ON CONTINUATION OF BOUNDARY VALUES FOR PARTIAL DIFFERENTIAL OPERATORS

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Let

$$L = \sum_{i=1}^{n} a_i(x)\partial/\partial x_i + b(x) \tag{1}$$

be a first order partial differential operator acting on m-component vector functions and defined in a bounded domain D with smooth boundary Γ. Suppose the $m \times m$-matrices $a_i(x)$ are hermitian symmetric and continuously differentiable in $D + \Gamma$. Further let the $m \times m$-matrix $b(x)$ be bounded and measurable over $D + \Gamma$.

Recently K. O. Friedrichs [3] has developed a theory of boundary value problems of the type

$$\begin{aligned} (L - \alpha)u &= f, \qquad x \in D \\ Tu &= 0, \qquad x \in \Gamma \end{aligned} \tag{2}$$

where α denotes a nonvanishing real constant and T a certain $m \times m$-matrix defined all over the boundary T and satisfying certain further conditions. Concurrently the author worked on the same type of boundary value problem from a different approach extending Friedrich's results to the case of nonlocal boundary conditions [1].

Study of these extensions showed that investigation of the following problem is of basic importance for the author's method:

The question is asked whether a given m-component vector function φ defined on the boundary Γ can be continued into the domain D to become a classical solution u of the equation

$$L(u) = f$$

where f is any arbitrary measurable function defined and squared integrable over D, which is not given in advance but may be defined after φ has been fixed.

Obviously this question is trivially answered "yes" if the boundary and the boundary function are sufficiently smooth. On the other hand if this is not the case, counter examples can be given. It is trivial to find counter examples for special nonelliptic systems but one also can find some for elliptic systems. For instance if the boundary functions u_0, v_0 on the periphery of the unit circle $x^2 + y^2 = 1$ are defined by

$$u_0 = \alpha(\vartheta) \sin \vartheta/2, \qquad v_0 = -\alpha(\vartheta) \cos \vartheta/2, \qquad 0 \leq \vartheta \leq 2\pi \tag{3}$$

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and if $\alpha(\vartheta)$ is piecewise continuous and has a jump for any $\vartheta_0 \neq 0, 2\pi$, then it will be shown in §4 that there does not exist any couple u, v of real or complex valued functions both being defined and continuously differentiable in the open unit disk $x^2 + y^2 < 1$ and such that

$$\text{(a)} \qquad -u_x + v_y = f, \qquad u_y + v_x = g$$

both are squared integrable over $x^2 + y^2 < 1$;

$$\text{(b)} \qquad u, v \text{ are uniformly bounded on } x^2 + y^2 < 1 \text{ and}$$

$$\text{(c)} \qquad \begin{gathered} \lim_{r\to 1} u(r\cos\vartheta, r\sin\vartheta) = u_0(\vartheta) \\ \lim_{r\to 1} v(r\cos\vartheta, r\sin\vartheta) = v_0(\vartheta) \end{gathered}$$

almost everywhere on $0 \leq \vartheta \leq 2\pi$.

Considering this problem more carefully it shows that the essential reason for this continuation to be impossible is the following:

The above problem can be connected with the differential operator

$$L = a_1\partial/\partial x + a_2\partial/\partial y \tag{5}$$

with a_1, a_2 being the matrices

$$a_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Using this operator notation we can say that the equation

$$L\varphi = \psi \tag{7}$$

with φ, ψ being two component vector functions has no classical solution, defined in the unit disk and achieving the boundary values defined by

$$\varphi(x, y) = (u_0(\vartheta), v_0(\vartheta)) \qquad x = \cos\vartheta, y = \sin\vartheta \tag{8}$$

in the sense of the conditions (a), (b), and (c) mentioned above.

If we define

$$A(\vartheta) = a_1\cos\vartheta + a_2\sin\vartheta \tag{9}$$

$$\tilde{A}(\vartheta) = a_2\cos\vartheta - a_1\sin\vartheta \tag{10}$$

then

$$L = A(\vartheta)\partial/\partial r + r^{-1}\tilde{A}(\vartheta)\partial/\partial\vartheta\ . \tag{11}$$

Hence $A(\vartheta)$ is the coefficient of the derivative in the direction normal to the boundary.

We note that $A(\vartheta)$ is a non-singular (even orthogonal) matrix for every ϑ. It will follow from our development that this is the reason why a continuation of discontinuous boundary values becomes impossible. If for some more general operator L the matrix which corresponds to $A(\vartheta)$ is singular on a point or on a set of points then this set can be allowed to contain discontinuities of certain types. And conversely it will be our main result that if φ_0 is bounded measurable only at the boundary and if in addition $A\varphi_0$ is Lipschitz continuous then a continuation in the above sense is possible.

The main result is stated in Theorem 3.1. Essentially we will obtain the continuation by use of the elementary solution of the parabolic equation

$$\nabla^2 u = \partial u/\partial t \ . \tag{12}$$

We shall use this for a kind of mollifier. In §§1 and 2 we prove some auxiliary results most of which will be known. In order to keep the paper as self contained as possible most of the facts required have been proved explicitely.

1. Auxiliary results. In this section we will establish some known results which have to be used essentially in the following. Let

$$s^2 = s_1^2 + s_2^2 + \cdots + s_p^2 \tag{1.1}$$

and let the function

$$\Phi(s; t) = \Phi(s_1, \cdots, s_p; t) \tag{1.2}$$

be defined by

$$\Phi(s; t) = (4\pi t)^{-p/2} \exp(-|s|^2/4t) \ . \tag{1.3}$$

It is known that this function $\Phi(s; t)$ is the elementary solution of the parabolic equation

$$\nabla^2 u = \sum_{i=1}^{p} \partial^2 u/\partial s_i^2 = \partial u/\partial t \ . \tag{1.4}$$

First we note

LEMMA 1.1.

$$\int |s|^{2k} e - |s|^{2k} e^{-|s|^2} ds = 2^{-k}\pi^{-p/2} p(p+2)(p+4)\cdots(p+2k-2) \ . \tag{1.5}$$

Here the integral extends over the whole $(s_1, \cdots, s_p)$-space.

The proof of Lemma 1.1 can be obtained by repeated application of Green's formula.

LEMMA 1.2. *Let*

$$f(s) = f(s_1, \cdots, s_p) \tag{1.6}$$

be a (scalar) complex valued bounded measurable function defined and nonnegative for

$$-\infty < s_j < \infty, \qquad j = 1, \cdots, p . \tag{1.7}$$

Let s_0 *be any point and let* Δ *denote the cube*

$$|s_j - s_j^0| \leq \delta, \qquad j = 1, \cdots, p . \tag{1.8}$$

Statement. If

$$\lim_{\delta \to 0} \delta^{-p} \int_\Delta f(s) ds = 0 \tag{1.9}$$

then

$$\lim_{t \to 0} \int_\Delta \Phi(s_0 - s'; t) f(s') ds' = 0 \tag{1.10}$$

the integral in (1.10) being taken over the whole s-space.

Proof. It is obvious that we can restrict ourself to the case $s_0 = 0$. Now, (1.9) being satisfied, let

$$\beta(\delta_0) = \sup_{0<\delta \leqq \delta_0} \left\{ \delta^{-p} \int_\Delta f(s) ds \right\}^{(p+1)^{-1}} \tag{1.11}$$

and let

$$\gamma(\delta) = \delta(\delta + \beta(\delta)) \tag{1.12}$$

$\gamma(\delta)$ is a strictly monotonically increasing function of δ, and $\gamma(0) = 0$. Hence the inverse function $\delta = \delta(\gamma)$ exists in some right neighborhood of $\gamma = 0$ and $\delta(0) = 0$. Also

$$\begin{aligned} \gamma^{-p} \int_\Delta f(s) ds &\leqq (\delta + \beta(\delta))^{-p} \beta(\delta)^{p+1} \\ &\leq \beta(\delta) \longrightarrow 0, \delta \longrightarrow 0 . \end{aligned} \tag{1.13}$$

