HIGHER DIMENSIONAL CYCLIC ELEMENTS

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Introduction. Whyburn, in 1934, introduced the higher dimensional cyclic elements [5]. He gave an analysis of the structure of the homology groups of a space in terms of its cyclic elements. His results were for finite dimensional spaces, and he used the integers modulo two as the coefficient group. Puckett generalized some of Whyburn's results to compact metric spaces [3]. Simon has shown that if E is a closed subset of a compact space M, which contains all the $(r-1)$-dimensional cyclic elements of M, then $H^r(E) \approx H^r(M)$[4]. He also obtained a direct sum decomposition of $H^r(M)$ using the cyclic elements of M. We will extend some of these results.

The properties of zero-dimensional cyclic elements in locally connected spaces, and the relation of these cyclic elements to monotone mappings, is basic in the applications of zero-dimensional cyclic element theory. We shall give some counter-examples concerning the generalization of these properties to higher dimensional cyclic elements.

1. Preliminaries. Throughout this paper M will always denote a compact Hausdorff space. We shall use the augmented Cech homology and cohomology with a field as coefficient group. Results stated in terms of cohomology may be given a dual expression in terms of homology by means of the dot product duality for the Cech theory.

DEFINITION 1.1. A T_r set in M is a closed subset T of M such that $H^r(K) = 0$, for all closed subsets K of T.

DEFINITION 1.2. An E_r set in M is a non-degenerate subset of M which is maximal with respect to the property that it can not be disconnected by a T_r set of M.

The proofs of Lemmas 1.3 through 1.9 can be found in the papers by Whyburn [5] and Simon [4]. The proofs given by Whyburn are for subsets of Euclidean space, but they can be carried over to our case without difficulty.

LEMMA 1.3. *Let K be a subset of M which can not be disconnected by a T_r set. If $M = M_1 \cup M_2$, T_r-separated (by this we mean M_1 and M_2 are proper closed subsets and $M_1 \cap M_2$ is a T_r set), then $K \subset M_1$ (or, $K \subset M_2$).*

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LEMMA 1.4. *If K is an E_r set, then K is closed and connected.*

LEMMA 1.5. *If K_1 and K_2 are both E_r sets and $K_1 \neq K_2$, then $K_1 \cap K_2$ is a T_r set. Any T_r set is also a T_{r+1} set.*

LEMMA 1.6. *If K is a non-degenerate subset of M, which can not be disconnected by a T_r set, then K is contained in a unique E_r set in M.*

DEFINITION 1.7. If $\gamma^r \in H^r(M)$ and D is a minimal, closed subset of M such that $i^*(\gamma^r) \neq 0$ (where i^*: $H^r(M) \to H^r(D)$ is the inclusion map), then D is called a *floor* for γ^r.

LEMMA 1.8. *If $\gamma^r \in H^r(M)$ and $\gamma^r \neq 0$, then there exists a floor for γ^r.*

LEMMA 1.9. *If D is a floor for γ^r, then D can not be disconnected by a T_{r-1} set.*

LEMMA 1.10. *If $\{E^1, \cdots, E^n\}$ is a finite collection of E_{r-1} sets in M, with $M \neq \bigcup_{i=1}^n E^i$, then there exist proper, closed subsets, M_1 and M_2, of M such that* (1) $M = M_1 \cup M_2$, (2) *$M_1 \cap M_2$ is the union of a finite number of T_{r-1} sets (therefore, $M_1 \cap M_2$ is a T_r set),* (3) $M_1 \supset \bigcup_{i=1}^n E^i$.

Proof. The proof will be by induction on n. The case $n = 1$ follows from Lemma 1.3.

