INTEGRAL BASES IN INDUCTIVE LIMIT SPACES

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1. The question of the continuity of the coefficient functionals associated with series bases has attracted the attention of several writers (for example Banach [3], p. 110, Newns [9]). Continuity was established by Banach for Banach spaces, by Newns for Fréchet spaces. The object of this paper is to extend these results in two ways. In the first place, we aim to replace Fréchet spaces by certain inductive limits of these. In the second place, we wish to replace the concept of series bases (discrete case) by that of integral bases (continuous case). The latter extension seems worthwhile inasmuch as it is the situation presented, for example, by continuous-spectrum eigenfunction expansions; the L^2 Fourier transform theory presents the simplest instance. Such examples indicate further that one should allow the basis elements to be chosen from a space larger than that formed by the elements admitting an expansion in terms of the basis elements.

Thus the situation will be roughly as follows. F will be a separated locally convex space, E a vector subspace of F endowed with its own topology finer than that induced by the topology of F. T will be a locally compact space, and the basis elements will be the values of a function $u: t \to u(t)$ mapping T into F. We shall admit a representation of each x in E as some sort of integral $\int_T u(t)d\mu(t)$, where μ is a measure (i.e. a Radon measure on T) depending upon x. It is too much to ask that this integral be weakly absolutely convergent: to demand this would be to exclude at the outset the original case of series bases. Instead, one will take a fixed increasing sequence (K_n) of compact subsets of T and interpret $\int_T u(t)d\mu(t)$ as the limit, in E, of the integrals $\int_{K_n} u(t)d\mu(t)$. (One might even "sum" the integral by a fixed sequence of "summation factors": such a modification would not require any essential change in the arguments to follow, and we shall not dwell further on this point.) The simpler case of weakly absolutely expansions will be dealt with in §5. In any case, the interpretation of the integral having been decided, we assume that μ lies in some topological vector space M of measures, and then examine when μ depends continuously on x. By analogy with the series case, this continuity of μ may be described by saying that u defines a *Schauder* integral basis in E.

All the hypotheses roughly described above require elaboration, and this will be undertaken in §2.

Our main result, Theorem 1, succeeds in extending the Banach-

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Newns Theorem in one of several possible natural ways. Another variation is given in Theorem 2, where the series prototype is a weak absolute basis. Theorem 3 deals with what may be called weak Schauder integral bases.

2. Elaboration of the hypotheses.

(A) Throughout the entire paper E and F are related in the manner described in §1. Moreover, we shall always assume that E is an inductive limit of Fréchet spaces. In its widest sense we shall interpret this in the following way (cf. [4], p. 61 et seq): there is a family (E_α) of Fréchet spaces and linear maps $\varphi_\alpha : E_\alpha \to E$ such that the topology $\mathscr{C}$ of E is the finest locally convex topology on E for which each φ_α is continuous. If each E_α is a vector subspace of E and φ_α is the injection map of E_α into E, we speak of an "internal" inductive limit; the continuity requirement then means that $\mathscr{C} | E_\alpha \leq \mathscr{C}_\alpha$ for each α, where $\mathscr{C}_\alpha$ is the topology on E_α.

Following a referee's suggestion, we have limited the statement of Theorem 1 to the case in which E is a *strict* inductive limit of Fréchet spaces, even though the theorem holds for more general spaces. Regarding this, see the proof of Theorem 1 and subsequent remarks in §4.

(B) Throughout the paper M is to be a vector space of (Radon) measures (on T). M is assumed to be a locally convex space, but the precise conditions will vary.

(C) *Concerning integrals.* F and E are visualised as imbedded in F'^*. (For any vector space G, G^* denotes the algebraic dual of G; for any topological vector space G, G' denotes the topological dual of G.) Let μ be a measure, K a μ-measurable subset of T, and $y : t \to y(t)$ a function mapping K into F. The WF-($=$weak-F) integral $\int_K y(t) d\mu(t)$ exists whenever $t \to \langle y(t), y' \rangle$ is μ-integrable over K for each y' in F' ($\langle\ ,\ \rangle$ is the bilinear form expressing the duality between F and F'); the value of this integral is then the element z of F'^* defined by

$$z(y') = \int_K \langle y(t), y' \rangle \, d\mu(t)$$

for all y' in F'. Naturally, z may or may not belong to F. (There is no call to use any of the so-called "strong" theories of the integral.)