Hence

$$\lim_{\gamma \to 0} \gamma^{-p} \int_\Delta f(s) ds = 0 . \tag{1.14}$$

Let

$$\pi = \delta / \gamma , \tag{1.15}$$

then

$$\lim_{\gamma \to 0} \tau(\gamma) = \infty . \tag{1.16}$$

Let Δ' be the cube $|s_j| \leq \gamma, j = 1, \cdots, p$. Then by (1.15) Δ can be written in the form

$$\Delta = \tau\Delta' \tag{1.17}$$

and (1.14) reads

$$\lim_{\gamma\to 0} \gamma^{-p}\int_{\tau\Delta'} f(s)ds = 0 . \tag{1.18}$$

Now for any given $t > 0$ set $\gamma = t^{1/2}$, then

$$\int\Phi(s_0 - s'; t)f(s')\, ds' = \int\Phi(s; t)f(s)ds = \int_{\tau\Delta'} + \int_{C(\tau\Delta')} \tag{1.19}$$

where $C(\tau\Delta')$ denotes the complement of the cube $\tau\Delta'$ with respect to the whole s-space. But remembering the definition of $\Phi(s; t)$ we obtain for the first integral

$$\leq (4\pi)^{-p/2}\gamma^{-p}\int_{\tau\Delta'} f(s)ds \tag{1.20}$$

and hence for $t \to 0$, i.e., $\gamma \to 0$ the first integral tends to zero by (1.18). On the other hand $f(s)$ is assumed to be uniformly bounded, hence the second integral can be estimated by

$$\begin{aligned} &c_0\int_{C(\tau\Delta')} \Phi(s; t)ds \\ &\leqq c_0(4\pi)^{-p/2}\gamma^{-p}\left\{\int_{|\sigma|\geqq\tau\gamma} e^{-\sigma^2/\gamma^2}d\sigma\right\}^p \\ &= c_0(4\pi)^{-p/2}\left\{\int_{|\sigma|\geqq\tau} e^{-\sigma^2}d\sigma\right\}^p . \end{aligned} \tag{1.21}$$

But by (1.16)

$$\pi = \tau(\gamma) = \tau(t^{1/2}) \tag{1.22}$$

tends to ∞ at $t \to 0$. Therefore the second integral also tends to zero. This proves the lemma.

LEMMA 1.3. *Let $\Phi(s; t)$ be as defined in* (1.3) *and let*

$$\Psi_i(s; t) = \partial/\partial s_i\Phi(s; t) . \tag{1.23}$$

Then

$$\int ds\Phi(s - s'; t)\Phi(s - s''; t) = (8\pi t)^{-p/2} \exp(-|s' - s''|^2/8t) \tag{1.24}$$

$$\begin{aligned} &\int ds\Psi_i(s - s'; t)\Psi_i(s - s''; t) \\ &= (4t)^{-1}(8\pi t)^{-p/2}(1 - (s_i' - s_i'')^2/4t) \exp(-|s' - s''|^2/8t) \end{aligned} \tag{1.25}$$

both integrals being taken over the whole $(s_1, \cdots, s_p)$-space.

Proof. We only remark that

$$\begin{aligned}(1.26)\qquad &\exp(-|s-s'|^2/4t)\exp(-|s-s''|^2/4t)\\ &= \exp(-|s'-s''|^2 8t)\exp(-|\hat{s}|^2/2t)\end{aligned}$$

where we denote

$$(1.27)\qquad \hat{s} = s - 1/2(s'+s'') .$$

Therefore the integral (1.24) equals to

$$(1.28)\qquad (4\pi t)^{-p}\exp(-|s'-s''|^2/8t)\int \exp(-|\hat{s}|^2/2t)d\hat{s}$$

and clearly

$$(1.29)\qquad \int \exp(-|\hat{s}|^2/2t)d\hat{s} = (2\pi t)^{p/2} .$$

This proves the first formula. For the second formula we note that

$$(1.30)\qquad \Psi_i(s;t) = -(2t)^{-1}(4\pi t)^{-p/2}s_i\exp(-|s|^2/4t) .$$

Now

$$(1.31)\qquad (s_i - s_i')(s_i - s_i'') = \hat{s}_i^2 - 1/4(s_i' - s_i'')^2 .$$

Hence the integral (1.25) gets the form

$$\begin{aligned}(1.32)\qquad &(2t)^{-2}(4\pi t)^{-p}\exp(-|s'-s''|^2/8t)\\ &\times\left\{\int \hat{s}\exp(-|\hat{s}|^2/2t)d\hat{s} - 1/4(s_i'-s_i'')^2\int\exp(-|\hat{s}|^2/2t)d\hat{s}\right\} .\end{aligned}$$

But

$$(1.33)\qquad \int \hat{s}_i^2\exp(-|\hat{s}|^2/2t)d\hat{s} = t(2\pi t)^{p/2} .$$

If we substitute (1.29) and (1.33) into (1.32) then we get

$$(1.34)\qquad = (4t)^{-1}(8\pi t)^{-p/2}(1-(s_i'-s_i'')^2/4t)\exp(-|s'-s''|^2/8t)$$

which completes the proof.

LEMMA 1.4. *Let*

$$(1.35)\qquad \Omega_1(s;t) = (2t)^{-1}(4\pi t)^{-p/2}\exp(-|s|^2/4t)$$

$$(1.36)\qquad \Omega_2(s;t) = |s|^2(2t)^{-2}(4\pi t)^{-p/2}\exp(-|s|^2/4t) .$$

Statement.

$$\int ds\Omega_1(s-s';t)\Omega_1(s-s'';t)\sum_{i=1}^{p}(s_i-s_i')(s_i-s_i'') \tag{1.37}$$
$$= -1/2d/dt((8\pi t)^{-p/2}\exp(-|s'-s''|^2/8t))$$

$$\int ds\Omega_2(s-s';t)\Omega_2(s-s'';t)\sum_{i=1}^{p}(s_i-s_i')(s_i-s_i'')$$
$$= -1/2d/dt[(8\pi t)^{-p/2}\{(8t)^{-2}|s'-s''|^4 \tag{1.38}$$
$$+ p(8t)^{-1}|s'-s''|^2 + 1/4(p+2)(p+4)\}\exp(-|s'-s''|^2/8t)] .$$

Proof. We introduce the notation

$$\hat{\sigma} = (2t)^{-1/2}(s - 1/2(s'+s'')), \qquad \sigma^* = (8t)^{-1/2}(s'-s'') \tag{1.39}$$

and we observe that

$$\sum_{i=1}^{p}(s_i-s_i')(s_i-s_i'') = 2t(|\hat{\sigma}|^2 - |\sigma^*|^2) . \tag{1.40}$$

Now if we substitute (1.36) and (1.40) into the integral (1.37) this integral equals

$$(2t)^{-1}(8\pi^2 t)^{-p/2}\exp(-|\sigma^*|^2)\int(|\hat{\sigma}|^2 - |\sigma^*|^2)\exp(-|\hat{\sigma}|^2 d\hat{\sigma}) \tag{1.41}$$
$$= (8\pi)^{-p/2}(p/4t^{-p/2-1} - 1/16|s'-s''|^2 t^{-p/2-2})\exp(-|s'-s''|^2/8t)$$

Here for the evaluation of

$$\int|\hat{\sigma}|^2\exp(-|\hat{\sigma}|^2)d\hat{\sigma} \tag{1.42}$$

Lemma 1.1 has been applied. Now (1.41) is equal to the derivative in (1.37) as can be proved by differentiation. Therefore (1.37) is proved. For the second integral we get in a similar way the expression

$$(2t)^{-1}(8\pi^2 t)^{-p/2}\exp(-|\sigma^*|^2) \tag{1.43}$$
$$\times\int|\hat{\sigma}-\sigma^*|^2|\hat{\sigma}+\sigma^*|^2(|\hat{\sigma}|^2 - |\sigma^*|^2)\exp(-|\hat{\sigma}|^2)d\hat{\sigma} .$$

Here we were using that

$$s - s' = (2t)^{1/2}(\hat{\sigma} - \sigma^*), \qquad s - s'' = (2t)^{1/2}(\hat{\sigma} + \sigma^*) . \tag{1.44}$$

