THE REPRESENTATION OF AN ANALYTIC FUNCTION BY GENERAL LAGUERRE SERIES

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1. Introduction. Hille [4] has solved the problem of finding necessary and sufficient conditions that a function be represented by Hermitian series in a strip. Pollard [7] has solved the analogous problem in a strip for Laguerre series of order zero. We propose to solve the problem for Laguerre series of order $\alpha(\alpha > -1)$ getting as a region of convergence a parabola instead of a strip. From this theorem the generalization of Pollard's result follows immediately.

We say that a function of a complex variable $f(z)$ where $z = x + iy = re^{i\theta}$ possesses a *Laguerre series of order* $\alpha(\alpha > -1)$ *or a general Laguerre series if*

$$f(z) \sim \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z) \qquad (n = 0, 1, 2, \cdots) \tag{1.1}$$

where

$$a_n^{(\alpha)} = \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\right\}^{-1}\int_0^{\infty} e^{-x}x^{\alpha}L_n^{(\alpha)}(x)f(x)\,dx \qquad (\alpha > -1) \tag{1.2}$$

$L_n^{(\alpha)}(x)$ is the Laguerre polynomial of order $\alpha > -1$ and degree n given by [8 p. 97 formula 5.1.6] and the above series converges. The series is said to be the *Laguerre expansion* of $f(z)$.

We define

$$d_\alpha \equiv -\limsup_{n} (2n^{\frac{1}{2}})^{-1} \log |a_n^{\alpha}| \tag{1.3}$$

and by the notation

$$z \in p(b) \quad b > 0\,; \qquad z \in \overline{p}(b) \tag{1.4}$$

we mean respectively that z lies in the open (closed) parabolic region

$$p(b):\ y^2 < 4b^2(x + b^2)\,; \qquad \overline{p}(b):\ y^2 \leqq 4b^2(x + b^2)\,.$$

If we select that branch of $z^{\frac{1}{2}}$ for which $(-z)^{\frac{1}{2}}$ is real and positive when $z < 0$ then $\Re(-z)^{\frac{1}{2}} = \{\frac{1}{2}(r - x)\}^{\frac{1}{2}} = b^2$ gives the equation $y^2 = 4b^2(x + b^2)$ of the parabola which is the boundary of the above regions.

The main result of this paper is the following.

THEOREM A. *In order that the function* $f(z)$ *possess a Laguerre series of order* α $(\alpha > -1)$ (or a general Laguerre series) *which converges to it*

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for $z \in p(d_\alpha)$ *where* $d_\alpha > 0$ (i.e. for every point z lying in the parabola $y^2 = 4d_\alpha^2(x + d_\alpha^2)$) *it is necessary and sufficient that* $f(z)$ *be analytic for* $z \in p(d_\alpha)$ *and that to every* b_α *such that* $0 \leqq b_\alpha < d_\alpha$ *there is a positive number* $B(\alpha, b)$ *such that*

$$(1.5) \qquad |f(z)| \leqq B(\alpha, b) \exp \{\tfrac{1}{2}x - |x|^{\frac{1}{2}}[b_2^\alpha - \tfrac{1}{2}(r - x)]^{\frac{1}{2}}\} \qquad (z \in \overline{p}(b_\alpha)) .$$

2. The necessity of Theorem A. By hypothesis we have $d_\alpha > 0$ where d_α is given by (1.3) and

$$(2.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z) \qquad (z \in p(d_\alpha) ;\ \alpha > -1) .$$

From Definition (1.3) it follows that

$$|a_n^{(\alpha)}| < m(\alpha, \varepsilon) \exp \{2n^{\frac{1}{2}}(-d_\alpha + \varepsilon)\} \qquad (\varepsilon > 0) .$$

We use the estimates

$$(2.2) \qquad \binom{n+\alpha}{n} \sim n^\alpha/\Gamma(\alpha+1) \quad (n \to \infty;\ \alpha \neq -1, -2, \cdots)$$

and $\log n < (4\,!\,n)^{\frac{1}{2}}$. By defining

$$c_\alpha \equiv \tfrac{1}{2}(b_\alpha + d_\alpha) > b_\alpha$$

and using the above estimate of $|a_n^\alpha|$ we can show that the series

$$(2.3) \qquad A^2(\alpha, b) = \sum_{n=0}^{\infty} |a_n^{(\alpha)}|^2 \exp (4c_\alpha n^{\frac{1}{2}}) \binom{n+\alpha}{n}$$

converges ($\alpha > -1$) by comparing it with the series $\sum_n 1/n^2$.

By direct calculation we can verify that series (2.1) can be put in the form

$$(2.4) \quad f(z) = \sum_{n=0}^{\infty} a_n^{(\alpha)} \exp \{2c_\alpha n^{\frac{1}{2}}\} \left\{\binom{n+\alpha}{n}\right\}^{\frac{1}{2}} L_n^{(\alpha)}(z) \exp \{-2c_\alpha n^{\frac{1}{2}}\} \left\{\binom{n+\alpha}{n}\right\}^{-\frac{1}{2}}.$$

From the integral

$$\exp (-s^{\frac{1}{2}}) = (1/2\pi^{\frac{1}{2}}) \int_0^\infty e^{-st} t^{-3/2} e^{-1/4t} dt$$

we obtain

$$\exp (-4c_\alpha n^{\frac{1}{2}}) = (4c_\alpha/2\pi^{\frac{1}{2}}) \int_0^\infty e^{-nq} q^{-3/2} \exp (-4c_\alpha^2/q) dq .$$

For polynomials $p_n(z)$ with real coefficients we have $|p_n(z)|^2 = p_n(\overline{z})p_n(z)$.

