n-PARAMETER FAMILIES AND BEST APPROXIMATION

PHILIP C. CURTIS, JR.

1. Introduction. Let $f(x)$ be a real valued continuous function defined on a closed finite interval and let F be a class of approximating functions for f. Suppose there exists a function $g_0 \in F$ such that $\|f - g_0\| = \inf_{g \in F} \|f - g\|$ where $\|f\| \equiv \sup_{x \in [a,b]} |f(x)|$. The problem of characterizing g_0 and giving conditions that it be unique is classical and has received attention from many authors. The well-known results for polynomials were generalized by Bernstein [2] to "Chebyshev" systems. Later Motzkin [10] and Tornheim [15] further extended these theorems to not necessarily linear families of continuous functions. The only essential requirement was that to any n-points in the plane with distinct abscissae lying in a finite interval $[a, b]$, there should be a unique function in the class F passing through the given points. Such a system F is called an n-parameter family. Constructive methods for determining the function from F of best approximation to f, due to Remes [14] in the polynomial case, were extended to the above situation by Novodvorskii and Pinsker [13]. In this paper and in the paper of Motzkin two apparently additional requirements were placed on the system F. One, a continuity condition, was shown by Tornheim to follow from the axioms of F. The other, a condition on the multiplicity of the roots of $f - g, f, g \in F$, also follows from the definitions as will be shown in § 2. In § 3 the characterization of g_0 is discussed. Methods for constructing g_0 are given in § 4. These are based on the maximization of a certain function of $n + 1$ variables. In § 5 it is shown that an n-parameter familiy has a unique function of best approximation to an arbitrary continuous function in the $L_{p,N}$ norm if and only if F is the translate of a linear n-parameter family. The problem of the existence of n-parameter families on general compact spaces S is discussed in § 6. Under additional hypotheses on F it is shown that S must be homeomorphic to a subset of the circumference of the unit circle. If n is even this subset must be proper.

2. n-parameter families functions. Following Tornheim we define, for a fixed integer $n \geq 1$, an n-parameter family of functions F to be a class of real valued continuous functions on the finite interval $[a, b]$ such that for any real numbers

$$x_1, \cdots, x_n, y_1, \cdots, y_n, a \leq x_1 < x_2 < \cdots < x_n \leq b$$

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there exists a unique $f \in F$ such that $f(x_i) = y_i$ $i = 1, \cdots, n$. For convenience we will usually take $[a, b]$ to be the interval $[0, 1]$. We will include the possibility that 0 and 1 are identified. Then of course $x_1 \neq x_n$, and the functions of F are periodic of period 1. We call such a family a periodic n-parameter family. If we wish to consider specifically the case when 0 and 1 are not identified, we will refer to F as an ordinary n-parameter family. If F is a linear vector space of functions then we will call F a linear n-parameter family (e.g., polynomials of degree $\leq n-1$). The following continuity theorem of Tornheim [15] is a generalization of a result of Beckenbach [1] for $n = 2$.

THEOREM 1. *Let F be an n-parameter family on* $[0, 1]$. *For*

$$k = 1, 2, \cdots, \textit{let } x_1^{(k)}, \cdots, x_n^{(k)}, y_1^{(k)}, \cdots, y_n^{(k)}, 0 \leq x_1^{(k)} < \cdots < x_n^{(k)} \leq 1$$

be given sequences of real numbers and let f_k be the unique function from F such that

$$f_k(x_i^{(k)}) = y_i^{(k)} \qquad i = 1, \cdots, n .$$

Suppose for each

$$i, \lim_{k\to\infty} x_i^{(k)} = x_i, \lim_{k\to\infty} y_i^{(k)} = y_i \textit{ and } 0 \leq x_1 < \cdots < x_n \leq 1 .^1$$

Let f be the unique function from F such that $f(x_i) = y_i$ $i = 1, \cdots, n$. Then $\lim_{k\to\infty} f_k = f$ uniformly on $[0, 1]$.

Proof. If 0 and 1 are not identified the proof is given in [15]. Therefore, let 0 and 1 be identified and the functions of F be periodic. Suppose f_k does not tend uniformly to f. For some $\varepsilon > 0$, there exists a sequence $\{u_k\} \subset [0, 1]$ such that for each k, $|f(u_k) - f_k(u_k)| \geq \varepsilon$. Since a subsequence of $\{u_k\}$ converges, we may assume $\{u_k\}$ does and let $u = \lim_{k\to\infty} u_k$. By a suitable rotation of $[0, 1]$ we may assume $u, x_1, \cdots, x_n$ all lie in the interior of an interval $[a, b]$, $0 < a < b < 1$. But F forms an ordinary n-parameter family on $[a, b]$ and hence $f_k \to f$ uniformly on $[a, b]$ which is a contradiction. This completes the proof.

We now verify that n-parameter families are unisolvent in the sense of Motzkin [10]. Let $f, g \in F$ and let x be an interior point of $[0, 1]$. If x is a zero of $f - g$ and if $f - g$ does not change sign in a suitably small neighborhood about x then we will say the zero x has multiplicity 2, otherwise we say x has multiplicity 1. If 0 and 1 are not identified and either is a zero of $f - g$, then the multiplicity is taken to be 1. We shall denote the sum of the multiplicities of the zeros of $f - g$ within an interval $[a, b]$ by $m_{a,b}(f, g)$. The following generalized con-

[1] If $0, 1$ are identified we assume $x_n^{(k)} < 1$ and $x_n < 1$,

vexity notion is also useful. A continuous function h will be said to be convex to F if h intersects no function of F at more than n points. The following result extends Theorems 2 and 3 of [15].

THEOREM 2. *Let F be an n-parameter family on $[0, 1]$ and let h be convex to F. Then for any $f, g \in F, m_{0,1}(f, h) \leq n$ and $m_{0,1}(f, g) \leq n-1$.*

Proof. We assume first that 0 and 1 are not identified and that F is an ordinary n-parameter family. We verify the first statement by induction on n. For $n = 1$ the result follows by [15] Theorem 2. Hence, let h be a continuous function convex to a $k+1$ parameter family F and assume the conclusion holds for all k-parameter families. For $f \in F$ let $x_i, i = 1, \cdots, m$, be the zeros of $f - h$ ordered from left to right and assume $m_{0,1}(f, h) > k + 1$. Choose a point u such that $x_1 < u < x_2$. If $F_1 = \{g \in F \mid g(x_1) = h(x_1)\}$, then F_1 is a k-parameter family on $[u, 1]$. $f \in F_1$ and h is convex to F_1. By our inductive assumption $m_{u,1}(f, h) \leq k$. Therefore x_1 must be a zero of $f - h$, and $m_{0,1}$ $(f, h) = k + 2$. By the same reasoning we may assume x_m is a double zero of $f - h$.