We observe that

$$|\hat{\sigma}-\sigma^*|^2|\hat{\sigma}+\sigma^*|^2 = (|\hat{\sigma}|^2 + |\sigma^*|^2)^2 - 4(\hat{\sigma}\sigma^*)^2 \tag{1.45}$$

and further that

$$\int(\hat{\sigma}\sigma^*)^2(|\hat{\sigma}|^2 - |\sigma^*|^2)\exp(-|\hat{\sigma}|^2)d\hat{\sigma} \tag{1.46}$$

$$= \sum_{i=1}^{p} \left\{ (\sigma_i^*)^2 \int (\hat{\sigma}_i)^2 (|\hat{\sigma}|^2 - |\sigma^*|^2) \exp(-|\hat{\sigma}|^2) d\hat{\sigma} \right\}$$
$$= 1/p |\sigma^*|^2 \int |\hat{\sigma}|^2 (|\hat{\sigma}|^2 - |\sigma^*|^2) \exp(-|\hat{\sigma}|^2) d\hat{\sigma} .$$

Here we used that

$$\int \hat{\sigma}_i \hat{\sigma}_k (|\hat{\sigma}|^2 - |\sigma^*|^2) \exp(-|\hat{\sigma}|_0^2) d\hat{\sigma} = 0, \ i \neq k . \tag{1.47}$$

Substituting (1.45) and (1.46) into (1.43) we get the expression

$$\begin{aligned} &(2t)^{-1}(8\pi^2 t)^{-p/2} e^{-|\sigma^*|^2} \int (|\hat{\sigma}|^4 \\ &\quad + |\sigma^*|^4 + 2(p-2)/p |\hat{\sigma}|^2 |\sigma^*|^2)(|\hat{\sigma}|^2 - |\sigma^*|^2) e^{-|\hat{\sigma}|^2} d\hat{\sigma} . \end{aligned} \tag{1.48}$$

Further

$$\begin{aligned} &(|\hat{\sigma}|^4 + |\sigma^*|^4 + 2(p-2)/p |\hat{\sigma}|^2 |\sigma^*|^2)(|\hat{\sigma}|^2 - |\sigma^*|^2) \\ &= |\hat{\sigma}|^6 + (p-4)/p |\hat{\sigma}|^4 |\sigma^*|^2 - (p-4)/p |\hat{\sigma}|^2 |\sigma^*|^4 - |\sigma^*|^6 . \end{aligned} \tag{1.49}$$

We substitute this into (1.38) and then use Lemma 1.2 to evaluate the integral, then this integral equals

$$\begin{aligned} &\pi^{-p/2}\{1/8p(p+2)(p+4) \\ &\quad + 1/4(p+2)(p-4)|\sigma^*|^2 - 1/2(p-4)|\sigma^*|^4 - |\sigma^*|^6\} . \end{aligned} \tag{1.50}$$

On the other hand by calculating the derivative (1.38) we get the expression

$$\begin{aligned} &-1/2(8\pi)^{-p/2}\{-1/2(p+4)t^{-p/2-1}|\sigma^*|^4 - 1/2p(p+2)t^{-p/2-1}|\sigma^*|^2 \\ &\qquad - 1/8p(p+2)(p+4)t^{-p/2-1}\} \exp(-|\sigma^*|^2) \\ &-1/(2t)(8\pi t)^{-p/2}\{|\sigma^*|^6 + p|\sigma^*|^4 + 1/4(p+2)(p+4)|\sigma^*|^2\} \exp(-|\sigma^*|^2) \\ &= -(2t)^{-1}(8\pi t)^{-p/2} \exp(-|\sigma^*|^2)\{|\sigma^*|^6 + 1/2(p-4)|\sigma^*|^4 \\ &\qquad -1/4(p+2)(p-4)|\sigma^*|^2 - 1/8p(p+2)(p+4)\} . \end{aligned} \tag{1.51}$$

If we substitute (1.50) into (1.49) and then compare the obtained expression with (1.51) we find that both are equal. Therefore formula (1.38) is proved.

2. Lemmata about special integral operators. The following lemma was used earlier by K. O. Friedrichs [2]. It can be considered to be a translation of a theorem about infinite matrices going back to I. Schur [6].

LEMMA 2.1. *Let*

$$X(s; s') = X(s_1, \cdots, s_p; s'_1, \cdots, s'_1, \cdots, s'_p) \tag{2.1}$$

be defined and continuous for $s, s' \in D_0$, D_0 being any region of $(s_1, \cdots, s_p)$-space, and let

$$\gamma = \sup_{s \in D_0} \int_{D_0} |X(s;s')| ds' \tag{2.2}$$

$$\delta = \sup_{t' \in D_0} \int_{D_0} |X(s;s')| ds . \tag{2.3}$$

Statement.

$$\int_{D_0} ds \left| \int_{D_0} X(s, s') u(s') ds' \right|^2 \leq \gamma^\delta \int_{D_0} |u(s)|^2 ds \tag{2.4}$$

holds for every complex valued measurable function $u(s)$ which is squared integrable over D_0.

Proof. By Schwarz' inequality

$$\begin{aligned} \int_{D_0} ds \left| \int_{D_0} X(s; s') u(s') ds' \right|^2 &\leq \int_{D_0} ds \left(\int_{D_0} |X(s; s')| \, |u(s')| \, ds' \right)^2 \\ &\leq \int_{D_0} ds \left\{ \int_{D_0} |X(s; s')| ds' \int_{D_0} |X(s; s')| \, |u(s')|^2 ds' \right\} \\ &\leq \gamma \int_{D_0} |u(s')|^2 \left(\int_{D_0} |X(s; s')| ds \right) ds' \leq \gamma^\delta \int_{D_0} |u(s')|^2 ds' . \end{aligned}$$

Now let $\Phi(s; t)$, $\Psi_i(s; t)$; $\Omega_1(s; t)$, $\Omega_2(s; t)$ be defined as in (1.1), (1.23), (1.35), and (1.36).

LEMMA 2.2.

$$\sum_{i=1}^{p} \iint dsdt \left| \int \Psi_i(s - s'; t) u(s') ds' \right|^2 \leq \int |u(s)|^2 ds \tag{2.5}$$

for every $u(s)$ squared integrable over the whole s-space and having a compact carrier. Here the integral $\int dt$ is taken over the interval $0 \leq t \leq 1$, the integrals $\int ds$ and $\int ds'$ are considered to be taken over the whole s-space.

Proof. First of all by Lemma 1.3:

$$\begin{aligned} &\sum_{i=1}^{p} \iint dsdt \left| \int \Psi_i(s - s'; t) u(s') ds' \right|^2 \\ &= \lim_{\varepsilon \to 0} \iint ds' ds'' \overline{u(s')} u(s'') \int_\varepsilon^1 dt \sum_{i=1}^{p} \int ds \Psi_i(s - s'; t) \Psi_i(s - s''; t) \\ &= \lim_{\varepsilon \to 0} \iint ds' ds'' \overline{u(s')} u(s'') \int_\varepsilon^1 dt (4t)^{-1} (8\pi t)^{-p/2} \\ &\quad \times (p - (4t)^{-1} |s' - s''|^2) \exp(-|s' - s''|^2 / 8t) . \end{aligned} \tag{2.6}$$

But as we saw in the proof of Lemma 1.4 (formula (1.51)) this integrand is equal to

$$(2.7) \qquad -1/2d/dt\{(8\pi t)^{-p/2}\exp(-|s'-s''|^2/8t)\}$$

and hence the right hand side equals to

$$\begin{aligned} &= -1/2\lim_{\varepsilon\to 0}\iint ds'ds''\overline{u(s')}u(s'') \\ &\quad\times \{(8\pi)^{-p/2}\exp(-|s'-s''|^2/8)-(8\pi\varepsilon)^{-p/2}\exp(-|s'-s''|^2/8)\} \\ &\leq 1/2\lim_{\varepsilon\to 0}\int ds'\overline{u(s')}\int ds''(8\pi\varepsilon)^{-p/2}\exp(-|s'-s''|^2/8\varepsilon)u(s'') \\ &\leq 1/2\lim_{\varepsilon\to 0}\left\{\int|u(s)|^2ds(8\pi\varepsilon)^{-p}\right. \\ &\quad\times\left.\int ds'\left|\int\exp(-|s'-s''|^2/8\varepsilon)u(s'')ds''\right|^2\right\}^{1/2}. \end{aligned}$$

Here we were using that the kernel $\exp(-|s'-s''|^2/8)$ is positive definite as can be easily seen by Lemma 1.3. Since

$$(2.8) \qquad \int\exp(-|s'-s''|^2/8\varepsilon)ds' = \int\exp(-|s'-s''|^2/8\varepsilon)ds = (8\pi\varepsilon)^{p/2}$$

Lemma 2.1 yields

$$(2.9) \qquad (8\pi\varepsilon)^{-p}\int ds'\left|\int\exp(-|s'-s''|^2/8\varepsilon)u(s''\right|^2 \leq \int|u(s)|^2ds\ .$$

This completes the proof of Lemma 2.2.