By applying Cauchy's inequality the above two equations and [8 p. 98 formula 5.1.15] equation (2.4) becomes

$$(2.5)\quad |f(z)|^2 \leqq \left[\sum_{n=0}^{\infty} |a_n^{(\alpha)}|^2 \exp(4c_\alpha n^{\frac{1}{2}})\binom{n+\alpha}{n}\right] \times\left[\sum_{n=0}^{\infty} |L_n^{(\alpha)}(z)|^2 \exp(-4c_\alpha n^{\frac{1}{2}})\left\{\binom{n+\alpha}{n}\right\}^{-1}\right]$$

$$= A^2(\alpha, b)\sum_{n=0}^{\infty} |L_n^{(\alpha)}(z)|^2\left\{\binom{n+\alpha}{n}\right\}^{-1}(4c_\alpha/2\pi^{\frac{1}{2}})\int_0^\infty e^{-nt}t^{-3/2}\exp(-4c_\alpha^2/t)dt$$

$$= A^2(\alpha, b)(4c_\alpha/2\pi^{\frac{1}{2}})\int_0^\infty\left[\sum_{n=0}^{\infty} |L_n^{(\alpha)}(z)|^2\left\{\binom{n+\alpha}{n}\right\}^{-1}e^{-nt}\right]t^{-3/2}\exp(-4c_\alpha^2/t)dt$$

$$= A^2(\alpha, b)(4c_\alpha/2\pi^{\frac{1}{2}})\int_0^\infty t^{-3/2}\exp(-4c_\alpha^2/t)\left[\sum_{n=0}^{\infty} L_n^{(\alpha)}(z)L_n^{(\alpha)}(\overline{z})\left\{\binom{n+\alpha}{n}\right\}^{-1}e^{-nt}\right]dt$$

$$= B(\alpha, b)\Gamma(\alpha+1)\int_0^\infty \frac{\exp(-2x/(e^t-1))J_\alpha(i|z|\operatorname{csch}\frac{1}{2}t)e^{\frac{1}{2}\alpha t}}{t^{3/2}\exp(4c_\alpha^2/t)(1-e^{-t})|z|^\alpha i^\alpha}\,dt$$

for $z \in \overline{p}(b_\alpha)$, $\alpha > -1$.

From [8 p. 197 formula (8.23.3)] we conclude that the following limit relation holds uniformly in any finite closed region S in the complex z-plane excluding the non-negative real axis

$$(2.6)\qquad n^{-\frac{1}{2}}\log|L_n^{(\alpha)}(z)| \to 2\{\tfrac{1}{2}(r-x)\}^{\frac{1}{2}} = 2r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta \qquad (n\to\infty)\,.$$

We also have the estimate:

$$(2.7)\qquad |L_n^{(\alpha)}(x)| \leqq \{\Gamma(n+\alpha+1)/\Gamma(\alpha+1)\Gamma(n+1)\}\exp(\tfrac{1}{2}x) \quad (x \geqq 0)$$

$$L_n^{(\alpha)}(x) = O(n^a) \quad a = \max(\tfrac{1}{2}\alpha - \tfrac{1}{4}, \alpha) \quad 0 \leqq x \leqq \omega$$

and α is aribitrary and real. The later formula is found in [8 p. 173 formula (7.6.11)]. Moreover from [7 p. 85 formula 2] it follows that

$$(2.8)\qquad \Gamma(n+\alpha+1)/\Gamma(n+1) \sim n^\alpha \qquad (n\to\infty)\,.$$

By use of (2.6), (2.2) (2.7) (2.8) and the Weierstrass M-test to show uniform convergence of the above series and the comparison test to show convergence of the series in the second line of (2.5) we can then apply the theorem of [9 p. 44] to justify the term by term integration of the series in (2.5).

From [8 p. 15 formula (1.71.6)] and p. 14 formulas (1.7.4) and (1.7.5)] we obtain

$$|J_\alpha(i\lambda)/i^\alpha| < (\tfrac{1}{2}y)^\alpha e^\lambda/\Gamma(\alpha+1) \quad (\alpha > -\tfrac{1}{2}, \lambda > 0)$$

from which we get by setting $\lambda = |z|\operatorname{csch}\frac{1}{2}t$

$$|J_\alpha(i|z|\operatorname{csch}\tfrac{1}{2}t)i^{-\alpha}| \leqq (|z|\operatorname{csch}\tfrac{1}{2}t)^\alpha\exp(|z|\operatorname{csch}\tfrac{1}{2}t)2^{-\alpha} \quad (\alpha > -\tfrac{1}{2})\,.$$

Using the above inequality to obtain an estimate of the last integral of (2.5) we obtain:

$$(2.9)\quad |f(z)|^2 \leqq B(\alpha, b)2^{-\alpha}\int_0^\infty \frac{\exp(-2x/(e^t-1)+2|z|/(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}))}{t^{3/2}\exp(4c_\alpha^2/t)(1-e^{-t})^{\alpha+1}}dt\,.$$