We now construct a set E of k points from $[0, 1]$ in the following manner. First choose an $\varepsilon > 0$ such that $x_i + 2\varepsilon < x_{i+1} - 2\varepsilon, i = 1, \cdots, m-1$. If x is a single zero of $f - h$ then let x belong to E. If x is a double zero of $f - h, x \neq x_1, x_m$ let $x + \varepsilon$, and $x - \varepsilon$ belong to E. We add the points $x_1 + \varepsilon, x_m - \varepsilon$. Since $m_{x_1+\varepsilon, x_m-\varepsilon}(f, h) = k - 2$ it is clear that E contains exactly k points. Choose a point $x', x_1 + \varepsilon < x' < x_2 - \varepsilon$. Let f_n be the unique function in F such that

$$f_n(x) = f(x), x \in E$$

$$f_n(x') = f(x') + \frac{1}{n}\operatorname{sgn}[f(x') - h(x')]$$

Now $f_n - f$ has k zeros which must all be simple by [15] Theorem 3. Within the interval $[x_1, x_m]$ $f_n - h$ has exactly k simple zeros since f_n was chosen so that at the points $x_i \pm 2\varepsilon, i = 2, \cdots, m-1, x_1 + 2\varepsilon, x_m - 2\varepsilon, f$ lies between f_n and h. Hence for $0 \leq x < x_1$ and $x_m < x \leq 1, f_n$ and h are on the same side of f (i.e., $\operatorname{sgn}[f_n(x) - f(x)] = \operatorname{sgn}[h(x) - f(x)]$. But by Theorem 1, f_n tends uniformly to f as $n \to \infty$. Hence for n sufficiently large $f_n - h$ must have at least $k + 2$ zeros which is a contradiction.

The case when 0 and 1 are identified and F is periodic causes no difficulty. For if $x_1, \cdots, x_m$ are the zeros of $f - h$, using a suitable rotation we may assume that there is an interval $[a, b]$, such that $0 < a < x_1 < \cdots < x_m < b < 1$. F is an ordinary n-parameter family on $[a, b]$ and $m_{0,1}(f, h) = m_{a,b}(f, h) \leq n$.

The verification of the second assertion is very similar to the above, and we leave the details to the reader.

COROLLARY. *There are no periodic n-parameter families when n is an even integer.*

Proof. Suppose false. Let F be a periodic n-parameter family and n an even integer. Let $f \in F$ and choose x_i $i = 1, \cdots, n$ such that $0 < x_1 < x_2 < \cdots < x_n < 1$. Choose $g \in T$ such that $g(x_i) = f(x_i)$ $i = 1, \cdots, n-1$, $g(x_n) = f(x_n) + 1$. By Theorem 2, $f - g$ changes sign at each of the points x_i, $i = 1, \cdots n-1$; and since $f - g$ can have no other zeros within $[0, 1]$, $g(1) > f(1)$. On the other hand $g(0) < f(0)$ which is a contradiction, since f, g are periodic of period 1.

3. Best approximation in the L_∞ norm.

If g is continuous on $[0, 1]$, $g \notin F$, then $\{g - f\}$ forms a new n-parameter family. Hence without loss of generality we may consider the characterization and construction of the function $\hat{f} \in F$ such that

$$\|\hat{f}\| = \inf_{f \in F} \|f\| \equiv \delta$$

We first adopt the following notation. If $S \subset [0, 1]$

$$\delta_S = \inf_{f \in F} \sup_{t \in S} |f(t)| .$$

Let T denote the class of vectors $\boldsymbol{u} = (u_1, \cdots, u_{n+1})$ satisfying the condition that $0 \leq u_1 < u_2 < \cdots u_{n+1} \leq 1$. The statements and proofs of the results of this section are valid when F consists of continuous periodic functions on $[0, 1]$. We shall assume, however, that F is an ordinary n-parameter family and leave the details in the periodic case to the reader.

The following two lemmas are appropriate generalizations of results of de la Vallee Poussin [6] for polynomials. Where possible we refer the reader to [13] for proofs.

LEMMA 1. *For any $\boldsymbol{u} = (u_1, \cdots, u_{n+1}) \in T$ there exists a unique $f \in F$ and unique real number λ such that $f(u_i) = (-1)^i \lambda \cdot i = 1, \cdots, n+1$. Moreover $|\lambda| = \delta_{\boldsymbol{u}}$ and f is the only function in F with the property that $\max_{i=1,\cdots,n+1} |f(u_i)| = \delta_{\boldsymbol{u}}$. In addition suppose for $k = 1, 2, \cdots$ that*

$$\boldsymbol{u}^{(k)} = (u_1^{(k)}, \cdots, u_{n+1}^{(k)}) \in T \text{ and } f_k\ (u_i^{(k)}) = (-1)^i \lambda^{(k)} .$$

Then if $\boldsymbol{u}^{(k)} \to \boldsymbol{u}$ and $\boldsymbol{u} \in T$, it follows that $f_k \to f$ uniformly on $[0, 1]$ and $\lambda^{(k)} \to \lambda$.

LEMMA 2. *Let $\boldsymbol{u} \in T$ and a sequence of non-negative numbers λ_i $i = 1, \cdots, n+1$ be given. If there exists an $f \in F$ such that*

$$f(u_i) = (-1)^i \lambda_i \; i = 1, \cdots, n+1 \text{ or } f(u_i) = (-1)^{i+1} \lambda_i \; i = 1, \cdots, n+1$$

then either $\min \lambda_i < \delta_{\boldsymbol{u}} < \max \lambda_i$ or $\lambda_i = \delta_{\boldsymbol{u}}$ $i = 1, \cdots, n+1$.

