=1}^{r-1} \left| \frac{1}{A_t} - \frac{1}{A_{t+1}} \right| < \frac{c}{(j-k)^\epsilon}$$

holds. Moreover, writing

$$C_t = \frac{1}{A_t} - \frac{1}{A_{t+1}}$$

and

$$\sum_{t=1}^{r-1} |C_t| = \sum_{t=1}^{r} C_t - 2\sum{}' C_t = \frac{1}{A_1} - \frac{1}{A_r} - 2 \sum{}' C_t,$$

where $\sum'$ is the sum over those t for which $C_t < 0$, we see that it suffices to show that

$$\sum{}' |C_t| < \frac{c}{(j-k)^\epsilon}.$$

We consider three cases. Suppose first that t is such that $\xi_{t+1} = z_m/j$ for some m, but that for no l is $\xi_{t+1} = z_l/k$. Then

$$A_t = \frac{j}{\delta(z_{m-1})} - \frac{k}{\delta(k\xi_t)}, \quad A_{t+1} = \frac{j}{\delta(z_m)} - \frac{k}{\delta(k\xi_t)},$$

so that $A_{t+1} \ge A_t$, and the term C_t does not occur in $\sum'$. If

$$\xi_{t+1} = z_m/j = z_l/k,$$

then $z_m > z_l$ and

$$C_t = \frac{1}{j/\delta(z_{m-1}) - k/\delta(z_{l-1})} - \frac{1}{j/\delta(z_m) - k/\delta(z_l)}$$

$$\geq \frac{-k(1/\delta(z_l) - 1/\delta(z_{l-1}))}{(j/\delta(z_{m-1}) - k/\delta(z_{l-1}))(j/\delta(z_m) - k/\delta(z_l))}.$$

Finally, if $\xi_{t+1} = z_l/k$ for some l, but $\xi_{t+1} \neq z_m/j$ for every m, then

$$C_t = \frac{-k(1/\delta(z_l) - 1/\delta(z_{l-1}))}{(j/\delta(j\xi_{t+1}) - k/\delta(z_{l-1}))(j/\delta(j\xi_{t+1}) - k/\delta(z_l))}.$$

Thus, writing $\delta(x^+)$ and $\delta(x^-)$ for $\lim_{\xi \to x^+}\delta(\xi)$ and $\lim_{\xi \to x^-}\delta(\xi)$, we have

$$\sum{}' |C_t| \leq k \sum{}'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j/\delta(j\xi_{t+1}^-) - k/\delta(z_{l-1}))(j/\delta(j\xi_{t+1}^+) - k/\delta(z_l))}$$

$$= \sum{}'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j/\delta(jz_l^-/k) - k/\delta(z_{l-1}))(j/\delta(jz_l^+/k) - k/\delta(z_l))},$$

where $\sum''$ denotes summation with respect to l with $z_l/k \in (a, b)$. But

$$\delta(jz_l^-/k) \leq \delta(z_{l-1})$$

and

$$\delta(jz_l^+/k) \leq \delta(z_l),$$

and so

$$\sum{}' |C_t| \leq k \sum{}'' \frac{1/\delta(z_l) - 1/\delta(z_{l-1})}{(j-k)^2/\delta(z_{l-1})\delta(z_l)}$$

$$= \frac{k}{(j-k)^2} \sum{}'' \{\delta(z_{l-1}) - \delta(z_l)\} \leq \frac{2k\,\delta(ka)}{(j-k)^2}.$$

If now $\delta(x) = O(1/x)$, then

$$\sum{}' |C_t| = O\left(\frac{1}{(j-k)^2}\right),$$

and the proof is complete.

5. **The** preceding result can be generalized considerably by using the following transfer theorem:

THEOREM 5. *Suppose that* $\{x_k\}$ *is u.d.* $(\bmod\ \Delta)$, *where* $\Delta = \{z_n\}$, *and that* f *is a function which is differentiable except possibly at the points* $z_1, z_2, \cdots$, *such that* $f(x) \uparrow \infty$ *as* $x \uparrow \infty$ *and*

$$\inf_{x \in (z_{n-1}, z_n)} f'(x) \sim \sup_{x \in (z_{n-1}, z_n)} f'(x). \tag{5}$$

Then the sequence $\{x_k^*\} = \{f(x_k)\}$ *is u.d.* $(\bmod\ \Delta^*)$, *where* $\Delta^* = \{f(z_n)\}$.