Since

$$0 \leqq e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \geqq t \qquad (t \geqq 0)$$

we have

$$\begin{aligned}
&- 2x/(e^t - 1) + 2|z|/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}) - 4c_\alpha^2/t \\
&= 2x(e^{\frac{1}{2}t} - 1)(e^{\frac{1}{2}t} + 1)(e^{\frac{1}{2}t} - 1) + (2|z| - 2x)/(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}) - 4c_\alpha^2/t \\
&\leqq 2x/(e^{\frac{1}{2}t} + 1) - 4\{c_\alpha^2 - \tfrac{1}{2}(r - x)\}/t.
\end{aligned}$$

If $q \equiv c_\alpha^2 - \frac{1}{2}(r - x)$ then by applying the last inequality to (2.9) we obtain

$$\begin{aligned}
|f(z)^2| &\leqq \int_0^\infty \exp\,(2x/(e^{\frac{1}{2}t} + 1) - 4q/t)t^{-3/2}(1 - e^{-t})^{-(\alpha+1)}\,dt \quad (\alpha > -\tfrac{1}{2}) \\
&= B(\alpha, b)e^x \int_0^\infty \exp\,\{-x[1 - 2/(e^{\frac{1}{2}t} + 1)\}\exp\,(-4q/t)t^{-3/2}(1 - e^{-t})^{-\alpha-1}dt\,.
\end{aligned}$$

Since $e^{\frac{1}{2}t} + 1 > 2 + \frac{1}{2}t$ $(t \geqq 0)$ we get from the last integral

(2.9.1)

$$\begin{aligned}
|f(z)|^2 &\leqq B(\alpha, b)e^x \int_0^\infty \exp\,\{-xt/(t + 4)\}\exp\,(-4q/t)t^{-3/2}(1 - e^{-t})^{-\alpha-1}dt \\
&= e^x\left[\int_0^{16q^{\frac{1}{2}}|x|^{\frac{1}{2}}} + \int_{16q^{\frac{1}{2}}/|x|^{\frac{1}{2}}}^\infty\right]\exp(-xt/(t + 4))\exp\,(-4q/t)t^{-3/2}(1 - e^{-t})^{-\alpha-1}dt \\
&= e^x(I_1 + I_2) \qquad (z \in \overline{p}(b_\alpha)\,;\ \alpha > -\tfrac{1}{2})
\end{aligned}$$

where I_1 and I_2 represent the first and second integrals respectively in the above line. If $x > 16c_\alpha^2 > 16q > 0$ then

$$-4q^{\frac{1}{2}}x^{\frac{1}{2}}/(1 + 4q^{\frac{1}{2}}x^{-\frac{1}{2}}) < -2q^{\frac{1}{2}}x^{\frac{1}{2}} < -2x^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}} \tag{2.10}$$

where $\tilde{q} \equiv b_\alpha^2 - \frac{1}{2}(r - x)$. Since $\exp\,(-xt/(t + 4))$ is a decreasing function of $t(t \geqq 0)$ we have

$$\begin{aligned}
I_2 \leqq \exp\,\{-x[16q^{1/2}|x|^{-\frac{1}{2}}/(16q^{\frac{1}{2}}|x|^{-\frac{1}{2}} + 4)]\} \\
\times \int_{16q^{\frac{1}{2}}/|x|^{\frac{1}{2}}}^\infty \exp\,(-4q/t)t^{-3/2}(1 - e^{-t})^{-\alpha-1}dt\,.
\end{aligned}$$

Since

$$(1 - e^{-t})^{-1} < 2/t \qquad (0 \leqq t \leqq 1) \tag{2.11}$$

and by applying inequality (2.10) to the above estimate of I_2 we have

$$\begin{aligned}
I_2 &\leqq \exp\,(-2x^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}})\{\Gamma(\alpha + 3/2)/4(c_\alpha^2 - b_\alpha^2)^{\alpha+3/2} \\
&\quad + 4\Gamma(\tfrac{1}{2})/4(c_\alpha^2 - b_\alpha^2)^{1/2}(1 - e^{-1})^{\alpha+1}\} = B(\alpha, b)\exp\,(-2x^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}}) \\
&\qquad (z \in \overline{p}(b_\alpha)\,;\ x > 16c_\alpha^2,\ \alpha > -\tfrac{1}{2})\,.
\end{aligned} \tag{2.11.1}$$

Consider the function $\mu(t) = xt/(t+4) + 4\bar{q}/t$. If we define

$$\bar{c}_\alpha^2 \equiv c_\alpha^2 - \tfrac{1}{2}(c_\alpha^2 - b_\alpha^2) = \tfrac{1}{2}(c_\alpha^2 + b_\alpha^2) > b_\alpha^2$$
$$\bar{q} \equiv \bar{c}_\alpha^2 - \tfrac{1}{2}(r - x)$$

and if t_0 represents the value of t for which $\mu(t)$ is a minimum then

$$t_0 = 4\bar{q}^{\frac{1}{2}}x^{-\frac{1}{2}}/(1 - \bar{q}^{\frac{1}{2}}x^{-\frac{1}{2}}) \qquad (x > \bar{c}_\alpha^2)$$

because $d\mu/dt = 0$; $d^2\mu/dt^2 > 0$ for $t = t_0$. Moreover

$$\mu(t_0) = 2x^{\frac{1}{2}}\bar{q}^{\frac{1}{2}} - \bar{q} \tag{2.12}$$

is the minimum value of $\mu(t)$. Also

$$4\bar{q}^{\frac{1}{2}}x^{-\frac{1}{2}})/(1 - \bar{q}^{\frac{1}{2}}x^{-\frac{1}{2}}) < 16q^{\frac{1}{2}}/x^{\frac{1}{2}} \qquad (x > 2(3)^{\frac{1}{2}}\bar{c}_\alpha/3)\ . \tag{2.13}$$