Proof. Lemma 2 is a restatement of Lemma 1 of [13]. Everything in Lemma 1 except the facts that $|\lambda| = \delta_{\boldsymbol{u}}$ and the function f satisfying $\max_{i=1,\ldots,n+1} |f(u_i)| = \delta_{\boldsymbol{u}}$ is unique is proved explicitly in [13]. To prove the latter statements observe that if there is a $g \in F$ satisfying $|g(u_i)| < |\lambda|$ then $f(u_i) - g(u_i) = (-1)^i \lambda_i$ $i = 1, \cdots, n+1$ where either $\lambda_i \geq 0$, $i = 1, 2, \cdots, n+1$ or $\lambda_i \leq 0$ $i = 1, 2, \cdots, n+1$. In either case by [12], Lemma 1, $f - g$ must have at least n zeros between u_1 and u_{n+1} counting multiplicity which is a contradiction.

For $\boldsymbol{u} \in T$ we will usually denote the function f of Lemma 1 by $f_{\boldsymbol{u}}$. Next we define a function $\delta(u_1, \cdots, u_{n+1})$ of $n+1$ variables.

$$\begin{aligned} \delta(\boldsymbol{u}) \equiv \delta(u_1, \cdots, u_{n+1}) &= \delta_{\boldsymbol{u}} \text{ if } \boldsymbol{u} = (u_1, \cdots, u_{n+1}) \in T \\ &= 0 \text{ otherwise} . \end{aligned}$$

If we restrict the points u_i to lie in some subset $S \subset [0, 1]$, then $\delta(u_1, \cdots, u_{n+1})$ will be denoted $\delta_S(u_1, \cdots, u_{n+1})$.

LEMMA 3. *$\delta(u_1, \cdots, u_{n+1})$ is continuous on R^{n+1}*

Proof. Assume that $\delta(u_1, \cdots, u_{n+1})$ is not continuous at some point $\boldsymbol{u} = (u_1, \cdots, u_{n+1})$. We may assume $0 \leq u_1 \leq u_2 \leq \cdots \leq u_{n+1} \leq 1$, and by Lemma 1 we may assume that $m (\leq n)$ of the points u_i are distinct. Consequently $\delta(u_1, \cdots, u_{n+1}) = 0$. Suppose there exists an $\varepsilon > 0$ and a sequence $\{\boldsymbol{u}_k\} \subset T$ such that $\boldsymbol{u}_k \to \boldsymbol{u}$ and $\delta_{\boldsymbol{u}_k} \geq \varepsilon$. Let $u_i^{(k)}$ be the ith coordinate of $\boldsymbol{u}_k$. Choose n points u'_i, $0 \leq u'_i < \cdots < u'_n \leq 1$ such that m of the points u'_i coincide with the m distinct points u_i. Let f_0 be the unique function in F such that $f_0(u'_i) = 0$. Choose η such that for any i $|u'_i - u_i| < \eta$ implies $|f_0(u_0)| < \varepsilon/2$. Choose k so large that all coordinates of $\boldsymbol{u}_k$ are within η neighborhoods of some coordinate of $\boldsymbol{u}'$. Then $f_{\boldsymbol{u}_k}(u_i^{(k)}) - f_0(u_i^{(k)}) = (-1)^i \lambda_i$ where $\operatorname{sgn} \lambda_i^{(k)} = \operatorname{sgn} \lambda_{i+i}^{(k)}$ $i = 1, \cdots, n$. As in the proof of Lemma 1 it follows that $f_{\boldsymbol{u}_k} - f_0$ must have at least n zeros within $[0, 1]$ which is a contradiction.

Using the function $\delta(u_1, \cdots, u_{n+1})$ one can give a simple proof of the Theorem of Motzkin and Tornheim characterizing the function $\hat{f}$ which has minimum deviation from zero.

THEOREM 3. *There exists a unique $\hat{f} \in F$ such that $\|\hat{f}\| = \inf_{f \in F} \|f\|$. $\hat{f}$ is uniquely characterized by the fact that for some $\boldsymbol{u} = (u_1, \cdots, u_{n+1}) \in T$*

$\|\hat{f}\| = \delta_u$. $\boldsymbol{u}$ *will have this property if and only if* $\delta(u_1, \cdots, u_{n+1})$ *is an absolute maximum, and then* $\hat{f} = f_u$.

Proof. Since $\delta(u_1, \cdots, u_{n+1})$ is a continuous function on a compact set, its maximum is attained for some $\boldsymbol{u} = (u_1, \cdots, u_{n+1}) \in T$. Assert $\|f_u\| = \delta_u$. If $\|f_u\| > \delta_u$, then there is a point x' in $[0, 1]$ for which $|f_u(x')| = \|f_u\|$. We form a new vector $\boldsymbol{u}' \in T$ by replacing one coordinate u_i of $\boldsymbol{u}$ by x' in the following way. If $u_i < x' < u_{i+1}$ $i = 1, \cdots, n$ and sgn $f_u(u_i) = \operatorname{sgn} f_u(x')$ then let $u'_j = u_j$, $j \neq i$, and $u'_i = x'$. If $\operatorname{sgn} f_u(u_i) = (-1) \operatorname{sgn} f_u(x')$ let $u'_j = u_j$ $j \neq i+1$ and $u'_{i+1} = x'$. If $x' < u_1 (x' > u_{n+1})$ and $\operatorname{sgn} f_u(u_1) = \operatorname{sgn} f_u(x')$ $(\operatorname{sgn} f_u(u_{n+1}) = \operatorname{sgn} f_u(x'))$ let $u'_j = u_j$ $j \neq 1$ $(j \neq n+1)$ and $u'_1 = x'$ $(u'_{n+1} = x')$. If $\operatorname{sgn} f_u(u_1) = (-1) \operatorname{sgn} f_u(x')$ $(\operatorname{sgn} f_u(u_{n+1}) = (-1) \operatorname{sgn} f_u(x'))$ then let $u'_1 = x'$, $u'_j = u_{j-1}$ $j = 2, \cdots, n+1$ $(u'_j = u_{j+1}, j = 1, \cdots, n, u'_{n+1} = x')$. Now either $f_u(u'_i) = (-1)^i \lambda_i$ $i = 1, \cdots, n+1$ or $f_u(u'_1) = (-1)^{i+1} \lambda_i$ $i = 1, \cdots, n+1$ where $\lambda_i = \delta_u$ or $\lambda_i = \|f_u\|$. Therefore by Lemma 2, $\delta_u < \delta_{u'} < \|f_u\|$ which contradicts the maximality of δ_u.