Assume the lemma is true up to $n - 1$. Since M is not an E_{r-1} set, we have $M = M_1 \cup M_2$, T_{r-1}-separated. Let $E = \bigcup_{i=1}^n E^i$. If $(M - E) \cap (M - (M_1 \cap M_2)) = \phi$, then the desired T_{r-1}-separation of M could be obtained by using the boundary of an open set in $M_1 \cap M_2$. Therefore, we can assume $(M - E) \cap (M - M_1) \neq \phi$. By Lemma 1.3, we can assume $\bigcup_{i=1}^s E^i \subset M_2$ and $\bigcup_{i=s+1}^n E^i \subset M_1$, where $1 \leq s < n$. We must have $E^i \subset (\overline{M - M_1})$, for $1 \leqq i \leqq s$. Otherwise, we could separate E^i by the T_{r-1} set $(\overline{M - M_1}) \cap (M_1 \cap M_2)$. Since $(M - E) \cap (M - M_1) \neq \phi$, $(\overline{M - M_1}) \neq \bigcup_{i=1}^s E^i$. Thus, by the induction assumption, $(\overline{M - M_1}) = M_4 \cup M_5$, where $\bigcup_{i=1}^s E^i$ is contained in M_4 and $M_4 \cap M_5$ is the union of a finite number of T_{r-1} sets. If we let $\tilde{M}_1 = M_1 \cup M_4$ and $\tilde{M}_2 = M_5$, then

(1) $M = \tilde{M}_1 \cup \tilde{M}_2$,

(2) $\tilde{M}_1 \cap \tilde{M}_2$ is the union of a finite number of T_{r-1} sets,

(3) $\bigcup_{i=1}^n E^i \subset \tilde{M}_1$,

(4) $\tilde{M}_1$ and $\tilde{M}_2$ are proper closed subsets of M.

2. Cyclic elements and the structure of M.

DEFINITION 2.1. A closed subset A of M is called a L_r set if every E_{r-1} set, whose intersection with A is not a T_r set, is contained in A.

The proofs of the following theorems are given below.

THEOREM 2.2. *If A is a L_r set, then $i^*: H^r(M) \to H^r(A)$ is onto. Thus, by duality, $i_*: H_r(A) \to H_r(M)$ is one-to-one.*

THEOREM 2.3. *Let A be a closed subet with the following property: if E is an E_{r-1} set and $H^r(E) \neq 0$, then E is contained in A. Then the map $i^*: H^r(M) \to H^r(A)$ is one-to-one and, by duality, $i_*: H_r(A) \to H_r(M)$ is onto.*

THEOREM 2.4. *Suppose there are only a finite number, say $\{E^1, \cdots, E^n\}$, of E_{r-1} sets such that $H^r(E^i) \neq 0$. Let $A = \bigcup_{i=1}^n E^i$. Then the mappings $i^*: H^r(M) \to H^r(A)$ and $i_*: H_r(A) \to H_r(A)$ are isomorphisms.*

REMARK. Theorem 2.4 can not be generalized to an infinite number of E_{r-1} sets, as the following example shows. In Euclidean space let $M = D \cup [\bigcup_{i=1}^\infty C_i]$, where $D = \{(x, y, z) | z = 0, x^2 + y^2 \leqq 1\}$ and $C_i = \{(x, y, z) | z = 1/i, x^2 + y^2 = 1\}$. We do not have $H_1(\bigcup_{i=1}^\infty C_i) \approx H_1(M)$, under the inclusion mapping.

THEOREM 2.5. *Let $\gamma^r \in H^r(M)$ and suppose U is an open set, such that if D is a floor for γ^r, then D is contained in U (see Definition 1.7). Then there exists a $\gamma_u^r \in H^r(M, M - U)$ such that $\gamma^r = j^*(\gamma_u^r)$, where $j^*: H^r(M, M - U) \to H^r(M)$.*

THEOREM 2.6. *Assume E is an E_{r-1} set in M and N is a closed subset of M, where $N \cap E = \phi$. Then the composite mapping $j_* i_*: H_r(E) \to H_r(M, N)$ is one-to-one. Here, $i_*: H_r(E) \to H_r(M)$ and $j_*: H_r(M) \to H_r(M, N)$ are the natural mappings.*

LEMMA 2.7. *Let (M, A) be a compact pair with $\gamma^r \in H^r(A)$. If $\delta^*(\gamma^r) \neq 0$, where $\delta^*: H^r(A) \to H^{r+1}(M, A)$, then there is a minimal closed set B such that $B \subset A$, and $\delta_B^*(\gamma_B^r) \neq 0$. Here, $\delta_B^*: H^r(B) \to H^{r+1}(M, B)$ and $\gamma_B^r = i^*(\gamma^r)$, where $i^*: H^r(A) \to H^r(B)$.*

LEMMA 2.8. *Let B be a minimal set defined in Lemma 2.7. There exists a minimal closed set N such that $\delta^*(\gamma_B^r) \neq 0$, where $\delta^*: H^r(B) \to H^{r+1}(N, B)$.*

Proof. The proof of these lemmas is obtained from the continuity of the Cech theory and Zorn's lemma.