If $\varepsilon_1 \equiv \frac{1}{2}(c_\alpha^2 - b_\alpha^2)$ then

$$I_1 = \int_0^{16q^{\frac{1}{2}}/x^{\frac{1}{2}}} \exp(-\, xt/(t+4)) \exp(-4\bar{q}/t) \exp(-4\varepsilon_1/t) t^{-3/2}(1 - e^{-t})^{-\alpha-1} dt\ .$$

By the estimate (2.11) this becomes

$$I_1 \leqq 2^{\alpha+1}\int_0^{16q^{\frac{1}{2}}/x^{\frac{1}{2}}} \exp(-\, xt/(t+4) - 4\bar{q}/t) \exp(-4\varepsilon_1/t) t^{-(\alpha+5/2)} dt$$

for $x > (16c_\alpha)^2$. By (2.12) and (2.13) the above becomes

$$\begin{aligned} I_1 &\leqq \exp(-\, 2x^{\frac{1}{2}}\bar{q}^{\frac{1}{2}} + \bar{q})\int_0^{16q^{\frac{1}{2}}/x^{\frac{1}{2}}} \exp(-\, 4\varepsilon_1/t) t^{-\alpha-5/2} dt \\ &< 2^{\alpha+1} \exp(-\, 2x^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}} + \bar{c}_\alpha^2)\Gamma(\alpha + 3/2)(2(c_\alpha^2 - b_\alpha^2))^{-\alpha-3/2} \\ &\leqq B(\alpha, b) \exp(-\, 2x^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}}) \end{aligned}$$
$$(z \in \bar{p}(b_\alpha)\ ;\ x > \max\{2(3)^{\frac{1}{2}}\bar{c}_\alpha/3\ ;\ \bar{c}_\alpha^2\ ;\ (16c_\alpha^2)^2\}\ ,$$

By (2.9.1), (2.11.1) and the last inequality we have

$$|f(z)|^2 \leqq B(\alpha, b) \exp(x - 2x^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}})$$
$$^*(z \in \bar{p}(b_\alpha)\ ; x > \max\{15c_\alpha^2\ ; 2(3)^{\frac{1}{2}}\bar{c}_\alpha/3\ ;\ \bar{c}_\alpha^2 \equiv x_0))$$

for $\alpha > -\frac{1}{2}$.

But since I_1 and I_2 are bounded functions of x for $-b_\alpha^2 \leqq x \leqq x_0$ we can choose $B(\alpha, b)$ in the last inequality large enough that the inequality holds for all points z in and on the parabola $y^2 = 4b_\alpha^2(x + b_\alpha^2)$. Hence

$$|f(z)| \leqq B(\alpha, b) \exp(\tfrac{1}{2}x - |x|^{\frac{1}{2}}\tilde{q}^{\frac{1}{2}}) \qquad (\alpha > -\tfrac{1}{2})$$

and for every point z in and on the parabola $y^2 = 4b_\alpha^2(x + b_\alpha^2)$.

Moreover by (1.3) (2.6) and (2.7) it can be shown that the Laguerre series (1.1) converges absolutely and uniformly in any closed region for

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* $(z \in \bar{p}(b_\alpha); x > \max\{16c_\alpha^2;\ 2(3)^{1/2}\bar{c}_\alpha/3;\ \bar{c}_\alpha^2 \equiv x_0\})$

which $\Re(-z)^{\frac{1}{2}} < d_\alpha (\alpha > -1)$ that is, inside the parabola of convergence $y^2 = 4d_\alpha^2(x + d_\alpha^2)$, and hence represents an analytic function there by [5 p. 74 Theorem 3]. For a proof of this see the dissertation of Evelyn Boyd [1].

This completes the proof of the necessity of Theorem A.

3. **The sufficiency of Theorem A.** Let $w = u + iv$. Then by the notation $w \in s(b)$; $w \in \bar{s}(b)$ where $b > 0$ we mean respectively that the point w lies in the open (closed) strip $s(b) : v^2 < b$. $\bar{s}(b) : v^2 \leqq b$.