It now follows immediately that $\|f_u\| = \inf_{f \in F} \|f\|$ and that f_u is the only such function with this property. For if $f_0 \in F$ and $\|f_0\| \leq \|f_u\|$ then $\|f_0\| \leq \delta_u$ which contradicts Lemma 1. Moreover the same argument shows that if there exists an $f_0 \in F$ and a $\boldsymbol{v} \in T$ such that $\|f_0\| = \delta_v$ then $\|f_0\| = \inf_{f \in F} \|f\|$. It is clear that $\delta(v_1, \cdots, v_{n+1})$ must be an absolute maximum.

In the above theorem if $\|f\|$ is replaced by $\|f\|_S = \sup_{t \in S} |f(t)|$ where S is any closed set of $[0, 1]$ containing at least $n+1$ points, then the same conclusions hold. Here of course, the function $\delta(u_1, \cdots, u_{n+1})$ is replaced by $\delta_S(u_1, \cdots, u_{n+1})$ and the points u_k are assumed to be in S. The following generalization of [11] Theorem 7.1 is therefore relevant.

THEOREM 4. *Let* S_k, S *be closed sets of* $[0, 1]$ *such that for each* k, S_k, *contains at least* $n+1$ *points*; S *contains infinitely many points, and* $S_k \subset S$. *Let* $\hat{f}_k, \hat{f}_0$ *be functions from* F *which minimize* $\|f\|_{S_k}, \|f\|_S$ *respectively. If for each* $\varepsilon > 0$ *there exists an integer* k_0 *such that for* $k > k_0$ *each point* $u \in S$ *is at a distance less than* ε *from some point of* S_k, *than* $\hat{f}_k \to \hat{f}_0$ *uniformly on* $[0, 1]$.

Proof. We assume $\delta_S > 0$. $S_k \subset S$ implies $\delta_{S_k} \leq \delta_S$. Choose $\boldsymbol{u} = (u_1, \cdots, u_{n+1}) \in T$, $u_i \in S$ such that $\delta_S(u_1, \cdots, u_{n+1})$ is an absolute maximum. Let $\boldsymbol{u}_k = (u_1^{(k)}, \cdots, u_{n+1}^{(k)}) \in T$, $u_j^{(k)} \in S_k$ be chosen such that $\boldsymbol{u}_k \to \boldsymbol{u}$. By Lemma 1, $\delta_{u_k} \to \delta_u$ and since $\delta_{u_k} \leq \delta_{S_k}$, $\delta_{S_k} \to \delta_u = \delta_S$. Let $\boldsymbol{v}_k = (v_1^{(k)}, \cdots, v_{n+1}^{(k)}) \in T$, $v_i^{(k)} \in S_k$ be chosen so that for each k, $\delta_{S_k}(v_1^{(k)}, \cdots, v_{n+1}^{(k)})$ is an absolute maximum. Extract any convergent subsequence $\boldsymbol{v}_{k_j}$ with limit $\boldsymbol{v}$.

If $\boldsymbol{v} = (v_1, \cdots, v_{n+1})$, then $v_i \in S$ and $\delta_v = \delta_S$. Also $\hat{f}_{k_j} = f_{v_{k_j}}$ tends uniformly to f_v, the function from F with minimum deviation on $\boldsymbol{v}$. But by the uniqueness of $f_v, f_v = \hat{f}_0$. The above argument shows that any subsequence of $\{\hat{f}_k\}$ contains a refinement which converges to $\hat{f}_0$. Hence $\lim_{k\to\infty} \hat{f}_k = \hat{f}_0$ uniformly on $[0, 1]$.

4. The estimation of f. In [13] Novodovorskii and Pinsker consider a direct method, due to Remes [14] in the polynomial case, for the estimation of $\hat{f}$. However the following Lemma shows that $\hat{f}$ is continuously dependent on estimates of the best approximation. Hence if $\boldsymbol{u}$ is a vector in T for which $\delta(\boldsymbol{u})$ is an estimate of $\inf_{h\in F} \|f\|$, then the solution of the equation $f(u_i) = (-1)^i\lambda$ $i = 1, \cdots, n+1$ is the appropriate estimate of $\hat{f}$.

LEMMA 4. *Let $\{\delta_n\}$ be a sequence of non-negative numbers converging to $\delta = \inf_{f\in F} \|f\|$ from below. If $\boldsymbol{u}_n$ are vectors in T for which $\delta(\boldsymbol{u}_n) = \delta_n$, then $\lim_{n\to\infty} f_{\boldsymbol{u}_n} = \hat{f}$ uniformly on* $[0, 1]$.

Proof. If the conclusion is false there exists a subsequence $\{\boldsymbol{u}_{k_j}\}$ and a number $\varepsilon > 0$ such that $\|\hat{f} - f^{u_{k_j}}\| \geq \varepsilon$. But $\{\boldsymbol{u}_{k_j}\}$ may be further refined to obtain a convergent subsequence of vectors. Calling this $\{\boldsymbol{u}_{k_j}\}$ and letting $\boldsymbol{u}_0 = \lim_{j\to\infty} \boldsymbol{u}_{k_j}$ we have by Lemma 1 $\delta(\boldsymbol{u}_0) = \lim_{j\to\infty} \delta(\boldsymbol{u}_{k_j})$. By Theorem 3 $f_{\boldsymbol{u}_0} = \hat{f}$ which is a contradiction.

We shall consider two algorithms for estimating δ and prove convergence of both.

Each of these algorithms can be used efficiently for actual numerical calculations. A detailed description of method 2 for polynomials on a finite point set can be found in [5]. Also for polynomials on an interval a maximization procedure has been announced by Bratton [3].