Put

$$N(\alpha, x) = \sum 1,\ N(1, x) = N(x),\ N^*(\alpha, x) = \sum\nolimits^* 1,\ N^*(1, x) = N^*(x),$$

where $\sum$ denotes summation with $x_k \le x$ and $\langle x_k \rangle_\Delta < \alpha$ and $\sum^*$ denotes summation with $x_k^* \le x$, $\langle x_k^* \rangle_{\Delta^*} < \alpha$. Since f is an increasing function,

$$N^*(f(x)) = \sum_{f(x_k) \le f(x)} 1 = \sum_{x_k \le x} 1 = N(x).$$

By assumption, the relation

$$\lim_{x \to \infty} \frac{N(\alpha, x)}{N(x)} = \alpha$$

holds for $\alpha \in [0, 1]$. So we need only show that $N^*(\alpha, f(x)) \sim N(\alpha, x)$ as $x \longrightarrow \infty$, and by Theorem 1 it suffices to prove this as x runs through the sequence $\{z_n\}$. But

$$N(\alpha, z_n) = \sum_{m=1}^{n} \{N(z_{m-1} + \alpha(z_m - z_{m-1})) - N(z_{m-1})\},$$

and so

$$N^*(\alpha, f(z_n)) = \sum_{m=1}^{n} \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N^*(z_{m-1}^*)\}$$

$$= N(\alpha, z_n) + \sum_{m=1}^{n} \{N^*(z_{m-1}^* + \alpha(z_m^* - z_{m-1}^*)) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\}.$$

Thus the problem reduces to showing that

$$\sum_{m=1}^{n} \{N^*(z^*_{m-1} + \alpha(z^*_m - z^*_{m-1})) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\} = o(N(\alpha, z_n)),$$

or what is the same thing, that

$$(6) \sum_{m=1}^{n} \{N(f^{-1}(z^*_{m-1} + \alpha(z^*_m - z^*_{m-1}))) - N(z_{m-1} + \alpha(z_m - z_{m-1}))\} = o(N(z_n)).$$

Put

$$f^{-1}(z^*_{m-1} + \alpha(z^*_m - z^*_{m-1})) = u_m(\alpha),$$
$$z_{m-1} + \alpha(z_m - z_{m-1}) = v_m(\alpha).$$

If it can be shown that

$$(7) \qquad |u_m(\alpha) - v_m(\alpha)| < \epsilon_m(z_m - z_{m-1}),$$

where $\epsilon_m \longrightarrow 0$ as $m \longrightarrow \infty$, then for every $\epsilon > 0$,

$$\sum_{m=1}^{n} \{N(u_m(\alpha)) - N(v_m(\alpha))\}$$
$$= O\left(\sum_{m=1}^{n} \{N(v_m(\alpha) + \epsilon(z_m - z_{m-1})) - N(v_m(\alpha))\}\right)$$
$$= O(N(\epsilon, z_n)) = O(\epsilon N(z_n)),$$

which implies (6).

Now

$$u_m(0) = v_m(0), u_m(1) = v_m(1),$$

and

$$u_m(\alpha) - v_m(\alpha) = f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1}))) - (z_{m-1} + \alpha(z_m - z_{m-1}));$$

hence

$$u_m'(\alpha) - v_m'(\alpha) = \frac{f(z_m) - f(z_{m-1})}{f'\{f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))\}} - (z_m - z_{m-1}).$$

To maximize $u_m(\alpha) - v_m(\alpha)$, we must have

$$f(z_m) - f(z_{m-1}) - (z_m - z_{m-1})\, f'\{f^{-1}(f(z_{m-1}) + \alpha(f(z_m) - f(z_{m-1})))\} = 0.$$

There is a $Z_0 \in (z_{m-1}, z_m)$ such that

$$\frac{f(z_m) - f(z_{m-1})}{z_m - z_{m-1}} = f'(Z_0),$$

and a corresponding $\alpha_0 \in (0, 1)$ such that

$$f(z_{m-1}) + \alpha_0(f(z_m) - f(z_{m-1})) = f(Z_0),$$

(so that $u_m'(\alpha_0) - v_m'(\alpha_0) = 0$) for which

$$|u_m(\alpha) - v_m(\alpha)| \le |u_m(\alpha_0) - v_m(\alpha_0)| = |Z_0 - v_m(\alpha_0)|$$