If $z = w^2$ then

$$z \in p(b) \rightleftarrows w \in s(b) \tag{3.1}$$

by virtue of (1.4) and the fact that

$$\tfrac{1}{2}(r - x) = v^2 \,. \tag{3.2}$$

For the function $h(w^2)$ the following two order conditions are equivalent

$$\begin{aligned} &|h(w^2)| < B_1(b) \exp\left((u^2 - v^2)/q - k|u^2 - v^2|^{\frac{1}{2}}(b^2 - v^2)^{\frac{1}{2}}\right) \\ &|h(w^2)| < B_2(b) \exp\left(u^2/q - k|u|(b^2 - v^2)^{\frac{1}{2}}\right) \\ &\qquad\qquad (w \in s(b) \,;\ q, k, B_1(b), B_2(b) > 0) \,. \end{aligned} \tag{3.3}$$

LEMMA 3.1. *We define*

$$g_\alpha(w) = w\int_0^1 t^\alpha f(tw^2)(1 - t)^{-\alpha - \frac{1}{2}} dt \qquad (w = u + iv \,;\ -1 < \alpha < \tfrac{1}{2}) \,. \tag{3.4}$$

We assume the hypothesis of the sufficiency Theorem A, then for every b_α, $0 \leqq b_\alpha < d_\alpha$ *there exists a positive number* $B(\alpha, b)$ *such that*

$$|g_\alpha(w)| \leqq B(\alpha, b) \exp\left(\tfrac{1}{2}u^2 - |u|(b_\alpha^2 - v^2)^{\frac{1}{2}}\right) \quad (w \in s(b_\alpha) \,;\ -1 < \alpha < \tfrac{1}{2}) \tag{3.5}$$

and g_α (w) *is analytic for* $w \in s(d_\alpha)$.

Proof. We make the transformation $z = w^2$ on the order condition of Theorem A. Then by equivalence relation (3.1) $w \in s(b_\alpha)$ so that in conjunction with (3.2) and (3.5) we conclude

$$|f(w^2)| \leqq B(\alpha, b) \exp\left(\tfrac{1}{2}(u^2 - v^2) - |u^2 - v^2|^{\frac{1}{2}}(b_\alpha^2 - v^2)^{\frac{1}{2}}\right)$$

for $w \in s(b_\alpha)$ and $-1 < \alpha$. Then by the equivalence relation (3.3) setting $q = 2$ and $k = 1$ we get from the above inequality

$$|f(w^2)| \leqq B(\alpha, b) \exp\left(\tfrac{1}{2}u^2 - |u|(b_\alpha^2 - v^2)^{\frac{1}{2}}\right) \quad (w \in s(b_\alpha) \,;\ -1 < \alpha) \,.$$

The change of variable $t = h^2$ in the integral of equation (3.4), expresses $|g_\alpha(w)|$ by

$$|g_\alpha(w)| = \left|2w\int_0^1 h^{2\alpha+1}f(hw^2)(1-h^2)^{-\alpha-\frac{1}{2}}dh\right|$$
$$\leqq 2|w|\{\max_{0\leq h\leq 1}|f\{(hw)^2\}|\}\left|\int_0^1(h^2/(1-h^2))^{\alpha+\frac{1}{2}}dh\right| .$$

The above integral converges for each value of α such that $\alpha < \frac{1}{2}$ and from Pollard's work (see [7 p. 362–363] in particular inequality (3.3)) we conclude that the function outside the integral satisfies the inequality (3.5).

From formula (3.4) by expanding a factor of the integrand into a power series we get

$$g(w) = w\int_0^1 t^\alpha f(tw^2)\left\{\sum_{n=0}^\infty\binom{n+\alpha-\frac{1}{2}}{n}t^n\right\}dt \qquad (t > 1)$$
$$= w\sum_{n=0}^\infty\binom{n+\alpha-\frac{1}{2}}{n}\int_0^1 t^{\alpha+n}f(tw^2)dt .$$

Using estimate (2.2) and the maximum modulus theorem on $f(tw^2)$ for the region

$$\bar{s}(d_\alpha, D):\ w\in\bar{s}(b_\alpha)\ ;\ u^2 < b_\alpha^2 + D \qquad (D > 0)$$

we get that the above series is in absolute value less then

$$K_1 w|f(z_0)|\sum_{n=0}^\infty n^{\alpha-3/2}$$

where z_0 is a point on the boundary of the region:

$$\overline{p}({}_\alpha d, D):\ y^2 \leqq 4b_\alpha^2(x+b_\alpha^2) \qquad x \leqq D .$$

The above series converges for $\alpha < \frac{1}{2}$. This not only justifies the above term by term integration (see [9 p. 45]) but shows that the series representing $g_\alpha(w)$ is absolutely and uniformly convergent in the closed region $\overline{s}(d_\alpha, D)$ and hence by [5 p. 73–74] the function $g_\alpha(w)$ is analytic if w lies in the simply connected region $s(d_\alpha)$.

LEMMA. 3.2. *Between Laguerre polynomials of different order there is the following relationship*

$$\int_x^\infty e^{-u}(u-x)^{-t}L_n^{(\alpha)}(u)du = e^{-x}L_n^{(\alpha+t-1)}(x)\Gamma(1-t) \quad (\alpha > -1\ ;\ t < 1\ ;\ x \geqq 0) ,$$

Proof. From [8 p. 97 formula (5.1.9)] we have

$$\sum_{n=0}^\infty L_n^{(\alpha)}(u)s^n = (1-s)^{-\alpha-1}\exp(-us/(1-s)) \qquad (|s| < 1) . \tag{3.6}$$

From which it follows that

$$(3.7)\qquad \int_x^\infty e^{-u}(u-x)^{-t}\Big[\sum_{n=0}^{\infty} L_n^{(\alpha)}(u)s^n\Big]du$$

$$= \int_x^\infty e^{-u}(u-x)^{-t}(1-s)^{-\alpha-1}\exp(-us/(1-s))du$$

$$= (1-s)^{-\alpha-1}e^{-x}\exp(-xs/(1-s))\int_0^\infty v^{-t}\exp(-v/(1-s))dv$$

$$= (1-s)^{(-\alpha-t+1)^{-1}}\exp(-xs/(1-s))e^{-x}\int_0^\infty e^{-T}T^{(-t+1)^{-1}}dT$$

$$= e^{-x}\sum_{n=0}^{\infty} L_n^{(\alpha+t-1)}(x)s^n\Gamma(1-t)\,.$$

The last line is obtained by substituting (3.6) in the previous line.