LEMMA 2.9. *The set N, in Lemma 2.8, can not be disconnected by a T_{r-1} set.*

Proof. Suppose $N = N_1 \cup N_2$, where $N_1 \cap N_2$ is a T_{r-1} set. We will show this to be impossible, unless $N = N_1$. Let B be as defined in Lemma 2.8, and define $B_i = N_i \cap B$ $(i = 1, 2)$. We will show that the mapping induced by inclusion

$$K^*\colon H^{r+1}(N, B) \to H^{r+1}(N_1, B_1) \oplus H^{r+1}(N_2, B_2)$$

is an isomorphism. We use the relative Mayer-Vietoris sequence given below; note that $T = N_1 \cap N_2$ is a T_{r-1} set [2].

$$\begin{tikzcd}
 & H^{r+1}(N_2, B_2) \oplus & H^{r+1}(N_1, B_1) \arrow[r, "K^*"] & H^{r+1}(N, B) & \\
H^r(N, N) \arrow[r] & H^{r+1}(N, N_1 \cup B) + \arrow[u, "i_1^*"'] & H^{r+1}(N, N_2 \cup B) \arrow[u, "i_2^*"'] \arrow[r, "\bar{K}^*"] & H^{r+1}(N, B \cup T) \arrow[u, "i^*"'] \arrow[r] & H^{r+1}(N, N) .
\end{tikzcd}$$

The mappings i_1^* and i_2^* are isomorphisms by excision, the map $\bar{K}^*$ by exactness. Using the three exact sequences given below we see that i^* is an isomorphism.

$$\begin{gathered}
H^{s-1}(B \cap T) \to H^s(B \cup T) \to H^s(B) \oplus H^s(T) \to H^s(B \cap T) \\
H^s(B \cup T) \to H^s(B) \to H^{s+1}(B \cup T, B) \to H^{s+1}(B \cup T) \to H^{s+1}(B) \\
H^r(B \cup T, B) \to H^{r+1}(N, B \cup T) \to H^{r+1}(N, B) \to H^{r+1}(B \cup T, B) .
\end{gathered}$$

The first is a Mayer-Vietoris sequence, the second is a sequence for a pair, the third is a sequence for a triple. Thus K^* is an isomorphism.

In the diagram below, since $\delta_N^*(\gamma_B^r) \neq 0$, we may assume $\delta_{N_1}^* \phi_1^*(\gamma_B^r) \neq 0$.

$$\begin{tikzcd}
H^{r+1}(N, B) \arrow[rr, "K^*"] & & H^{r+1}(N_1, B_1) \oplus & H^{r+1}(N_2, B_2) \\
 & H^{r+1}(M, B_1) \arrow[ur, "i_1^*"] & & \\
H^r(B) \arrow[uu, "\delta_N^*"] \arrow[rr, "\phi_1^* \oplus \phi_2^*"] & & H^r(B_1) \oplus \arrow[ul, "\delta_1^*"'] \arrow[uu, "\delta_{N_1}^*"] & H^r(B_2) \arrow[uu, "\delta_{N_2}^*"]
\end{tikzcd}$$

We now have $\delta_1^* \phi_1^*(\gamma_B^r) \neq 0$, since $i_1^* \delta_1^* \phi_1^*(\gamma_B^r) = \delta_{N_1}^* \phi_1^*(\gamma_B^r) \neq 0$. This implies $B_1 = B$, by the definition of B. Therefore, $\phi_1^*(\gamma_B^r) = \gamma_B^r$ and $\delta_{N_1}^* \phi_1^*(\gamma_B^r) = \delta_{N_1}^*(\gamma_B^r) \neq 0$. Since N is minimal, we must have $N_1 = N$. Thus, N can not be disconnected by a T_{r-1} set.