for all $\alpha \in (0, 1)$. But

$$\begin{aligned} v_m(\alpha_0) &= z_{m-1} + \frac{f(Z_0) - f(z_{m-1})}{f(z_m) - f(z_{m-1})}\,(z_m - z_{m-1}) \\ &= z_{m-1} + \frac{f(Z_0) - f(z_{m-1})}{f'(Z_0)}, \end{aligned}$$

so that

$$Z_0 - v_m(\alpha_0) = Z_0 - z_{m-1} - \frac{f(Z_0) - f(z_{m-1})}{f'(Z_0)}$$

and

$$|u_m(\alpha) - v_m(\alpha)| \le \sup_{Z \in \delta_m} \left(|Z - z_{m-1}| \left| 1 - \frac{f(Z) - f(z_{m-1})}{(Z - z_{m-1})\, f'(Z)} \right| \right),$$

whence

$$\left| \frac{u_m(\alpha) - v_m(\alpha)}{z_m - z_{m-1}} \right| \le \sup_{\substack{Z \in \delta_m \\ W \in \delta_m}} \left| 1 - \frac{f'(W)}{f'(Z)} \right|,$$

and this last upper bound is $o(1)$ as $m \longrightarrow \infty$. Thus (7) holds, and the proof is complete.

If the f of Theorem 5 is taken to be an arbitrary increasing polygonal function, with vertices on the abscissas $x = z_1, z_2, \cdots$, then the condition (5) on the derivative is trivially satisfied. Such a transformation merely represents a change of scale inside each interval δ_n, and the distribution modulo Δ of any sequence $\{x_k\}$ is identical with the distribution of $\{f(x_k)\}$ modulo Δ^*.

In case f' is monotone, (5) can be replaced by the simpler condition

$$f'(z_{n-1}) \sim f'(z_n) \qquad \text{as } n \longrightarrow \infty. \tag{5'}$$

Combining this version of Theorem 5 with Theorem 4, we have:

THEOREM 6. *The sequence* $\{f(k\theta)\}$ *is u.d.* (mod Δ) *for almost all* $\theta > 0$ *if* $f(x) \uparrow \infty$, f' *is monotonic, and*

$$f^{-1}(z_n) - f^{-1}(z_{n-1}) \searrow 0,$$

$$f^{-1}(z_n) - f^{-1}(z_{n-1}) = O\left(\frac{1}{f^{-1}(z_n)}\right),$$

$$f'(f^{-1}(z_n)) \sim f'(f^{-1}(z_{n-1})),$$

where f^{-1} *is the function inverse to* f.

COROLLARY. *The sequence* $\{\alpha^k\}$ *is u.d.* (mod Δ) *for almost all* $\alpha > 1$ *if* $z_n = g(n)$, *where* g *is an increasing function with monotonic logarithmic derivative such that*

$$\frac{g'(x)}{g(x)} = O(x^{-1/2}). \tag{8}$$

For writing α^k as $e^{k \log \alpha}$, we see that we can take the f of Theorem 6 to be the exponential function, and the conditions displayed there become

$$\log z_n - \log z_{n-1} \searrow 0,$$

$$\log z_n - \log z_{n-1} = O\left(\frac{1}{\log z_n}\right),$$

$$z_n \sim z_{n-1}.$$

Of these, the third is implied by the first. Since

$$\frac{d}{dx}\log g(x) \searrow 0,$$

it is clear that $\log g(n) - \log g(n-1) \searrow 0$. From the extended law of the mean,

$$\frac{G(x) - G(x-1)}{H(x) - H(x-1)} = \frac{G'(X)}{H'(X)}, \quad X \in (x-1, x),$$

it follows that if $G'(x) = O(H'(x))$, then

$$G(x) - G(x-1) = O(H(x) - H(x-1)).$$

Taking

$$G(x) = \log g(x), \quad H(x) = \log e^{\sqrt{x}} = \sqrt{x},$$

we have by (8) that

$$\log g(n) - \log g(n-1) = O(n^{-1/2}).$$

But it also follows from the relation $G'(x) = O(H'(x))$ that $G(x) = O(H(x))$; hence

$$\log g(x) = O(x^{1/2}), \quad n^{-1/2} = O((\log g(n)^{-1}),$$

and the proof is complete.

For sufficiently smooth g, (8) can be replaced by the condition $g(x) = O(\exp\sqrt{x})$.

References

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