LEMMA 2.3. *Let*

$$(2.10) \qquad \Omega(s;t) = d/dt\Phi(s;t)$$

and let $v(s)$ be Lipschitz continuous over the whole $(s_1, \cdots, s_p)$-space and with compact carrier.

Statement.

$$(2.11) \qquad \iint dsdt\left|\int ds'\Omega(s-s';t)v(s)\right|^2 \leq p\int\sum_{i=1}^{p}|\partial v/\partial s_i|^2ds\ .$$

Proof. Since $\Phi(s;t)$ is a solution of the parabolic equation (1.29) we get

$$(2.12) \qquad \Omega(s;t) = \sum_{i=1}^{p}\partial/\partial s_i\Psi_i(s;t)$$

and hence by Green's formula

$$\int ds'\Omega(s-s';t)v(s') = \sum_{i=1}^{p}\int \Psi_i(s-s';t)v_i(s')ds' \tag{2.13}$$

where we denote

$$v_i(s) = \partial/\partial s_i(v(s)) . \tag{2.14}$$

Consequently

$$\begin{aligned} &\iint dsdt\left|\int ds'\Omega(s-s';t)v(s')\right|^2 \\ &\quad\leq p\sum_{i=1}^{p}\iint dsdt\left|\int ds'\Psi_i(s-s';t)v_i(s')\right|^2 \\ &\quad\leq p\sum_{i=1}^{p}\left(\sum_{k=1}^{p}\iint dsdt\left|\int ds'\Psi_k(s-s';t)v_i(s')\right|^2\right) \\ &\quad\leq p\sum_{i=1}^{p}\int|\partial v/\partial s_i|^2 ds \end{aligned} \tag{2.15}$$

which prove the lemma.

In the following c always denotes a constant not depending on $u(s)$.

LEMMA 2.4.

$$\iint dsdt\left|\int ds'\Omega(s-s';t)(s_i-s'_i)u(s')\right|^2 \leq c\int|u(s)|^2 ds \tag{2.16}$$

for any arbitrary $u(s)$ with compact carrier and squared integrable over the s-space.

Proof. Clearly

$$\begin{aligned} \Omega(s;t) &= d/dt\Phi(s;t) \\ &= (4\pi t)^{-p/2}(|s|^2/(4t)^2 - p/(2t))\exp(-|s|^2/4t) \\ &= \Omega_2(s;t) - p\Omega_1(s;t) . \end{aligned} \tag{2.17}$$

Hence the integral in (2.16) can be estimated by

$$\begin{aligned} &2\sum_{i=1}^{p}\iint dsdt\left|\int ds'\Omega_2(s-s';t)(s_i-s'_i)u(s')\right|^2 \\ &\quad+ 2p^2\sum_{i=1}^{p}\iint dsdt\left|\int ds'\Omega_1(s-s';t)(s_i-s'_i)u(s')\right|^2 . \end{aligned} \tag{2.18}$$

Now this can be written in the form

$$\begin{aligned} &2\lim_{\varepsilon\to 0}\iint ds'ds''\overline{u(s')}u(s'') \\ &\quad\times\int_{\varepsilon}^{1}dt\sum_{i=1}^{p}\int ds\Omega_2(s-s';t)\Omega_2(s-s'';t)(s_i-s'_i)(s_i-s''_i) \\ &+ 2p^2\lim_{\varepsilon\to 0}\iint ds'ds''\overline{u(s')}u(s'') \end{aligned} \tag{2.19}$$

$$\times \int_\varepsilon^1 dt \sum_{i=1}^p \int ds \Omega_1(s-s';t)\Omega_1(s-s'';t)(s_i-s_i')(s_i-s_i'') .$$

We apply Lemma 1.4 and this equals

$$(2.20) \qquad -\lim_{\varepsilon\to 0}\int ds'ds''\overline{u(s')}u(s'') \times \{(8\pi)^{-p/2}\Xi_1(|s-s'|^2/8)-(8\pi\varepsilon)^{-p/2}\Xi_1(|s'-s''|^2/8\varepsilon)\} \times \exp(-|s'-s''|^2/8\varepsilon)$$

where $\Xi_1(\alpha)$ means a certain polynomial in α with constant coefficients and of degree two, the coefficients only depending on p. By a treatment similar to the last expression of Lemma 2.3 we get the final statement.

LEMMA 2.5.

$$(2.21) \qquad \iint dsdt\left|\int ds'|\Omega(s-s';t)||s-s'|^{1+\varepsilon}u(s')\right|^2 \leq c(\varepsilon)\int|u(s)|^2ds$$

for any positive ε *and for any arbitrary* $u(s)$ *with compact carrier and squared integrable over the whole space,* $c(\varepsilon)$ *being a constant independent of* $u(s)$.

Proof. Clearly it suffices to prove the corresponding inequality with $\Omega(s-s';t)$ replaced by $\Omega_j(s-s';t)$, $j=1,2$. In order to achieve these estimates we again use the notation (1.49) and estimate as follows:

$$(2.22) \qquad \int ds|\Omega_1(s-s';t)||\Omega_1(s-s'';t)|[|s-s'|^2|s-s''|^2]^{(1+\varepsilon)/2}$$
$$= (2t)^{-1+\varepsilon}(8\pi^2t)^{-p/2}\exp(-|s'-s''|^2/8t)\int d\hat\sigma e^{-|\hat\sigma|^2} \times \{(|\hat\sigma|^2+|\sigma^*|^2)^2-4(\hat\sigma\sigma^*)^2\}^{(1+\varepsilon)/2}$$
$$= (2t)^{-1+\varepsilon}(8\pi^2t)^{-p/2}\exp(-|s'-s''|^2/8t)J(s'-s''|^2/8t)$$

where

$$(2.23) \qquad J(|\sigma^*|^2) = \int d\hat\sigma e^{-|\hat\sigma|^2}\{(|\hat\sigma|^2+|\sigma^*|^2)^2-4(\hat\sigma\sigma^*)^2\}^{(1+\varepsilon)/2}$$
$$\leq \int d\hat\sigma \exp(-|\hat\sigma|^2)\{|\hat\sigma|^2+|\sigma^*|^2\}^{(1+\varepsilon)}$$
$$\leq 2^\varepsilon\int d\hat\sigma\exp(-|\hat\sigma|^2)|\hat\sigma|^{2+2\varepsilon}+2^\varepsilon|\sigma^*|^{2+2\varepsilon}\int d\hat\sigma\exp(-|\hat\sigma|^2)$$
$$\leq \gamma_1(\varepsilon)t^{-1+\varepsilon}(8\pi t)^{-p/2}\exp(-|s'-s''|^2 8/t)$$
$$+\gamma_2(\varepsilon)t^{-1+\varepsilon}(8\pi t)^{-p/2}[|s'+s''|^2/8t]^{1+\varepsilon}\exp(-|s'-s''|^2/8t) .$$

Here Hoelders inequality has been employed. Hence (2.22) can be estimated as follows:

$$\begin{aligned}
&\iint dsdt \left| \int ds' \left| \Omega_1(s-s';t) \right| |s-s'|^{1+\varepsilon} u(s') \right|^2 \\
&+ \gamma_1(\varepsilon) \int_0^1 dt t^{\varepsilon-1} \iint ds' ds'' \overline{u(s')} u(s'') (8\pi t)^{-p/2} \exp(-|s'-s''|^2/8t) \\
&+ \gamma_2(\varepsilon) \int_0^1 dt t^{\varepsilon-1} \iint ds' ds'' \overline{u(s')} u(s'') \\
&\quad \times (8\pi t)^{-p/2} (|s'-s''|^2/8t)^{1+\varepsilon} \exp(-|s'-s''|^2/8t) \\
&\leq \gamma(\varepsilon) \int_0^1 dt t^{\varepsilon-1} \int |u|^2 ds = \gamma(\varepsilon)^{\varepsilon-1} \int |u(s)|^2 ds \ .
\end{aligned}$$

Here again Lemma 2.1 and Lemma 1.1 were employed. A quite analogous argument is possible for $\Omega_2(s-s';t)$; therefore Lemma 2.5 is proved.