Proof of Theorem 2.2. We will show $\delta^*(\gamma^r) = 0$, for all $\gamma^r \in H^r(A)$, where $\delta^*: H^r(A) \to H^r(M, A)$. Suppose not; then choose N and B according to Lemma 2.8. Then there exists an E_{r-1} set containing N, by Lemma 1.6. Let E denote this E_{r-1} set. Since E contains N, we have $E \cap A \supset B$. Since $H^r(B) \neq 0$, B is not a T_r set. Therefore $E \subset A$, because A is an L_r set. This implies that N is contained in A. But this is impossible, as the diagram below shows. By the definition of the pair (N, B), $\delta^* i^*(\gamma^r) \neq 0$.

$$\begin{array}{ccc} H^r(A) & \xrightarrow{\delta_1^*} & H^{r+1}(A, B) \\ \downarrow{\scriptstyle i*} & & \downarrow \\ H^r(B) & \xrightarrow{\delta^*} & H^{r+1}(N, B) . \end{array}$$

Proof of Theorem 2.3. Consider the exact sequence:

$$H^r(M, A) \xrightarrow{j^*} H^r(M) \xrightarrow{i^*} H^r(A) .$$

Suppose $j^*(\gamma^r) \neq 0$, where $\gamma^r \in H^r(M, A)$. By Lemmas 1.9 and 1.6 there is an E_{r-1} set which contains a floor for $j^*(\gamma^r)$. Let E be this E_{r-1} set. Since E contains a floor for $j^*(\gamma^r)$, $H^r(E) \neq 0$. Therefore, $E \subset A$; which implies $i^* j^*(\gamma^*) \neq 0$, since E contains a floor for $j^*(\gamma^r)$. Therefore j^* is a trivial map and i^* is one-to-one.

Proof of Theorem 2.4. By Theorem 2.3, $i^*: H_r(A) \to H_r(M)$ is onto. If $i_*(Z_r) = 0$, for some $Z_r \in H_r(A)$; then there is a minimal set K such that

(1) $K \supset A$, and

(2) $i_*^K(Z_r) = 0$, where $i_*^K : H_r(A) \to H_r(K)$ [2]. If $K \neq A$; then, by Lemma 1.10, we have $K = K_1 \cup K_2$, T_r-separated. The Mayer-Vietoris sequence below implies $i_*^{K_1}(Z_r) = 0$, where $i_*^{K_1} : H_r(A) \to H_r(K_1)$.

$$H_r(K_1 \cap K_2) \to H_r(K_1) \oplus H_r(K_2) \to H_r(K) .$$

Therefore, $K = A$ and $Z_r = 0$, or i_* is one-to-one.

Proof of Theorem 2.5. Consider the exact sequence,

$$H^r(M, M - U) \xrightarrow{j^*} H^r(M) \xrightarrow{i^*} H^r(M - U) .$$

We will show $i^*(\gamma^r) = 0$, where γ^r is the element of $H^r(M)$ given in the theorem. Suppose $i^*(\gamma^r) \neq 0$; then, by Lemma 1.8, there exists a floor for $i^*(\gamma^r)$ contained in $M - U$. If D is this floor, then D is a floor for γ^r, since $i_D^* = i_{DU}^* i^*$. Here, $i_D^*: H^r(M) \to H^r(D)$ and $i_{DU}^*: H^r(M - U) \to H^r(D)$ are inclusion mappings. Therefore, by the definition of U, D is contained in U. This is impossible, hence $i^*(\gamma^r) = 0$.

Proof of Theorem 2.6. Let $\phi_* = j_* i_*$, and suppose $\phi_*(Z_r) = 0$, for some $Z_r \in H_r(E)$. Then there exists a minimal closed set K in M such that $K \supset E$ and $\phi_*^K(Z_r) = 0$, where $\phi_*^K\colon H_r(E) \to H_r(K, K \cap N)$ is analogous to ϕ_* defined above. This follows from Zorn's lemma and continuity. We will assume $K \neq E$. Since E is an E_{r-1} set, we can write $K = K_1 \cup K_2$, T_{r-1}-separated. Also, we can assume $E \subset K_1$. Consider the following commutative diagram:

$$\begin{tikzcd}
H_{r-1}(K_1 \cap K_2 \cap N) & & H_{r-1}(K_1 \cap K_2) & & \\
H_r(K \cap N) \arrow[u] \arrow[rr] & & H_r(K) \arrow[u] \arrow[rr, "j_*^K"] & & H_r(K, K \cap N) \\
& & & H_r(E) \arrow[ul, "i_*^K"'] \arrow[ur, "\phi_*^K"'] \arrow[dl, "i^*_{K_1}"] \arrow[dr, "\phi_*^{K_1}"] & \\
H_r(K_1 \cap N) \arrow[uu, "\phi_*^1"'] & & H_r(K_1) \arrow[uu, "\phi_*^2"'] \arrow[rr, "j_*^{K_1}"] & & H_r(K_1, K_1 \cap N) \\
\oplus & \overset{i_*^1}{\oplus} & \oplus & & \\
H_r(K_2 \cap N) \arrow[rr, "i_*^2"] & & H_r(K_2) & & \\
H_r(K_1 \cap K_2 \cap N) \arrow[u] & & H_r(K_1 \cap K_2) \arrow[u] & &
\end{tikzcd}$$

The two vertical sequences are Mayer-Vietoris sequences. Also, the two horizontal sequences are exact. We have

$$H_{r-1}(K_1 \cap K_2) = H_{r-1}(K_1 \cap K_2 \cap N) = H_r(K_1 \cap K_2) = H_r(K_1 \cap K_2 \cap N) = 0 ,$$

since $K_1 \cap K_2$ is a T_{r-1} set. Since $\phi_*^K(Z_r) = j_*^K i_*^K(Z_r) = 0$, there exists a $Z_r^3 \in H_r(K \cap N)$ such that $i_*^3(Z_r^3) = i_*^K(Z_r)$. There exists

$$(Z_r^1, Z_r^2) \in H_r(K_1 \cap N) \oplus H_r(K_2 \cap N)$$

such that $\psi_*^1(Z_r^1, Z_r^2) = Z_r^3$. By commutativity,

$$\psi_*^2(i_*^1(Z_r^1), i_*^2(Z_r)) i_*^2 \psi_*^1(Z_r^1, Z_r^2) = i_*^3(Z_r^3) = i_*^K(Z_r) ,$$

and

$$\psi_2^*(i_*^{K_1}(Z_r), 0) = i_*^K(Z_r) .$$

By exactness, ψ_*^2 is an isomorphism, hence $i_*^1(Z_r^1) = i_*^{K_1}(Z_r)$. Therefore, $j_*^{K_1} i^{K_1}(Z_r) = j_*^{K_1} i_*^1(Z_r^1) = 0$. But this is impossible, since K is minimal. Thus, $K = E$ and ϕ^* is one-to-one.

3. **Cyclic elements in locally connected spaces.** The zero-dimensional cyclic elements in a locally connected continuum have several useful properties. For example, if the continuum M is locally connected, then the zero-dimensional cyclic elements of M are also locally connected and these cyclic elements form a null sequence. Also, the simple 0-links (definition below) are identical with the E_0 sets in an lc^0 space [6]. The examples below show that these properties do not generalize.

DEFINITION 3.1. A non-degenerate subset K of M is called a simple r-link of M, if K is maximal with respect to the following property: if $M = M_1 \cup M_2$, T_r-separated, then $K \subset M_1$ (or $K \subset M_2$). In other words, K is a maximal subset which can not be separated by a T_r set that also separates M.

LEMMA 3.2. *All simple r-links in M are closed. If K_1 and K_2 are two distinct simple r-links in M, then $K_1 \cap K_2$ is a T_r set. If L is a non-degenerate subset of M that is not disconnected by any T_r set which also disconnects M, then L is contained in a simple r-link of M.*

Proof. The proof is similar to those for the corresponding lemmas for cyclic elements.

EXAMPLE. We will construct an lc^r space M in which the collection of E_r sets does not form a null sequence. This example will also show that, in an lc^r space, the simple r-links need not be the same as the E_r sets.

For each positive integer n, let R_n be a solid, three dimensional rod of height one and diameter $1/2^n$. In Euclidean three-space, define I by $I = \{(x, y, z) | x = 0, y = 0, 0 \leqq z \leqq 1\}$. Imbed R_n in three-space so that R_n is tangent to R_{n+1} and the sequence of sets R_n converges to I (i.e. $R_n = \{(x, x, z) | x^2 + (y - 3/2^{n+1})^2 \leqq 1/2^{2n+2}, 0 \leqq z \leqq 1\}$). Let M be the set $[\bigcup_{n=1}^{\infty} R_n] \cup I$. Then M is a compact $1c^1$ space, each R_n is an E_1 set in M, but the collection $\{R_n\}$ is not a null sequence. Also, I is a simple 1-link, but is not an E_1 set.