In equation (3.7) we get the third line from the second by change of variable $u-x=v$ and the fourth from the third by $T=v/(1-s)$.

That the integrand in the first line of (3.7) is absolutely and uniformly convergent (fixed $\alpha>-1$; fixed t, $|s|<1$; $0\leqq x<D_1\leqq u\leqq D_2$) we can show by comparing it with the series $C_\alpha M\sum_{n=0}^{\infty} n^\alpha s^n$ which can be done by using estimates (2.7) and (2.8). Thus the integrand of (3.15) can be integrated term by term over the interval $0<D_1\leqq u\leqq D_2$. We obtain

$$0\leqq \sum_{n=0}^{\infty}\int_x^\infty e^{-u}(u-x)^{-t}|L_n^{(\alpha)}(u)|s^n du$$

$$\leqq c_\alpha e^{-\frac{1}{2}x}2^{1-t}\Gamma(1-t)\sum_{n=0}^{\infty} n^\alpha s^n \qquad (t<1,\ |s|<1,\ \alpha>-1,\ x\geqq 0)$$

by using estimates (2.7) and (2.8) on the first series in the above and then making the change of variable $\frac{1}{2}(u-x)=p$. Thus the above series converges and since the series of (3.7) is uniformly convergent by [9] p. 45 can integrate the first series of (3.7) term by term:

$$\sum_{n=0}^{\infty} s^n\int_x^\infty e^{-u}(u-s)^{-t}L_n^{(\alpha)}(u)du$$

$$= \int_x^\infty e^{-u}(u-x)^{-t}\Big[\sum_{n=0}^{\infty} L_n^{(\alpha)}(u)s^n\Big]du = e^{-x}\sum_{n=0}^{\infty} L_n^{(\alpha+t-1)}(x)s^n\Gamma(1-t)\,.$$

Equating the coefficients of the two power series in s, we get Lemma 3.2. If in Lemma 3.2 we set $\alpha=\frac{1}{2}$ and if in the resulting equation we set $\alpha=t-\frac{1}{2}<\frac{1}{2}$ (since $t<1$) we obtain

$$(3.8)\qquad \int_x^\infty e^{-u}(u-x)^{-\alpha-\frac{1}{2}}L_n(\tfrac{1}{2})(u)du = \Gamma(-\alpha+\tfrac{1}{2})e^{-x}L_n^{(\alpha)}(x)$$

$$(-1<\alpha<\tfrac{1}{2};\ x\geqq 0)\,.$$

LEMMA 3.3 (Hille's lemma). *By Lemma* 3.1 *if $f(z)$ satisfies the order condition* (1.5) *of Theorem* A *and is analytic then $g_\alpha(w)$ is analytic and satisfies Hille's condition for an analytic function to possess a convergent*

Hermitian series, i.e. *condition* (3.5) (see [4 p. 81 Theorem 1]). *Hence we may write*

$$g_\alpha(w) = \sum_{n=0}^{\infty} (2^n n\,!\,\pi^{\frac{1}{2}})^{-\frac{1}{2}} f_n H_n(w)$$

where

$$(2^{2n+1}(2n+1)\,!\,\pi^{\frac{1}{2}})^{\frac{1}{2}} f_{2n+1} = \int_{-\infty}^{\infty} \exp(-q^2) H_{2n+1}(q) g_\alpha(q)\,dq\ . \tag{3.9}$$

Pick any $b < d$, where $2d$ is the width of the strip of convergence of the above series. Select $\overline{b}$ such that $b < \overline{b} < d$. Then by [4 p. 90] we have that there exists a positive number $B(b)$ such that

$$|f_{2n+1}| < B(b) \exp(-\overline{b}(4n+3)^{\frac{1}{2}})\ . \tag{3.10}$$

LEMMA 3.4. *Let $d > 0$ and suppose $f(z)$ is analytic for $z \in p(d)$. Moreover suppose for every b such that $0 \leqq b < d$ there exists a positive number $B(\alpha, b)$ such that*

$$|f(z)| \leqq B(\alpha, b) \exp(\tfrac{1}{2}x - |x|^{\frac{1}{2}}(b^2 - \tfrac{1}{2}[r - x])^{\frac{1}{2}}) \qquad (z \in p(b))\ , \tag{3.11}$$

then

$$|a_n^\alpha| \leqq B(b) \exp(-2n^{\frac{1}{2}}b) \qquad (-1 < \alpha < \tfrac{1}{2})\ . \tag{3.12}$$