For both methods the following notation is convenient. For $\boldsymbol{u} = (u_1, \cdots, u_{n+1}) \in T$ define for $j = 1, \cdots, n+1$.

$$\begin{aligned}\delta_{\boldsymbol{u}}^{(j)}(x) &= \delta(u_1, \cdots, u_{j-1}, x, u_{j+1}, \cdots, u_{n+1}) \text{ if } u_{j-1} \leq x \leq u_{j+1} \\ &= 0 \text{ otherwise}\end{aligned}$$

where we take $u_0 = 0$, $u_{n+2} = 1$. We now form $\eta_{\boldsymbol{u}}(x) \equiv \max_{j=1,\dots,n+1} \delta_{\boldsymbol{u}}^j(x)$. From the continuity of $\delta(u_1, \cdots, u_{n+1})$ it follows that for each j, $\delta_u^{(j)}(x)$ is continuous, and hence $\eta_{\boldsymbol{u}}(x)$ is continuous. Therefore there exists a point x', $0 \leq x' \leq 1$ and integer $1 \leq m \leq n+1$ such that

$$\delta_{\boldsymbol{u}}^m(x') = \max_{j=1,\dots,n+1} \|\delta_{\boldsymbol{u}}^j\| = \|\eta_{\boldsymbol{u}}\| .$$

For a given vector $\boldsymbol{u}$ we define $\boldsymbol{u}' = (u_1', \cdots, u_{n+1}')$ by setting $u_j' = u_j$, $j \neq m$, $u_m' = x'$.

THEOREM 5. *If vectors $\boldsymbol{u}_k$ are defined inductively in the above fashion with $\boldsymbol{u}_1 \in T$ chosen arbitrarily, then $\lim_{k\to\infty}\delta(\boldsymbol{u}_k)$ exists and there exists $\boldsymbol{u}_0 \in T$ such that $\delta(\boldsymbol{u}_0) = \lim_{k\to\infty}\delta(\boldsymbol{u}_k)$. Furthermore $\delta(\boldsymbol{u}_0)$ is an absolute maximum of the function $\delta(\boldsymbol{u})$.*

Proof. $\{\delta(\boldsymbol{u}_k)\}$ is a monotonically increasing, bounded sequence hence convergent. If $\delta = \lim_{k\to\infty}\delta(\boldsymbol{u}_k)$, then a suitable subsequence $\{\boldsymbol{u}_{k_j}\}$, converges to $\boldsymbol{u}_0$ and $\delta(\boldsymbol{u}_0)=\delta$. We now assert $\eta_{\boldsymbol{u}_{k_j}}(x)$ converges uniformly to $\eta_{\boldsymbol{u}_0}(x)$. It suffices to assume $u_i \le x \le u_{i+1}$. Then

$$\begin{aligned}|\eta_{\boldsymbol{u}_0}(x) - \eta_{\boldsymbol{u}_{k_j}}(x)| &= |\max(\delta^i_{\boldsymbol{u}_0}(x), \delta^{i+1}_{\boldsymbol{u}_0}(x),)) - \max(\delta^i_{\boldsymbol{u}_{k_j}}(x), \delta^{i+1}_{\boldsymbol{u}_{k_j}}(x))| \\ &\le |\delta^i_{\boldsymbol{u}_0}(x) - \delta^i_{\boldsymbol{u}_{k_j}}(x)| + |\delta^{i+1}_{\boldsymbol{u}_0}(x) - \delta^{i+1}_{\boldsymbol{u}_{k_j}}(x)|\ .\end{aligned}$$

Since $\delta(\boldsymbol{u})$ is a uniformly continuous function the latter expression tends to zero uniformly in x.
Hence

$$\|\eta_{\boldsymbol{u}_0}\| = \lim_{j\to\infty}\|\eta_{\boldsymbol{u}_{k_j}}\|\ .$$

But

$$\|\eta_{\boldsymbol{u}_{k_j}}\| = \delta(\boldsymbol{u}_{k_j+1}) \le \delta(\boldsymbol{u}_{k_{j+1}}) \le \|\eta_{\boldsymbol{u}_{k_{j+1}}}\|$$

Therefore $\|\eta_{\boldsymbol{u}_0}\| = \lim_{j\to\infty}\delta(\boldsymbol{u}_{k_j}) = \delta(\boldsymbol{u}_0)$. It now follows by the same argument as in the proof of Theorem 3 that $\|f_{\boldsymbol{u}_0}\| = \delta(\boldsymbol{u}_0)$ and by Theorem 3, $\delta(\boldsymbol{u}_0)$ is a maximum.

For the second method of estimation of f we alter slightly our definition of $\delta^1_{\boldsymbol{u}}(x)$ and $\delta^{n+1}_{\boldsymbol{u}}(x)$. We now define

$$\begin{aligned}\delta^1_{\boldsymbol{u}}(x) &= \delta(x, u_2, \cdots, u_{n+1}) \text{ if } 0 \le x \le u_2\ . \\ &= \delta(u_2, u_3, \cdots, u_{n+1}, x) \text{ if } u_{n+1} \le x \le 1\end{aligned}$$

$$\begin{aligned}\delta^{n+1}_{\boldsymbol{u}}(x) &= \delta(u_1, \cdots, u_n, x) \text{ if } u_n \le x \le 1 \\ &= \delta(x, u_1, \cdots, u_n) \text{ if } 0 \le x \le u_1\ .\end{aligned}$$

The algorithm proceeds as follows. First let $\varepsilon > 0$ be chosen. Select an arbitrary vector $\boldsymbol{u} \in T$. Maximize $\delta^2_{\boldsymbol{u}}(x)$ over its domain of definition. Let x' be a point for which $\delta^2_{\boldsymbol{u}}(x)$ is a maximum. If $\delta^2_{\boldsymbol{u}}(x') \ge (1+\varepsilon)\delta(\boldsymbol{u})$, replace u_2 by x' forming a new vector u'. If not, let $\boldsymbol{u}' = \boldsymbol{u}$. We now maximize $\delta^3_{\boldsymbol{u}'}(x)$ and continue inductively. Special attention is necessary for $\delta^{n+1}_{\boldsymbol{u}}(x)$ and $\delta^1_{\boldsymbol{u}}(x)$. If x' is a point for which $\delta^{n+1}_{\boldsymbol{u}}(x)$ is a maximum and $\delta^{n+1}_{\boldsymbol{u}}(x) \ge (1+\varepsilon)\delta(\boldsymbol{u})$, then $\boldsymbol{u}'$ is formed in the following way. If $x' \ge u_n$ then $u'_i = u_i$, $i = 1, \cdots, n$, $u'_{n+1} = x'$; if $x' \le u_1$ then $u'_1 = x'$ $u'_i = u_{i-1}$ $i=2, \cdots, n+1$. In the latter case, the next function maximized is $\delta^2_{u'}(x)$. If the first case occurs then $\delta^1_{u'}(x)$ is maximized. Let x'' be a point for which $\delta^1_{u'}(x)$.