THEOREM 3.3. *If M is $s - lc$ and E is an E_r set of M, where $s \geqq r$, then E is $s - lc$.*

Proof. Given any $x \in E$, and an open set U^0 of E containing x, then there exists an open set U of M such that $U \cap E = U^0$. Since M is $s - lc$, there exists an open set V, containing x, such that $\bar{V} \subset U$ and any compact s-cycle in V bounds on a compact subset of U. Let Z_s be a compact cycle on $V \cap E = V^0$. Then there exists a minimal closed set K in M such that $\bar{V}^0 \subset K \subset U$, and Z_s bounds on K. By using the Mayer-Vietoris sequence, as it was used in the proof of Theorem 2.4, we can show $K \subset U^0$. Therefore Z_s bounds in U^0 and E is $s - lc$.

EXAMPLE. We will construct a compact lc^r space which contains an E_r set which is not lc^r. Consider the following curve in three-space:

$$x = 0, y = t, z = \sin(\pi/t), \text{ for } 0 < t \leqq 1 .$$

Expand this curve slightly so that it becomes a solid, three dimensional figure, which oscillates as it approaches the origin. Let N be this space, along with its limiting line segment on the z-axis. Let $P = \{(x, y, z) \mid x = 0, 0 \leqq y \leqq 1, -1 \leqq z \leqq 1\}$; then define $M = P \cup N$. Thus N is an E_1 set in M and M is lc^1 but N is not $0 - lc$.

4. Cyclic elements and monotone mappings. A very basic property of the zero-dimensional cyclic element theory is the following: if $f: M \to N$ is a monotone mapping (*i.e.* the inverse image of any point is connected), M and N are lc^0, and E_N is an E_0 set in N; then there is an E_0 set in M whose image under f contains E_N. This result does not hold in higher dimensions, as the example below demonstrates. The best result we have obtained in this direction is Theorem 4.2.

DEFINITION 4.1. A mapping f: $M \to N$ is r-monotone, if $H^s(f^{-1}(y)) = 0$, for all $y \in N$ and $0 \leqq s \leqq r$.

THEOREM 4.2. *Let f be an $(r-1)$-monotone mapping of M onto N, where M and N are compact Hausdorff spaces. If D_N is a floor for $\gamma_N^r \in H^r(N)$, then there exists a floor D_M for $f^*(\gamma_N^r)$ such that $f(D_M) = D_N$.*

Proof. Since f is $(r-1)$-monotone, f^*: $H^r(N) \to H^r(M)$ is a one-to-one mapping [1]. Therefore, $f^*(\gamma_N^r) \neq 0$. Consider the commutative diagram below. The vertical mappings are inclusion mappings; and D_M is defined below.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow{\scriptstyle i_M} & & \uparrow{\scriptstyle i_N} \\ f^{-1}(D_N) & \xrightarrow{f_1} & D_N \\ \uparrow{\scriptstyle j_M} & & \uparrow{\scriptstyle j_N} \\ D_M & \xrightarrow{f_2} & f(D_M) \end{array} .$$

The mapping f_1 is the restriction of f to $f^{-1}(D_N)$. Therefore, f_1 is $(r-1)$-monotone. Since D_N is a floor for γ_N^r, $i_N^*(\gamma_N^r) \neq 0$. Since

f_1^*: $H^r(D_N) \to H^r(f^{-1}(D))$ is one-to-one, $i_M^* f^*(\gamma_N^r) = f_1^* i_N^*(\gamma_N^r) \neq 0$. Therefore, $f^{-1}(D_N)$ contains a floor for $f^*(\gamma_N^r)$. Denote this floor by D_M and let f_2 be the restriction of f to D_M. By the definition of a floor, $j_M^* i_M^* f^*(\gamma_M^r) \neq 0$. Since $j_M^* i_M^* f^*(\gamma_N^r) = f_2^* j_N^* i_N^*(\gamma_N^r)$, we have $j_N^* i_N^*(\gamma_N^r) \neq 0$. This implies $f(D_M) = D_N$, since D_N is a floor for γ_N^r.