It is perhaps convenient to recall here some simple sufficient conditions ensuring that z shall exist and belong to F. Consider the following conditions:

(C_1) *F is complete;*

(C_2) *$t \to y(t)$ is almost separably-valued*[1];

[1] This means that y, when restricted to the complement of a suitable μ-null set, has a range lying in some separable subspace of F.

(C_3) *$t \to \langle y(t), y'\rangle$ is μ-measurable on K for each y' in F';*

(C_4) $\int_K^* p(y(t))d\mu(t)^2 < +\infty$ *for each continuous seminorm p on F.*

It will be shown in Appendix 3 that: *if conditions* (C_1)-(C_4) *are fulfilled, then the WF-integral* $z = \int_K y(t)d\mu(t)$ *exists and belongs to F.*

(D) *The main hypotheses.* The function $u : t \to y(t)$ mapping T into F, and the increasing sequence (K_n) of compact subsets of T, are suppose a given.

(I) $\int_{K_n} u(t)d\mu(t) \in F$ *for each n and each μ in M.*

(II) *There exists a subset S of F' which separates points of F and which is such that, for each n and each y' in S, the linear form* $\mu \to \int_{K_n} \langle u(t), y'\rangle d\mu(t)$ *is continuous on M.*

(III) *For each x in E there exists a unique $\mu = \mu_x \in M$ such that* $s_n(x) = \int_{K_n} u(t)d\mu_x(t)$ *belongs to E for each n, and* $x = \mathscr{C}\text{-}\lim_{n\to\infty} s_n(x)$.

All the integrals appearing in (I)-(III) are to be *WF*-integrals.

The hypotheses (I)-(III) formulate in precise terms one of several possible senses in which the mapping u may be said to generate an integral basis for the space E. Unlike the case of series bases, the ambient space F appears in a relatively important role; the series case is discussed in §7.

3. The main theorem.

This amounts to a more-or-less direct analogue and extension of the Banach-Newns Theorem.

THEOREM 1.[3] *Suppose that E is the strict inductive limit of Fréchet spaces E_α, that conditions* (I)-(III) *hold, and that M is a Fréchet space. Then:*

(i) *the mapping $X : x \to \mu_x$ is continuous from E into M;*

(ii) *the mappings $s_n (n = 1, 2, \cdots)$ are equicontinuous from E into itself.*

Proof. The proof we give is rather lengthy. It is perhaps not the shortest possible, but it is thought to be more illuminating than the alternatives considered. We give first an outline of the main steps.

The idea is to define vector subspaces G_α of E, and to equip each G_α with a topology $\mathscr{G}_\alpha$ which makes G_α into a Fréchet space in such a way that $E = \bigcup_\alpha G_\alpha$ and $\mathscr{C}$ is the inductive limit of the $\mathscr{G}_\alpha$. These G_α will, when compared with the E_α, have the advantage that

[2] $*$ signifies the upper integral.

[3] This is the "continuous" analogue of Theorem 10 of [**1**]; the proof has been remodelled and adapted.

the $\mathscr{G}_\alpha$-continuity of $X|G_\alpha : G_\alpha \to M$, and the $\mathscr{G}_\alpha$-equicontinuity of the $s_n|G_\alpha : G_\alpha \to E$, are both obvious because of the way in which the G_α are chosen. Then (i) and (ii) will follow immediately, For example, given any convex neighbourhood V of 0 in E, the set U of x in E, such that $s_n(x) \in V$ for all n, is convex and will have the property that $U \cap G_\alpha$ is a neighbourhood of 0 in G_α (by equicontinuity of the $s_n|G_\alpha$); hence U will be a neighbourhood of 0 in E, and (ii) will be established. (Alternatively: continuity of $s_n|G_\alpha$ for a fixed n and all α implies the continuity of s_n; the s_n are pointwise bounded, and E is a t-space, and so their equicontinuity follows ([5], p. 27, Théorème 2).)

Before proceeding to the details, we wish to recall two properties of strict inductive limits of Fréchet spaces which contribute essentially to the arguments, namely:

(1) *$\mathscr{C}|E_\alpha = \mathscr{C}_\alpha$, so that each E_α is closed in E;*

(2) *each bounded subset of E is bounded in E_α, for some α depending on the set in question.*

Now these imply in particular the properties (A1)-(A3) below, denoted collectively by (A^*):

(A1) *E is the internal inductive limit of a countable increasing directed family (E_α) $(\alpha = 1, 2, \cdots)$ of Fréchet spaces, each E_α being sequentially closed in E.*

(A2) *As (2) above.*

(A3) *If (x_n) is bounded in E_α and Cauchy in E, then it is convergent in E to some $x \in E_\alpha$, and $p_\alpha(x) \leq \operatorname{Sup}_n p_\alpha(x_n)$ holds for each p_α of a defining family of seminorms for $\mathscr{C}_\alpha$.*

Inspection of the proof shows that (A1)-(A3) suffice to ensure the the validity of Theorem 1. There are some familiar spaces (spaces of functions holomorphic on a closed set, for example) which satisfy (A^*) without being strict inductive limits of Fréchet spaces. For further remarks, see §4.