is a maximum and $\delta^1_{u'}(x'') \geq (1+\varepsilon)\delta(\boldsymbol{u}')$. If $x'' \leq u'_2$ then $u''_1 = x''$ and $u''_i = u'_i$ $i = 2, 3, \cdots, n+1$. If $x'' \geq u'_{n+1}$ then $u''_i = u_{i+1}$ $i = 1, \cdots, n$ and $u''_{n+1} = x''$. For the first case the next function maximized is $\delta^2_{u''}(x)$; the second case, $\delta^{(1)}_{u''}(x)$. If

$$\delta^{n+1}_{\boldsymbol{u}}(x') < (1+\varepsilon)\delta(\boldsymbol{u})\,(\delta^1_{u'}(x'') < (1+\varepsilon)\delta(\boldsymbol{u}'))$$

then we take $\boldsymbol{u}' = \boldsymbol{u}\,(\boldsymbol{u}'' = u')$. When there have been $n+1$ consecutive maximizations with no change in the vector $\boldsymbol{u}$, ε is now replaced by $\varepsilon/2$ and the process is repeated. We now continue inductively and pass to the limit as $\varepsilon/2^k \to 0$.

THEOREM 6. *The conclusions of Theorem* 5 *hold if the sequence* $\{\boldsymbol{u}_k\}$ *is chosen inductively in accordance with the above algorithm.*

Proof. As before, $\lim_{k\to\infty}\delta(\boldsymbol{u}_k) = \delta$ exists. We choose a particular convergent subsequence $\{\boldsymbol{u}_{k_j}\}$ of $\{\boldsymbol{u}_k\}$. For each j let $\boldsymbol{u}_{k_j}$ be a vector of $\{\boldsymbol{u}_k\}$ such that for each i, $i = 1, \cdots, n+1$ and all appropriate x, $\delta^i_{\boldsymbol{u}_{k_j}}(x) < (1+\varepsilon/2^j)\delta(\boldsymbol{u}_{k_j})$. The algorithm guarantees that for each integer j such a vector $\boldsymbol{u}_{k_j}$ exists in the sequence $\{\boldsymbol{u}_k\}$. Since a refinement of this sequence is convergent, we assume $\{\boldsymbol{u}_{k_j}\}$ converges. Then if $\boldsymbol{u}_{k_j} \to \boldsymbol{u}_0$, $\delta(\boldsymbol{u}_0) = \delta$. Suppose $\delta(\boldsymbol{u}_0)$ is not a maximum of $\delta(\boldsymbol{u})$, then $\|f_{\boldsymbol{u}_0}\| > \delta(\boldsymbol{u}_0)$. Choose x' so that $|f_{\boldsymbol{u}}(x')| = \|f\|$, and form $\boldsymbol{u}'$ by replacing one point, the ith say, of $\boldsymbol{u}_0$ by x' in the manner of the proof of Theorem 3. Form $\boldsymbol{u}'_{k_j}$ by replacing the ith coordinate of $\boldsymbol{u}_{k_j}$ by x' Then $\boldsymbol{u}'_{k_j} \to \boldsymbol{u}'$ and $\delta(\boldsymbol{u}'_{k_j}) \to \delta(\boldsymbol{u}')$. Therefore for j sufficiently large, since $\delta(\boldsymbol{u}') > \delta$,

$$\delta(\boldsymbol{u}'_{k_j}) > \frac{\delta(\boldsymbol{u}') + \delta}{2}$$

On the other hand for each j there is a point x and an integer m such that

$$\delta(\boldsymbol{u}'_{k_j}) = \delta^m_{\boldsymbol{u}_{k_j}}(x) \leq \left(1 + \frac{\varepsilon}{2^j}\right)\delta(\boldsymbol{u}_{k_j}) \leq \left(1 + \frac{\varepsilon}{2^j}\right)\delta\,.$$

For j sufficiently large this is a contradiction, therefore $\|f_{\boldsymbol{u}_0}\| = \delta(\boldsymbol{u}_0)$ and $\delta(\boldsymbol{u}_0)$ is an absolute maximum.

5. Approximation in $L_{p,N}$ norm. For $N \geq n$ let $x_1, \cdots, x_N$ be N distinct points of $[0,1]$. In place of the sup norm let $\|f\| = \{\sum_{i=1}^N |f(x_i)|^p\}^{1/p}$ and assume $p > 1$. The fundamental problem to be considered here is to give necessary and sufficient conditions that the function $\hat{f} \in F$ for which $\|\hat{f}\| = \inf_{f\in F}\|\hat{f}\|$ is unique. Now the image of F under the mapping $f \to (f(x_1), \cdots, f(x_N))$ is a closed set in N dimensional Euclidean space. By a theorem of Motzkin [9] as generalized by Busemann [4], to each point $x \in E_N$ there will exist a unique nearest point in a given set $S \subset E_N$ with respect to a strictly convex metric if and only if S is closed and convex. Hence $\hat{f}$ will be unique if and only if F is convex, but for n-parameter families we can say more.[2]

THEOREM 7. *An n-parameter family F is convex if and only if F is the translate of a linear n-parameter family.*

Proof. If F is the translate of a linear n-parameter family, i.e., there exists a continuous g on $[0, 1]$ and a linear n-parameter family F_0 such that each $f \in F$ can be written uniquely as $f = g + f', f \in F_0$, then F is obviously convex. Conversely suppose F is convex. Choose n distinct points $x_1, \cdots, x_n$ in $[0, 1]$. Let $f_0, f_1, \cdots, f_n$ be the unique functions of F such that $f_0(x_j) = 0, j = 1, \cdots, n; f_k(x_j) = \delta_{kj}$ for $k, j = 1, \cdots, n$ where δ_{kj} is the Kronecker delta. We assert that each $f \in F$ has a representation as

$$f = f_0 + \sum_{k=1}^{n} \lambda_k(f_k - f_0) \text{ where } \lambda_k = f(x_k) .$$