Proof. By estimate (2.7) and inequality (3.11) the following integral converges

$$\begin{aligned} a_n^{(\alpha)} &= \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\right\}^{-1}\int_0^\infty t^\alpha e^{-t} f(t) L_n^{(\alpha)}(t)\,dt \\ &= \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\Gamma(-\alpha+\tfrac{1}{2})\right\}\int_0^\infty t^\alpha f(t)\,dt \\ &\quad \times \int_t^\infty e^{-s} L_n^{(\frac{1}{2})}(s)(s-t)^{-\alpha-\frac{1}{2}}\,ds \quad \text{(by (3.8))} \\ &= \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\Gamma(-\alpha+\tfrac{1}{2})\right\}^{-1}\int_0^\infty e^{-s} L_n^{(\frac{1}{2})}(s)\,ds \\ &\quad \times \int_0^s t^\alpha f(t)(s-t)^{-\alpha-\frac{1}{2}}\,dt \end{aligned} \tag{3.13}$$

The reversal of integration will be justified later. Making two successive changes of variable $t = ps$; $s = q^2$ we obtain

$$\begin{aligned} a_n^\alpha = &\left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\Gamma(-\alpha+\tfrac{1}{2})\right\}^{-1} \\ &\times \int_0^\infty 2\exp(-q^2) L_n^{(\frac{1}{2})}(q^2) q^2\,dq \int_0^1 p^\alpha f(pq^2)(1-p)^{-\alpha-\frac{1}{2}}\,dp \end{aligned}$$

$$= 2\left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\Gamma(-\alpha+\tfrac{1}{2})(-1)^n 2^{2n+1} n\,!\right\}^{-1}$$
$$\times \int_0^\infty \exp(-q^2) H_{2n+1}(q) g_\alpha(q) dq\ .$$

From [8 p. 102 formula (5.6.1)] we have the formula relating Laguerre and Hermite polynomials

$$qL_n^{(\frac{1}{2})}(q^2) = \{(-1)^n 2^{2n+1} n\,!\}^{-1} H_{2n+1}(q) \qquad (q \text{ real})\ .$$

By this relationship and (3.4) the above expression for $a_n^{(\alpha)}$ becomes

$$a_n^{(\alpha)} = \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\Gamma(-\alpha+\tfrac{1}{2})(-1)^n 2^{2n+1} n\,!\right\}^{-1}$$
$$\times \int_{-\infty}^\infty \exp(-q^2) H_{2n+1}(q) g_\alpha(q) dq$$

since the integrand is an even function. By (3.9)

$$a_n^\alpha = \{2^{2n+1}(2n+1)\,!\,\pi^{\frac{1}{2}}\}^{\frac{1}{2}} f_{2n+1}\left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\,\Gamma(-\alpha+\tfrac{1}{2})(-1)^n 2^{2n+1} n\,!\right\}^{-1}$$
$$\leqq \{2^{2n+1}(2n+1)\,!\,\pi^{\frac{1}{2}}\}^{\frac{1}{2}} B(\overline{b})$$
$$\times \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\Gamma(-\alpha+\tfrac{1}{2})(-1)^n 2^{2n+1} n\,!\right\}^{-1} \exp(-\overline{b}(4n+3)^{\frac{1}{2}}).$$

This follows from (3.10). Using Stirling's theorem (see [9 formula (4) p. 58])

$$n\,! \sim (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n} \qquad (n \to \infty)$$

and the relation $\Gamma(n) = (n-1)\Gamma(n-1)$ as well as the estimate (2.2) we get from the above inequality

$$\begin{aligned} a_n^\alpha &\leqq K_1 n^{-\alpha+\frac{1}{4}} 2\Gamma(-\alpha+\tfrac{1}{2}) \exp(-\overline{b}(4n+3)^{\frac{1}{2}}) \\ &\leqq B(\overline{b}) \exp\{(-\alpha+\tfrac{1}{4})\log n + 2n^{\frac{1}{2}}(b-\overline{b})\} \exp(-2bn^{\frac{1}{2}}) \\ &< B(\overline{b}) \exp(-2bn^{\frac{1}{2}}) \qquad (-1<\alpha<\tfrac{1}{2}) \end{aligned}$$

since $\overline{b} > b$. We can justify the above interchange of order of integration by proving that the integral is absolutely convergent. (To use [9 p. 55] consider a function which is zero over part of the rectangular region).

Since $L_n^{\frac{1}{2}}(s)$ is a polynomial of degree n we have for a fixed n

$$|e^{-\frac{1}{2}s} L_n^{(\frac{1}{2})}(s)| < c \qquad (0 \leqq s < \infty)\ .$$

From the above estimate and (3.11) the integral in the second line of (3.13) is in absolute value less than

$$B(\alpha, b)\int_0^\infty t^\alpha \exp(\tfrac{1}{2}t - t^{\frac{1}{2}}b)\,dt\int_t^\infty e^{-\frac{1}{2}s}(s-t)^{-\alpha-\frac{1}{2}}ds \qquad (-1 < \alpha < \tfrac{1}{2}) .$$

By change of variable $s - t = 2p$ on the inside integral and then $q = t^{\frac{1}{2}}b$ on the outside integral this becomes

$$B(\alpha, b)2^{-\alpha+3/2}\Gamma(-\alpha+\tfrac{1}{2})b^{-2\alpha-2}\int_0^\infty q^{2\alpha+2-1}e^{-q}dq = B(\alpha, b)\Gamma(2\alpha+2) \qquad (-1 < \alpha < \tfrac{1}{2}) .$$

Conclusion. Since by hypothesis $f(z)$ is analytic for $z \in p(d_\alpha)$ the function $f(x)$ is continuous for $x \geqq 0$ and hence integrable for $0 \leqq x \leqq D$ $(D > 0)$. Moreover since by hypothesis inequality (1.5) holds we have $|f(x)| < Ae^{x/2}$ for $x \geqq 0$. Then by a theorem of Caton and Hille [2 p. 227] series (1.1) is Abel summable to $f(x)$ for almost all x, including the points of continuity and hence in this case for all points x for which $x \geqq 0$.