We shall omit the proofs of Lemmas 4.3 and 4.5.

LEMMA 4.3. *Let N_1 and N_2 be subsets of M which can not be disconnected by a T_r set. Suppose that $\bar{N}_1 \cup \bar{N}_2$ is not a T_r set. Then $\bar{N}_1 \cup \bar{N}_2$ can not be disconnected by a T_r set.*

LEMMA 4.4. *Let $f: M \to N$, and suppose $T \subset N$ is a T_r set such that $f^{-1}(T)$ is also a T_s set. Also, assume f is a homeomorphism of $M - f^{-1}(T)$ onto $N - T$. Then, if T^N is a T_r set in N, $f^{-1}(T^N)$ is a T_r set in M.*

Proof. Let K be a closed subset of $f^{-1}(T^N)$. Denote $f^{-1}(T)$ by T^{-1}. In the commutative diagram below f_1^* is an isomorphism, by excision. Therefore, by exactness, $H^r(K) = 0$.

$$\begin{array}{ccccc} H^r(K, K \cap T^{-1}) & \to & H^r(K) & \to & H^r(K \cap T^{-1}) \\ \uparrow f_1^* & & \uparrow f^* & & \\ H^r(f(K), f(K \cap T^{-1})) & \to & H^r(f(K)) & \to & \end{array}$$

LEMMA 4.5. *Assume f is a mapping of M onto N such that the inverse image of any T_r set in N is a T_r set in M. If $K \subset M$ can not be disconnected by a T_r set in M, then $f(K)$ can not be disconnected a T_r set in N.*

EXAMPLE. If f is an r-monotone mapping of M onto N, where M and N are lc^∞ spaces and E^N is an E_r set in N; there may not be an E_r set, E^M, in M such that $f(E^M) \supset E^N$.

We will construct the example in three space. Consider the following solid cylinders:

$$M_1 = \{(x, y, z) \mid x^2 + y^2 \leqq 1,\ 0 \leqq z \leqq 1\}$$
$$M_2 = \{(x, y, z) \mid x^2 + (y-2)^2 \leqq 1,\ 0 \leqq z \leqq 1\}\ .$$

The cylinders M_1 and M_2 are tangent along $I = \{(x, y, z) \mid x = 0, y = 1, 0 \leqq z \leqq 1\}$. Let M_3 be an arc joining the endpoints of I, which does not meet $M_1 \cup M_2$ except at these endpoints. Let $M = \bigcup_{i=1}^3 M_i$. We will define a decomposition of M, and will let $f: M \to N$ be the decomposition mapping.

To form N, identify all the points in M_3 into a single point. Then the mapping $f: M \to N$ is r-monotone for all r and the restriction of f to $M - M_3$ is a homeomorphism.

We will show that N is an E_1 set. First, neither M_1 nor M_2 can be disconnected by a T_1 set. Lemmas 4.4 and 4.5 imply that neither $f(M_1)$ nor $f(M_2)$ can be disconnected by a T_1 set. By Lemma 4.3, $N = f(M_1) \cup f(M_2)$ can not be disconnected by a T_1 set, since $f(M_1) \cup f(M_2)$ contains an essential 1-cycle. If K is a closed subset of M such that $f(K) \supset N$, then $K \supset M_1 \cup M_2$. Then K can be disconnected by a T_1 set, namely $M_1 \cap M_2$. Therefore, there is no E_1 set in M whose image is N.

Note that M is obviously lc^r for all r. Therefore N is also lc^r, for all r, since f is r-monotone, for all r.

References

1. E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Annals of Math., (2) **51** (1950), 534-542.

2. Samuel Eilenberg and Norman Steenrod, *Foundations of Algebraic Topology*, Princeton, 1952.

3. W. J. Puckett, Jr., *A theorem concerning homologies in a compact space*, Fundamentae Mathematicae, **31-32** (1938), 32.

4. Arthur Simon, *N-Cyclic elements*, Duke Math. Journal, **24** (1957), 1-7.

5. G. T. Whyburn, *Cyclic elements of higher order*, Amer. J. Math., **56** (1934), 133-146.

6. ———, *Analytic Topology*, Amer. Math. Soc. Coll. Pub., 1942.

7. R. L. Wilder, *Topology of Manifolds*, Amer. Math. Soc. Coll. Pub. 1949.

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