LEMMA 2.6.

$$\iint t^2 dsdt \left| \int \Omega(s-s';t) u(s') ds' \right|^2 \leq c \int |u(s)|^2 ds \tag{2.25}$$

for arbitrary $u(s)$ with compact carrier squared integrable over the whole s-space.

Proof. Again it suffices to prove this inequality for Ω replaced by Ω_2 and Ω_2. Now

$$\begin{aligned}
&\int ds \Omega_1(s-s';t) \Omega_1(s-s'';t) \\
&\quad = (2t)^{-2} (8\pi^2 t)^{-p/2} \exp(-|s'-s''|^2/8t) \int d\hat{\sigma} \exp(-|\hat{\sigma}|^2) \\
&\quad = (2t)^{-2} (8\pi t)^{p/2} \exp(-|s'-s''|^2/8t) \ .
\end{aligned} \tag{2.26}$$

Hence by Lemma 2.1:

$$\begin{aligned}
&\iint ds' ds'' \overline{u(s')} u(s'') \int ds \Omega_1(s-s';t) \Omega_1(s-s'';t) \\
&\quad \leq (2t)^{-2} \int |u(s)|^2 ds \ .
\end{aligned} \tag{2.27}$$

Consequently

$$\begin{aligned}
&\iint t^2 dsdt \left| \int \Omega_1(s-s';t) u(s') ds' \right|^2 \\
&\quad \leq 1/4 \int |u(s)|^2 ds \ .
\end{aligned} \tag{2.28}$$

Again a similar argument proves the corresponding inequality for Ω_2; therefore Lemma 2.6 is proved.

We finally use the preceding lemmata to establish

LEMMA 2.7. *Let*

$$A(s;t) = ((a_{ik}(s;t))) \tag{2.29}$$

be an $m \times m$-matrix with coefficients $a_{ik}(s;t)$ having uniformly Hoelder continuous and uniformly bounded first partial derivatives in the domain

$$D_0 = \{s_1, \cdots, s_p; t \ni -\infty < s_k < +\infty, k = 1, \cdots, p; 0 < t < 1\} . \tag{2.30}$$

Let

$$u(s) = (u_1(s), \cdots, u_m(s)) \tag{2.31}$$

be an m-component vector function having a compact carrier and being squared integrable over the whole $(s_1, \cdots, s_p)$-space. Let the vector function

$$A(s;0)u(s) = v(s) \tag{2.32}$$

be Lipschitz continuous over the whole $(s_1, \cdots, s_p)$-space.

Statement. There exist two constants c_1, c_2 which are independent of $u(s)$ such that

$$\begin{aligned} &\iint dsdt \left| A(s;t) \int ds' \Omega(s-s';t) u(s') \right|^2 \\ &\qquad \leq c_1 \int |u(s)|^2 ds + c_2 \sum_{i=1}^{p} \int |\partial v / \partial s_i|^2 ds . \end{aligned} \tag{2.33}$$

Proof. We decompose as follows:

$$\begin{aligned} A(s;t) \int ds' \Omega(s-s';t) u(s') &= \int \Omega(s-s';t) v(s') ds' \\ &+ (A(s;t) - A(s;0)) \int \Omega(s-s';t) u(s') ds' \\ &+ \sum_{i=1}^{p} \int \Omega(s-s';t)(s_i - s'_i) u_i(s') ds' \\ &+ \int \Omega(s-s';t)[A(s;0) - A(s';0) \\ &- \sum_{i=1}^{p} (s_i - s'_i) \partial / \partial s'_i A(s';0)] u(s') ds' \end{aligned} \tag{2.34}$$

where

(2.35) $$v(s) = A(s;0)u(s),\ u_i(s) = [\partial/\partial s_i(A(s;0))]u(s)\ .$$

By our assumption for $A(s;t)$ we get

(2.36) $$|(A(s;t) - A(s;0))w| \leq ct|w|$$

and

(2.37) $$|[A(s;0) - A(s';0) - \sum_{i=1}^{p}(s_i - s'_i)\partial/\partial s_i(A(s';0))]u(s')| \leq c|s - s'|^{1+\varepsilon}|u(s')|\ .$$

Therefore we can use the Lemmata 2.3, 2.4, 2.5, and 2.6 respectively to estimate the integrals in (2.33) for the succeeding terms in (2.34) by either $c\int|u(s)|^2\,ds$ or $\int|\partial v/\partial s_i|^2 ds$. Hence Lemma 2.7 is proved.

LEMMA 2.8. *Let $u(s)$ be a bounded measurable m-component vector function defined in the whole s-space and let it have a compact carrier. Further, with the notations of Lemma 2.7, let*

(2.38) $$v(s) = A(s;0)u(s)$$

be Lipschitz continuous over the whole s-space.

Let

(2.39) $$u(s;t) = \int\Phi(s - s';t)u(s')ds'\ .$$

Then

(2.40) $$\lim_{t\to 0} u(s;t) = u(s) \text{ almost everywhere}$$

and

(2.41) $$v(s;t) = A(s;t)u(s;t)$$

is continuous all over in the domain D_0 defined in (2.30) and its boundary.

Proof. Let $\varepsilon > 0$ be given. Since $u(s)$ is bounded and measurable, by Lusin's theorem a measurable set E_ε of p-dimensional measure $m(E_\varepsilon)$ less than ε exists such that $u(s)$ is continuous on the complement $C(E_\varepsilon)$ of E_ε with respect to the s-space. If $\chi(s)$ denotes the characteristic function of E_ε and if Δ denotes the cube with sides 2δ defined in (1.8), then by well known facts

(2.42) $$\lim_{\delta\to 0}\delta^{-p}\int_\Delta \chi(s)ds = 0$$

for every $s_0 \in C(E_\varepsilon + N_\varepsilon)$ where N_ε denotes a certain nullset. We will show that for every $s_0 \in C(E_\varepsilon + E_\varepsilon)$ relation (2.40) holds. This will prove the first statement of the lemma, because then obviously it is possible to construct a monotonically decreasing sequence of sets which converges toward a nullset and such that after exempting any set of the sequence the statement (2.40) holds.

Now, $s_0 \in C(N_\varepsilon + E_\varepsilon)$ being given, decompose as follows:

$$\int \Phi(s_0 - s'; t)u(s')ds' = \int_{C(E_\varepsilon)\cap\Delta_0} + \int_{E_\varepsilon\cap\Delta_0} + \int_{C(\Delta_0)} \tag{2.43}$$

where Δ_0 denotes the cube (1.8) with side $\delta = \delta_0$. Then

$$\int_{C(E_\varepsilon)\cap\Delta_0} = \mu_{\Delta_0}\int_{C(E_\varepsilon)\cap\Delta_0} \Phi(s_0 - s'; t)ds' \tag{2.44}$$

where μ_{Δ_0} denotes a mean value of $u(s)$ in the cube Δ_0.