If such a representation exists it is obviously unique. Also the vector space spanned by $f_1 - f_0, \cdots, f_n - f_0$, is obviously an n-parameter family and the theorem is proved. To prove the assertion let

$$F_k = \{f \in F \mid f(x_{k+1}) = f(x_{k+2}) = \cdots = f(x_n) = 0\}$$
$$F'_k = \{f \in F \mid f(x_j) = 0 \ j \neq k\} .$$

From the convexity of F, F'_k is a convex one parameter family on a suitably small interval containing x_k. We assert $f \in F'_k$ implies $f = f_0 + \lambda_k(f_k - f_0)$ where $\lambda_k = f(x_k)$. By convexity this is obviously true for $0 \leq \lambda_k \leq 1$. For $\lambda_k > 1$ if $f \in F'_k, f(x_k) = \lambda_k$ then by convexity

$$f_k = \frac{1}{\lambda_k} f + \left(1 - \frac{1}{\lambda_k}\right) f_0$$

or $f = f_0 + \lambda_k(f_k - f_0)$. If $\lambda_k < 0$,

$$f_0 = \frac{1}{1 - \lambda_k} f + \frac{(-\lambda_k)}{1 - \lambda_k} f_k$$

or $f = f_0 + \lambda_k(f_k - f_0)$. To finish the proof we apply an induction. Assume $f \in F_k$ implies that $f = f_0 + \sum_{j=1}^{k} \lambda_j(x_j - x_0)$ where $f(x_j) = \lambda_j$ and

[2] For a discussion of related results see the article by Motzkin in the Symposium on Numerical Approximation, University of Wisconsin Press, 1959.

suppose $g \in F_{k+1}$ and $g(x_j) = \mu_j$, $j = 1, \cdots, k+1$. Then if $g_1 = f_0 + \sum_{j=1}^{k} 2\mu_j(f_j - f_0)$, $g_2 = f_0 + 2\mu_{k+1}(f_{k+1} - f_0)$ it follows that

$$g' = \frac{g_1 + g_2}{2} \in F_{k+1}$$

and $g'(x_j) = \mu_j$, $j = 1, \cdots, k+1$. Therefore

$$g = g' = f_0 + \sum_{j=1}^{k+1} \mu_j(f_j - f_0) .$$

6. The existence of n-parameter families on compact space. Let $f_1, \cdots, f_n$, be n linearly independent real valued continuous functions defined on a compact set S in finite dimensional Euclidean space. Let V be the span of the functions $f_1, \cdots, f_n$. In 1918 Haar [7] showed that to each continuous real valued function g defined on S, there is a unique $\hat{f} \in V$ satisfying $\|\hat{f} - g\| = \inf_{f \in V} \|f - g\|$ where $\|f\| = \sup_{s \in S} |f(s)|$ if and only if no non-zero function in V vanished at more than $n-1$ points of S. Haar noted that the existence of such a set of functions V placed a severe restriction on the set S. In 1956 Mairhuber [8] proved that if V satisfied the above condition of Haar then S is a homeomorphic image of a subset of the circumference of the unit circle. If n is even this subset must be proper. It is clear that V satisfies the condition of Haar if and only if V is a linear n-parameter family. The characterization of those compact Hausdorff spaces on which there exist n-parameter families F for $n > 1$ seems to be quite difficult. One can give a characterization if one imposes a rather strong local condition on F. The result presented here includes the one of Mairhuber, and is proved by somewhat different means. The following fundamental lemma is perhaps of independent interest.

LEMMA 5. *Let S be a compact connected Hausdorff space with the property that for each point $x \in S$ there exists a neighborhood U_x and continuous real valued functions f_1, f_2 defined on U_x such that for $y, z \in U_x$, $y \neq z$*

$$\begin{vmatrix} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{vmatrix} \neq 0 . \tag{1}$$

Then S may be embedded homeomorphically into the circumference C of the unit circle.

Proof. Without loss of generality we assume U_x is a closed, therefore compact neighborhood of x. f_1, f_2 never vanish simultaneously on U_x and therefore f_1/f_2 defines a continuous mapping of U_x into the compactified real line. (1) guarantees that the mapping is one to one and $\phi_x(u) = \text{Arctan}\,(f_1/f_2)(u)$ gives a homeomorphism of U_x into C.

We next verify that S is locally connected. To do this it suffices to show that for each $x \in S$ there exists a connected neighborhood which can be mapped homeomorphically into C. In fact if ϕ_x is the homeomorphism for a point $x \in S$ constructed above, and if $C_x = \phi_x(U_x)$, it is enough to show that there exists a connected neighborhood V_x in C_x of $\lambda_x \equiv \phi_x(x)$. For then $\phi_x^{-1}(V_x)$ is a connected neighborhood of x contained in U_x. But C_x is a compact subset of C. Therefore let I_x be the component of λ_x in C_x. I_x is a compact connected subset of C. I_x is then either an interval or all of C. If I_x is the latter we are through. Also if I_x is an interval and λ_x an interior point (relative to C) then $\phi_x^{-1}(I_x)$ is the required neighborhood. Hence assume that λ_x is an end point of I_x. This will include that degenerate case when I_x is just one point. We may also assume that there does not exist a suitably small connected neighborhood N of λ_x in C such that $N \cap C_x \subset I_x$. For then $\phi_x^{-1}(N \cap N_x)$ is an appropriate neighborhood of x. Therefore it now must follow that for any connected neighborhood N of λ_x in C there exists λ_1, λ_2 in the interior of N such that $\lambda_1, \lambda_2 \notin C_x$ and $(\lambda_1, \lambda_2) \cap C_x \neq \phi$. If we let $F = \phi_x^{-1}[(\lambda_1, \lambda_2) \cap C_x]$ and $G = \phi_x^{-1}[C_x \sim (\lambda_1, \lambda_2)]$ then $F \cup (S \sim U_x)$ and G separate S which is a contradiction.