By virtue of (3.12) and (2.6) it follows that the limit (1.3) exists that is $d_\alpha > 0$ and series (1.1) converges for $z \in p(d_\alpha)$ and for $-1 < \alpha < \frac{1}{2}$. We now want to generalize this result to the condition $-1 < \alpha$.

From (1.2) and [8 p. 98 formula (5.1.14)] and [9 p. 55 formula (2)] we have $a_n^{(\alpha+1)} = a_n^{(\alpha)} - a_{n+1}^{(\alpha)}$. Let

$$s_n^\alpha = \sum_{k=0}^{n} L_n^{(\alpha)}(z) \qquad (n = 0, 1, 2, \cdots)$$

then

$$\sum_{k=0}^{n} a_k^{(\alpha)}L_k^{(\alpha)}(z) = \sum_{k=0}^{n-1}(a_k^{(\alpha)} - a_{k+1}^{(\alpha)})s_k^\alpha + a_n^{(\alpha)}s_n^\alpha = \sum_{k=0}^{n-1} a_n^{(\alpha+1)}L_k^{(\alpha+1)}(z) + a_n^\alpha L_n^{(\alpha+1)}(z) .$$

By (1.3) (2.6) and (2.7) we have for a fixed z_0, $|a_n^{(\alpha)}L_n^{(\alpha+1)}(z_0)| \to 0$ for $n \to \infty$ for $z_0 \in p(d_\alpha)$ and for $-1 < \alpha < \frac{1}{2}$. Hence

$$\sum_{n=0}^{\infty} a_n^{(\alpha)}L_n^{(\alpha)}(z) = \sum_{n=0}^{\infty} a_n^{(\alpha+1)}L_n^{(\alpha+1)}(z) \qquad (z \in p(d^\alpha)\,;\ -1 < \alpha < \tfrac{1}{2}) .$$

Hence, by mathematical induction, the range of α for which the series (1.1) converges has been extended to $-1 < \alpha$.

Since the Abel sum of series (1.1) is $f(x)$ for $x \geqq 0$ and also the same as the Cauchy sum, the Cauchy sum of series (1.1) is $f(x)$ for $x \geqq 0$ and $-1 < \alpha$.

We remarked at the end of § 2 that series (1.1) is an analytic function inside its parabola of convergence. Hence by the identity theorem for analytic functions (see [5 p. 87]) since both $f(z)$ and series (1.1) are analytic for $z \in p(d_\alpha)$ and identical along the real axis they must be identical for $z \in p(d_\alpha)$ that is inside their common region of analyticity which is the parabola of convergence of the series: $y^2 = 4b^2_\alpha(x + b_\alpha^2)$ $(0 \leqq b_\alpha < d_\alpha)$.

This completes the proof of Theorem A.

4. The equivalence of Theorems A and B. We now note that Theorem A is equivalent to a generalization of Pollard's Theorem A [7]. We state the generalization as follows.

THEOREM B. *In order that $g(w)$ possess a Laguerre series of order alpha* ($\alpha > -1$) *such that*

$$f(z) = f(w^2) = g(w) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(w^2) \qquad (w \in s(d_\alpha)\,;\ d_\alpha > 0)$$

where $a_n^{(\alpha)}$ is given by (1.2) *it is necessary and sufficent that $g(w)$ be analytic and even for $w \in s(d_\alpha)$ and that to every b_α with $0 \leqq b_\alpha < d_\alpha$ there correspond a positive number $B(\alpha, b)$ such that*

$$|g(w)| \leqq B(\alpha, b) \exp\left(\tfrac{1}{2}u^2 - |v|\,(b^2 - v^2)^{\frac{1}{2}}\right) \qquad (w \in s(d_\alpha), \alpha > -1)\,. \tag{4.1}$$

THEOREM C. *Theorems* A *and* B *are equivalent.*

Proof. For

$$z = w^2 \tag{4.2}$$

the equation of Theorem B becomes

$$f(z) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z) \qquad (z \in p(d_\alpha)) \tag{4.3}$$

and conversely by (3.1).

Since $f(z) = f(w^2) = g(w)$ the function $g(w)$ will be analytic if $f(z)$ is. Moreover since $f(z) = g(z^{\frac{1}{2}})$ we get the converse by the same reasoning except at $z = 0$, since $z^{\frac{1}{2}}$ is analytic except at the origin. In a neighborhood of $w = 0$ we have since $g(w)$ is even and analytic

$$f(z) = f(w^2) = g(w) = \sum_{n=0}^{\infty} a_n w_n^{2n} = \sum_{n=0}^{\infty} a_n z^n$$

so that $f(z)$ is analytic at $z = 0$ also.

Applying the transformations (4.2) and (3.2) and the equivalence relations (3.3) to the inequalities (4.1) and (1.5) we get their equivalence because of (3.1).

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