But since u is continuous in $C(E_\varepsilon) \cap \Delta$ it follows that

$$|\mu_{\Delta_0} - u(s_0)| < \varepsilon' \tag{2.45}$$

if $\delta_0 > 0$ is sufficiently small. Also

$$\int_{C(E_\varepsilon)\cap\Delta_0} \Phi(s_0 - s'; t)ds' \leq \int \Phi(s_0 - s'; t)ds' = 1 . \tag{2.46}$$

Consequently, using (2.44) and (2.46) we get

$$\begin{aligned} &\left|\int_{C(E_\varepsilon)\cap\Delta_0} \Phi(s_0 - s'; t)u(s')ds' - u(s_0)\right| \\ &\leq |\mu_{\Delta_0} - u(s_0)| + c\int_{\Delta_0} \Phi(s_0 - s'; t)\chi(s')ds' \\ &+ c\int_{C(\Delta_0)} \Phi(s - s'; t)ds' \end{aligned} \tag{2.47}$$

with $c = \sup|u(s)|$. Finally for the second and third integral in (2.43) we obtain estimates

$$\left|\int_{E_\varepsilon\cap\Delta_0}\right| \leq c\int_{\Delta_0} \Phi(s - s'; t)\chi(s')ds' \tag{2.48}$$

and

$$\left|\int_{C(\Delta_0)}\right| \leq c\int_{C(\Delta_0)} \Phi(s - s'; t)ds' . \tag{2.49}$$

Hence by (2.43), (2.47), (2.48), and (2.49)

$$\begin{aligned} &\left|\int \Phi(s - s'; t)u(s')ds' - u(s_0)\right| \\ &\leq |\mu_{\Delta_0} - u(s_0)|\, 2c\int_{\Delta_0} \Phi(s - s'; t)\chi(s')ds' \\ &+ 2c\int_{C(\Delta_0)} \Phi(s - s'; t)\, ds' . \end{aligned} \tag{2.50}$$

Choosing first δ_0 sufficiently small the first term can be made arbitrarily small; then keeping δ_0 fixed by Lemma 1.2 and (2.42) the second term also can be made arbitraily small by choosing t small. Also the last term for fixed δ_0 becomes arbitrarily small if t tends to zero. Hence formula (2.40) is proved.

In order to prove the continuity of (2.41) we decompose

$$v(s;\ t) = \int \Phi(s - s';\ t) v(s')\, ds' + \int \Phi(s - s';\ t)\,(A(s;\ t) - A(s';\ 0)) u(s')\, ds' \ . \tag{2.51}$$

Since $v(s)$ is assumed to be Lipschitz continuous, the first term obuiously is a continuous function in D_0. The second term is also continuous for every $t > 0$. But since $u(s)$ is assumed to be bounded we get

$$\begin{aligned} &\int \Phi(s - s';\ t)\,(A(s;\ t) - A(s';\ 0)) u(s')\, ds' \\ &\leq ct \int \Phi(s - s';\ t)\, ds' + c' \int \Phi(s - s';\ t)\, |\, s - s'\, |\, ds' \\ &= c''t + c'/\, t^{1/2} \longrightarrow 0,\ t \longrightarrow 0 \ . \end{aligned} \tag{2.52}$$

Therefore the continuity is also proved for $t = 0$. This proves the lemma.

3. **A continuation theorem.** Let D be a bounded domain of the $(x_1, \cdots, x_n)$-space with a twice continuously differentiable boundary Γ which consits of a finite number of simple nonintersecting hyper surfaces. More specifically we assume that the boundary Γ has second derivatives satisfying a uniform Hoelder condition. Let

$$a_i(x) = ((a^i_{jl}(x))),\ i = 1, \cdots, n,\ b(x) = ((b_{ik}(x))) \tag{3.1}$$

be $m \times m$-matrices with complex coefficients defined in $D + \Gamma$. Let $a_i(x)$ be hermitian symmetric and its coefficients be continuously differentiable in $D + \Gamma$ and let the derivatives satisfy a uniform Hoelder condition in $D + \Gamma$. Let $b(x)$ have continuous coefficients in $D + \Gamma$. Let $A(x)$, $x \in D + \Gamma$ be any hermitian symmetric $m \times m$-matrix having continuously differentiable coefficients in $D + \Gamma$ and such that

$$A(x) = \sum_{i=1}^{n} a_i(x)\, \nu_i(x),\ x \in \Gamma \tag{3.2}$$

where $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ denotes the exterior normal on Γ. We define the differential operator L_1 in $\mathfrak{D}_{L_1}$ by

$$L_1 u = \sum_{i=1}^{n} a_i(x) \partial u / \partial x_i + b(x) u(x) \tag{3.3}$$

for complex valued m-component vector functions

$$u(x) = (u_1(x), \cdots, u_n(x)) \tag{3.4}$$

where $\mathfrak{D}_{L_1}$ is the space of all $u(x)$ satisfying the following conditions:

(a) u, $\partial u/\partial x_i$, $i = 1, \cdots, n$, continuous in D.
(b) $u(x)$ uniformly bounded in D.
(c) $\lim_{\varepsilon\to 0} u(x - \varepsilon\nu) = u(x)$ for every $x \in \Gamma$, except possibly on an n-1-dimensional null set.
(d) $v(x) = A(x)\,u(x)$ is continuous on $D + \Gamma$
(e) $\int_D |L_1 u|^2 \, dx < \infty$.

We prove the following

THEOREM 3.1. *Let $u_0(x)$ be an m-component vector function which is defined measurable and bounded on Γ and for which*

$$v_0(x) = A(x)u_0(x) \tag{3.5}$$

is Lipschitz continuous on Γ.

Then there exists a function $u(x) \in \mathfrak{D}_{L_1}$ such that

$$u(x) = u_0(x) \text{ on } \Gamma\,. \tag{3.6}$$

Proof. We consider any arbitray point $x_0 \in \Gamma$. There is a certain neighborhood

$$U_{x_0} = \{x \ni |x - x_0| \leq \varepsilon\} \tag{3.7}$$

which can be mapped by a twice Hoelder continuously differentiable one to one mapping

$$y = y(x) \tag{3.8}$$

onto a bounded region in the $(y_1, \cdots, y_n)$-space in such a way that the point x_0 goes into the origin $y = (0, \cdots, 0)$, the intersection

$$\Gamma_x = \Gamma_0 \cap U_{x_0} \tag{3.9}$$

into a certain neighborhood of $(0, \cdots, 0)$ on the hyperplane $y_1 = 0$, and the intersection

$$D_{x_0} = (D + \Gamma) \cap U_{x_0} \tag{3.10}$$

into a certain half neighborhood of $(0, \cdots, 0)$ satisfying $y_1 \geq 0$. We also can assume that the Jacobian does not vanish.

$$\det\,((\partial y_i/\partial x_k)) \neq 0,\ y \in D_{x_0} + \Gamma_{x_0}\,. \tag{3.11}$$

The image $y(D_{x_0})$ of D_{x_0} under this transformation contains a cube of the type

(3.12) $\quad \mathfrak{Q}_{x_0} = \{y \in 0 \leq y_1 \leq \eta(x_0),\ |y_\nu| \leq 1/2\eta(x_0),\ \nu = 2, \cdots, n\}$.

We denote the intersection of $\mathfrak{Q}_{x_0}$ with the hyperplane $y_1 = 0$ by $\mathfrak{q}_{x_0}$ and we set

(3.13) $$x(\mathfrak{Q}_{x_0}) = \mathfrak{Q}'_{x_0},\ x(\mathfrak{q}_{x_0}) = \mathfrak{q}'_{x_0}$$

where $x = x(y)$ denotes the inverse transformation of (3.8). There is a hypersphere

(3.14) $$U'_{x_0} = \{x \ni |x - x_0| \leq \eta'(x_0)\}$$

such that

(3.15) $$D'_{x_0} = D_{x_0} \cap U'_{x_0} \subset \mathfrak{Q}'_{x_0}$$

and such that the same inclusion still holds for $\eta'(x_0)$ being replaced by a somewhat larger number.

This construction can be employed for every $x_0 \in \Gamma$. Since Γ is a bounded closed set, the whole Γ can be covered by the interior points of a finite number of spheres

(3.16) $$U'_{x_\nu},\ \nu = 1, \cdots, N.$$

There is a decomposition of the identity, i. e., a set of N functions

(3.17) $$\varphi_\nu(x),\ \nu = 1, \cdots, N$$

being defined and infinitely differentiable in the whole $(x_1, \cdots, x_n)$-space and such that

(3.18) $$\varphi_\nu(x) = 0 \text{ outside of } U'_{x_\nu}$$

and

(3.19) $$\sum_{\nu=1}^{N} \varphi_\nu(x) = 1 \text{ on } \Gamma .$$

Now any vector function $u_0(x)$ being given which satisfies the conditions of the Theorem 3.1, define

(3.20) $$u_{\nu,0}(x) = u_0(x)\varphi_\nu(x),\ x \in \Gamma,\ \nu = 1, \cdots, N .$$

Clearly $u_{\nu,0}(x)$ also satisfies the assumptions of Theorem 3.1, especially because

(3.21) $$A(x)u_{\nu,0}(x) = (A(x)u_0(x))\varphi_\nu(x) .$$

We will prove that every $u_{\nu,0}(x)$ can be continued to a function $u_\nu(x) \in \mathfrak{D}_{L_1}$

in the sense of the assertion. This obviously will prove Theorem 3.1, because the sum of all $u_\nu(x)$ will be the desired continuation of $u_0(x)$.