We note that S is certainly a separable metric since a finite number of homeomorphic images of subsets of C cover S. Hence by [16] Theorem 5.1, S is arc wise connected.

We now assert S is homeomorphic to a subset of C. Let $U_1, \cdots, U_n$ be a finite collection of connected neighborhoods covering S each of which is homeomorphic to a subset of C. By a suitable rearrangement we may assume that $U_2 \cap U_1 \neq \phi$ and $U_2 \not\subset U_1$. Let $x_1 \in U_1 \sim U_2$, $x_2 \in U_2 \sim U_1$ $x \in U_1 \cap U_2$. Let A be the maximal subset of $U_1 \cup U_2$ connecting x_1, x, x_2. This must be all of $U_1 \cup U_2$, for if $y \in U_1 \cup U_2$ and $y \notin A$, then y may be connected to any point in A by an arc in $U_1 \cup U_2$. If y is connected to A at an end point of A, this is an enlargement of A which contradicts maximality. If y is connected to A at a point other than an end point, then no neighborhood of this point is homeomorphic to a subset of C. This also is a contradiction. If $U_1 \cup U_2$ is not all of S then $U_1 \cup U_2$ is homeomorphic to an arc, and by induction the homeomorphism may be extended to all of S.

THEOREM 8. *For $n > 1$ let F be an n-parameter family of functions defined on a compact Hausdorff space S. Suppose in addition that to each point $x \in S$ there exists a neighborhood N_x and functions $f_1, f_2 \in F$ such that*

$$\begin{vmatrix} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{vmatrix} \neq 0$$

for $y, z \in N_x$, $y \neq z$. *Then there exists a homeomorphism of* S *into the circumference of the unit circle. If* n *is even the image of* S *must be a proper subset of* C.

Proof. First we note that S cannot have a proper subset W homeomorphic to C. If n is even this follows directly from the Corollary to Theorem 2. If n is odd, choose $x \in S \sim W$ and let $F' = \{f \in F \mid f(x) = 0\}$; then F' is an $n-1$ parameter family defined on W. Since $n-1$ is even this is a contradiction. We may therefore assume that if n is even S is not homeomorphic to C.

If I is a component of S then by Lemma 5 there exists a homeomorphism ϕ of I onto the closed interval $[0, 1]$ considered as a subset of C. We assert that if I is not all of S, then ϕ can be extended to an open and closed set $U \supset I$. U and its complement then separate S. If I is itself open in S then we take $U = I$. If not, let $x = \phi^{-1}(0)$, $y = \phi^{-1}(1)$. Let N_x, N_y be compact neighborhoods of x and y respectively and let ϕ_x, ϕ_y be homeomorphisms of N_x and N_y respectively into C. We may assume $\phi_x(x) = 0$, $\phi_y(y) = 1$ and

$$\phi_x(N_x \cap I) \subset [0, 1] \text{ and } \phi_y[N_y \cap I] \subset [0, 1] .$$

If we define ϕ' by

$$\begin{aligned} \phi'(z) &= \phi(z) \quad \text{if } z \in I \\ &= \phi_x(z) \text{ if } z \in N_x \sim I \\ &= \phi_y(z) \text{ if } z \in N_y \sim I \end{aligned}$$

then ϕ' is a homeomorphism of $N_x \cup N_y \cup I \equiv N$ into C. Also int. $N \supset I$. Now $[0, 1] = \phi'(I)$ is the maximal connected subset of $\phi'(N)$ containing $\phi'(I)$. Therefore there exist sequences $\{\lambda_n\}$, $\{\mu_n\}$ of real numbers tending monotonically to 0 from below, and monotonically to 1 from above, respectively such that $\{\lambda_n\} \cap \phi'(N) = \phi$ and $\{\mu_n\} \cap \phi'(N) = \phi$. Choose n large enough that $\phi'^{-1}[\lambda_n, 0] \subset$ interior of N_x and $\phi'^{-1}[1, \mu_n] \subset$ interior of N_y. Clearly $J_n = \phi'^{-1}[\lambda_n, \mu_n]$ is a closed set containing I. J_n is open in the interior of N. Hence J_n is open in S.

Let T be the class of open sets O of S which can be mapped homeomorphically into C. We partially order T in the following way. If $O_1, O_2 \in T$ then $O_1 \leq O_2$ if $O_1 \subset O_2$ and if there exist homeomorphisms ϕ_1, ϕ_2 of O_1, O_2 respectively into C such that ϕ_2 agrees with ϕ_1 on O_1. By Zorn's lemma there exists a maximal element O of T. We assert $O = S$. If not, let $x \in S \sim O$. Then there exists an open and closed set $U \ni x$ and mapping ϕ such that ϕ maps U homeomorphically into C.

$O \cap U$ and $O \sim U$ are separated open sets of S. Hence if ϕ' is any homeomorphism of O into C such $\phi'(O) \cap \phi(U) = \phi$. ϕ'' defined by $\phi''(x) \equiv \phi(x)$, $x \in O \cap U$, $\phi''(x) \equiv \phi'(x)$, $x \in O \sim U$ is also a homeomorphism of O into C. ϕ'' has an obvious extension to $U \cup O$ which contradicts the maximality of O.

COROLLARY. *If F is a linear n-parameter family $(n > 1)$ defined on the compact Hausdorff space S, then S is homeomorphic to a subset of C. If n is even the subset must be proper.*

Proof. We assume S contains more than n points. For a given $x \in S$ choose $n-2$ distinct points $x_1, \cdots, x_{n-2}$ of S outside a suitably small compact neighborhood N_x of x. If $F_x = \{f \in F \mid f(x_i) = 0, i = 1, \cdots, n-2\}$ then F_x is a linear 2-parameter family defined on N_x. Therefore, for any two linearly independent functions f_1, f_2 in F_x,

$$\begin{vmatrix} f_1(y) & f_1(z) \\ f_2(y) & f_2(z) \end{vmatrix} \neq 0 \text{ for } y, z \in N_x, y \neq z .$$

We now apply the theorem.

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UNIVERSITY OF CALIFORNIA LOS ANGELES AND
YALE UNIVERSITY