Now, if we apply the mapping just defined in each particular neighborhood D_{x_ν} then the vector function $u_{\nu,0}(x)$ will be transformed into a certain function

$$(3.22) \qquad w_{\nu,0}(y) = u_{\nu,0}(x(y))$$

defined and measurable on $y(\Gamma_{x_\nu})$ which contains the cube $\mathfrak{q}_{x_\nu}$. Since by definition $u_{\nu,0}(x) = 0$ outside of D'_{x_ν} and since

$$(3.23) \qquad y(D'_{x_\nu}) \subset \mathfrak{Q}_{x_\nu}$$

holds, the function $w_{\nu,0}(y)$ is defined for $y \in \mathfrak{q}_{x_\nu}$ and has its carrier in the interior of this n-1-dimensional cube. We can consider $w_{\nu,0}(y)$ as being defined on the whole hyperplane $y_1 = 0$ by setting it equal to zero outside of $\mathfrak{q}_{x_\nu}$. We would like to apply the various lemmata of § 2. In order to do this we first transform the operator L_1 to the new variables y.

$$(3.24) \qquad L_1 = \sum_{i=1}^{n} \tilde{a}_i(y)\partial/\partial y_i + \tilde{b}(y) \ , y \in y(D_{x_\nu})$$

where

$$(3.25) \qquad \tilde{a}_i(y) = \sum_{k=1}^{n} \partial y_i/\partial x_k a_k(x(y)); \ \tilde{b}(y) = b(x(y)) \ .$$

Further we define

$$(3.26) \qquad \tilde{A}(y) = A(x(y)), \ y \in y(D_{x_\nu}) \ ,$$

Clearly it is possible to continue the matrix $\tilde{A}(y)$ to a matrix function being defined, bounded and continuously differentiable on the whole semispace

$$(3.27) \qquad y_1 \geq 0, \ -\infty < y_\nu < +\infty, \ \nu = 2, \cdots, n \ ;$$

its first derivatives satisfying a uniform Hoelder condition in every compact subregion. Now we remark that for

$$(3.28) \qquad y_1 = t, \ y_2 = s_1, \ y_3 = s_2, \cdots, y_n = s_p; \ p = n-1$$

the functions $w_{\nu,0}(y)$ and $\tilde{A}(y)$ satisfy every assumption necessary for application of Lemma 2.2, Lemma 2.7, and Lemma 2.8. Hence the function

$$(3.29) \qquad w_\nu(y) = \int \Phi(s-s'; \ y_1) w_{\nu,0}(s')\, ds'$$

satisfies the following conditions:

(α) w_ν, $\partial w_\nu/\partial y_i$ continuous for $y_1 > 0$.
(β) w_ν uniformly bounded for $y_1 \geq 0$.
(γ) $\lim_{\varepsilon\to 0} w_\nu(y - \varepsilon z)$ exists for every y with $y_1 = 0$ and every vector $z_1 = 1$, $z_j = 0$, $j = 2, \cdots, n$ with the possible exemption of a set of n-1-dimensional measure zero which is contained in $\mathfrak{q}_{x_\nu}$.
(δ) $v_\nu(y) = \tilde{A}(y)w_\nu(y)$ is continuous for $y_1 \geq 0$.
(ε)

$$\int_{y_1\geq 0}\left\{| w_\nu(y) |^2 + | \tilde{A(y)}\partial w_\nu/\partial y_1 |^2 + \sum^n | \partial w_\nu/\partial y_i |^2\right\}dy < \infty \ . \tag{3.30}$$

Finally take any infinitely differentiable function $\tilde{\varphi}_\nu(y)$ being $= 1$ on $y(D'_{x_\nu})$ and having its carrier in $y(D_{x_\nu})$ and take

$$\tilde{w}_\nu(y) = \tilde{\varphi}_\nu(y)w(y) \ . \tag{3.31}$$

Clearly $\tilde{w}_\nu(y)$ also has the properties (α), $\cdots$, (ε). Transform this function back to the old variables and continue it zero outside of $D_{x_\nu}(x)$. Call the new function $u_\nu(x)$. Then it is clear that

$$u_\nu(x) = u_{\nu,0}(x) \text{ on } \Gamma \ . \tag{3.32}$$

Also $u_\nu(x)$ satisfies the conditions (a), (b), (c), and (d). Since

$$| L_1 u_\nu |^2 \leq c\left[| \tilde{A}(y)\partial u_\nu/\partial y_1 |^2 + \sum_{i=2}^n | \partial u_\nu/\partial y_i |^2 + | u_\nu |^2\right] \tag{3.33}$$

(3.30) yields the condition (e) too. Hence $u_\nu(x)$ is the desired continuation and Theorem 3.1 is proved.

4. A counterexample. Let D be the unit circle $x_1^2 + x_2^2 < 1$ and accordingly Γ be the periphery of the unit circle $x_1^2 + x_2^2 = 1$. In D we consider the operator defined in formula (5) of the introduction

$$L_1 = a_1\partial/\partial x_1 + a_2\partial/\partial x_2 \tag{4.1}$$

with

$$a_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{4.2}$$

Then the equation

$$L_1 u = f \tag{4.3}$$

for the 2-component vector functions

$$u = \{u_1, u_2\}, \ f = \{f_1, f_2\} \tag{4.4}$$

defined in $D + \Gamma$ is equivalent to the system

$$(4.5)\qquad \begin{aligned} -\partial u_1/\partial x_1 + \partial u_2/\partial x_2 &= f_1 \\ \partial u_1/\partial x_2 + \partial u_2/\partial x_1 &= f_2 \,. \end{aligned}$$

Hence for real valued u_1, u_2 we get

$$(4.6)\qquad \int_D (f_1^2 + f_2^2)\,dx = \int_D (\partial u_1/\partial x_1 - \partial u_2/\partial x_2)^2 + (\partial u_1/\partial x_2 + \partial u_2/\partial x_1)^2\,dx$$
$$= \int_D [(\partial u_1/\partial x_1)^2 + (\partial u_1/\partial x_2)^2 + (\partial u_2/\partial x_1)^2 + (\partial u_2/\partial x_2)^2]\,dx$$
$$+ 2\int_D (\partial u_1/\partial x_2 \partial u_2/\partial x_1 - \partial u_1/\partial x_1 \partial u_2/\partial x_2)\,dx \,.$$

Now, assuming u twice continuously differentiable in $D + \Gamma$ we can apply Green's formula to the last integral:

$$(4.7)\qquad \int_D (\partial u_1/\partial x_2 \partial u_2/\partial x_1 - \partial u_1/\partial x_1 \partial u_2/\partial x_2)\,dx$$
$$= \int_\Gamma u_1(x_2 \partial u_2/\partial x_1 - x_1 \partial u_2/\partial u_2/\partial x_2)\,d\sigma \,.$$

Hence the last integral in (4.6) is equal to

$$(4.8)\qquad 2\int_\Gamma u_1(x_2\partial u_2/\partial x_1 - x_1 \partial u_2/\partial x_2)\,d\sigma = -2\int_0^{2\pi} u_1 \partial u_2/\partial\vartheta\,d\vartheta$$

where

$$(4.9)\qquad \vartheta = \text{arc } tg\; x_2/x_1 \,.$$

Now we impose on u the condition

$$(4.10)\qquad u_1 \sin\vartheta/2 + u_2\cos\vartheta/2 = 0 \,.$$

Then

$$(4.11)\qquad -2\int_0^{2\pi} u_1\partial u_2/\partial\vartheta d\vartheta = \int_0^{2\pi} [\partial/\partial\vartheta(u_2^2)] \cot\vartheta/2$$
$$= -\int_0^{2\pi} u_2^2 \partial/\partial\vartheta(\cot\vartheta/2)\,d\vartheta = 1/2\int_0^{2\pi} u_2^2 \sin^{-2}\vartheta/2\,d\vartheta \,.$$

This integration by parts is legitimate because the condition (4.10) implies $u_2 = 0$ at $\vartheta = 0,\ 2\pi$. Since u is supposed to have continuous first derivatives it follows that $u_2^2 \sin^{-2}\vartheta/2$ remains bounded also for $\vartheta = 0,\ 2\pi$. Consequently

$$(4.12)\qquad \int_D |L_1 u|^2\,dx = \int_D |f|^2\,dx$$
$$= \int_D [(\partial u_1/\partial x_1)^2 + (\partial u_1/\partial x_2)^2 + (\partial u_2/\partial x_1)^2 + (\partial u_2/\partial x_2)^2]\,dx$$
$$+ 1/2\int_0^{2\pi} u_2^2 \sin^{-2}\vartheta/2\,d\vartheta \,.$$

Since the last integral is nonnegative we obtain

$$\int_D |L_1 u|^2 dx \geq \int_D [(\partial u_1/\partial x_1)^2 + (\partial u_1/\partial x_2)^2 + (\partial u_2/\partial x_1)^2 + (\partial u_2/\partial x_2)^2]\, dx\ . \tag{4.13}$$

Next assume $\varphi = \{\varphi_1, \varphi_2\}$ to be some function satisfying the conditions (a), (b), (c), and (e), of Theorem 3.1 applied to the special operator L_1 defined in (4.1). Also assume that on the boundary Γ:

$$\varphi_1 = \alpha(\vartheta)\cos\vartheta/2,\ \ \varphi_2 = -\,\alpha(\vartheta)\sin\vartheta/2,\ \ 0 \leq \vartheta \leq 2\pi\ . \tag{4.14}$$

Let $\alpha(\vartheta)$ be real valued and piecewise continuous but not continuous. Then we will show that this leads to a contradiction.

First of all the vector function φ can be assumed to be real valued in $D + \Gamma$ because any complex valued such φ being given, $1/2(\varphi + \bar{\varphi})$ would satisfy the same conditions as φ and would be real valued.

Now, if L_- in $\mathfrak{D}_{L_-}$ denotes the restriction of the operator L_1 in $\mathfrak{D}_{L_1}$ to the space $\mathfrak{D}_{L_-}$ of all functions twice continuously differentiable in $D + \Gamma$ and satisfying the boundary conditions (4.10) then we obtain a dissipative operator in the sense of R. S. Phillips [4], which is characterized by local boundary conditions. For the matrix

$$A = \sum_{i=1}^{2} a_i \nu_i = a_1 \cos\vartheta + a_2 \sin\vartheta \tag{4.15}$$

we get the representation

$$A(\vartheta) = \begin{pmatrix} -\cos\vartheta & \sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} = \begin{pmatrix} \sin^2\vartheta/2, & \sin\vartheta/2\cos\vartheta/2 \\ \sin\vartheta/2\cos\vartheta/2, & \cos^2\vartheta/2 \end{pmatrix} - \begin{pmatrix} \cos^2\vartheta/2, & -\sin\vartheta/2\cos\vartheta/2 \\ -\sin\vartheta/2\cos\vartheta/2, & \sin^2\vartheta/2 \end{pmatrix} \tag{4.16}$$

and it is easy to see that the two matrices of this last decomposition are identical with the matrices P_0 and N_0 respectively which project orthogonally onto the spaces of all eigenvectors corresponding to the eigenvalues $+1$ and -1 respectively. The boundary condition

$$P_0 u = 0 \ \text{on}\ \Gamma \tag{4.17}$$

obviously is equivalent to the condition (4.10). Hence the inner product $\bar{u}Au$ is ≤ 0 for all u satisfying the condition (4.17) (or (4.10)). Hence

$$Q(u,\ u) = 2Re\int_D \bar{u} L_1 u\, dx = \int_\Gamma \bar{u} A u\, d\sigma \leq 0\ , \tag{4.18}$$

which proves that L_- in $\mathfrak{D}_{L_-}$ is dissipative. On the other hand in the sense of K. O. Kriedrichs [3] this boundary condition is "admissible", because

$$A = P_0 - N_0,\ P_0 \geq 0,\ N_0 \geq 0 . \tag{4.19}$$

Also the rank of A is constantly equal to two.

Hence if L_-^* in $\mathfrak{D}_{L_-^*}$ denotes the adjoint of L_- in $\mathfrak{D}_{L_-}$ with respect to the inner product

$$\langle u,\ v\rangle = \int_D \bar{u}v\, dx \tag{4.20}$$

and if L_+ in $\mathfrak{D}_{L_+}$ denotes the operator analogous to L_- in $\mathfrak{D}_{L_-}$ with the boundary condition (4.17) replaced by $N_0 u = 0,\ x \in \Gamma$, then

$$L_-^{**} = L_+^* . \tag{4.21}$$

But φ is a function of L_+^* because from the conditions (a), (b), (c) and (e) it follows immediately that

$$\langle \varphi,\ Lu\rangle + \langle L\varphi,\ u\rangle = \int_\Gamma \bar{\varphi} A u\, d\sigma = 0 \tag{4.22}$$

for all $u \in \mathfrak{D}_{L_+}$. Hence (4.21) implies

$$\varphi \in \mathfrak{D}_{L_-^{**}} . \tag{4.23}$$

Therefore a sequence $\varphi^n \in \mathfrak{D}_{L_-}$ exists such that

$$\langle \varphi^n - \varphi,\ \varphi^n - \varphi\rangle \longrightarrow 0,\ n \longrightarrow \infty \tag{4.24}$$

$$\langle L_1(\varphi^n - \varphi),\ L_1(\varphi^n - \varphi)\rangle \longrightarrow 0,\ n \longrightarrow \infty . \tag{4.25}$$

Now (4.25) implies

$$\langle L_1(\varphi^n - \varphi^m),\ L_1(\varphi^n - \varphi^m)\rangle \longrightarrow 0,\ n,\ m \longrightarrow \infty . \tag{4.26}$$

Let

$$\varphi^{n,m} = \varphi^n - \varphi^m \tag{4.27}$$

then (4.13) yields

$$\langle \partial\varphi^{nm}/\partial x_1,\ \partial\varphi^{nm}/\partial x_1\rangle + \langle \partial\varphi^{nm}/\partial x_2,\ \partial\varphi^{nm}/\partial x_2\rangle \longrightarrow 0,\ n,\ \mathrm{m} \to \infty . \tag{4.28}$$

Hence $\partial\varphi^n/\partial x_1$, $\partial\varphi^n/\partial x_2$ converges in the square mean. Let

$$\partial\varphi^n/\partial x_1 \longrightarrow \psi,\ n \longrightarrow \infty , \tag{4.29}$$

and let u be any vector function continuously differentiable in $D + \Gamma$ and vanishing outside of some circle $|x| \leq r < 1$. Then

$$\langle \partial\varphi^n/\partial x_1,\ u\rangle = - \langle \varphi^n,\ \partial u/\partial x_1\rangle . \tag{4.30}$$

For $n \to \infty$ we get

$$\langle \varphi, u \rangle = - \langle \varphi, \partial u / \partial x_1 \rangle \ . \tag{4.31}$$

But φ is continuously differentiable for $|x| < 1$. Hence, using the special properties of u, we get

$$\langle \psi, u \rangle = - \langle \varphi, \partial u / \partial x_1 \rangle = \langle \partial \varphi / \partial x_1, u \rangle \ . \tag{4.32}$$

Or

$$\langle \psi - \partial \varphi / \partial x_1, u \rangle = 0 \tag{4.33}$$

for all u with the above properties. But the set of all such u is dense in the space L_2; hence

$$\psi = \partial \varphi / \partial x_1 \ . \tag{4.34}$$

In the same manner we obtain the relation

$$\partial \varphi^n / \partial x_2 \longrightarrow \partial \varphi / \partial x_2 \ . \tag{4.35}$$

Hence the derivatives $\partial \varphi / \partial x_1$, $\partial \varphi / \partial x_2$ are squared integrable and the Dirichlet-integral of φ exists.

But it is a well known fact that a function φ with the properties (a), (b), (c) which is piecewise continuous on the periphery of the unit circle and has a jump, cannot have the Dirichlet integral existing.

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