We now proceed to the details of the proof of Theorem 1.

(a) *Definition of G_α and $\mathscr{G}_\alpha$.* We define G_α to be the vector subspace of E formed of those x for which the $s_n(x) (n = 1, 2, \cdots)$ form a $\mathscr{C}_\alpha$-bounded subset of E_α. By (A1) and (III), $G_\alpha \subset E_\alpha$; and by (A2), $E = \bigcup_\alpha G_\alpha$. We shall topologize G_α as follows: Let $p_{\alpha,r} (r = 1, 2, \cdots)$ be a system of seminorms defining $\mathscr{C}_\alpha$, chosen so that each satisfies (A3); and let $q_r (r = 1, 2, \cdots)$ be a system of seminorms defining the topology of M. The new topology $\mathscr{G}_\alpha$ on G_α shall then be defined by the seminorms

$$(1) \qquad N_{\alpha,r}(x) = q_r(\mu_x) + \operatorname*{Sup}_n p_{\alpha,r}(s_n(x)) \qquad (r = 1, 2, \cdots) .$$

Since plainly $q_r(X(x)) \leq N_{\alpha,r}(x)$ for x in G_α, the continuity of $X|G_\alpha$ is clear. The inequality $\operatorname{Sup}_n p_{\alpha,r}(s_n(x)) \leq N_{\alpha,r}(x)$ likewise exhibits the $\mathscr{G}_\alpha$-

equicontinuity of the s_n. Moreover, (A3) and (III) yield $p_{\alpha,r}(x) \leq N_{\alpha,r}(x)$ for x in G_α, so that

$$(a_1) \qquad \mathscr{C}_\alpha | G_\alpha \leq \mathscr{G}_\alpha .$$

It is also easily seen that

$$(a_2) \qquad E_\alpha \subset E_\beta \text{ implies } G_\alpha \subset G_\beta \text{ and } \mathscr{G}_\beta | G_\alpha \leq \mathscr{G}_\alpha .$$

(b) *$\mathscr{G}_\alpha$-completeness of G_α.* Let (x_i) be a $\mathscr{G}_\alpha$-Cauchy sequence in G_α and put temporarily $\mu_i = \mu_{x_i} = X(x_i)$. Reference to (1) shows that (μ_i) is Cauchy in M and so $\mu = \lim_i \mu_i$ exists in M. At the same time, (1) gives for each r:

$$(2) \qquad p_{\alpha,r}(s_n(x_i) - s_n(x_j)) \leq \varepsilon_{\alpha,r,i,j} ,$$

where $\lim_{i,j\to\infty} \varepsilon_{\alpha,r,i,j} = 0$; in (2) there is uniformity with respect to n. The sequence $(s_n(x_j))$ (j varying) is thus Cauchy in E and is $\mathscr{C}_\alpha$-bounded. By (A3), therefore, this sequence converges in E to a limit s_n, s_n is in E_α, and

$$(3) \qquad p_{\alpha,r}(s_n(x_i) - s_n) \leq \eta_{\alpha,r,i} ,$$

where $\lim_{i\to\infty} \eta_{\alpha,r,i} = 0$; once again there is uniformity with respect to n. From this it is easily deduced that (s_n) is $\mathscr{C}$-Cauchy and $\mathscr{C}_\alpha$-bounded. So, by (A3) once more, $x = \mathscr{C}\text{-}\lim s_n$ exists and belongs to E_α; moreover, letting $n \to \infty$ in (3), we derive

$$(4) \qquad p_{\alpha,r}(x_i - x) \leq \eta_{\alpha,r,i} ,$$

so that $x = \mathscr{C}_\alpha\text{-}\lim x_i$.

On the other hand, since $\mu_i \to \mu$ in M, (II) shows that for y' in S one has

$$\langle s_n, y' \rangle = \lim_j \langle s_n(x_j), y' \rangle = \lim_j \int_{K_n} \langle u, y' \rangle d\mu_j = \int_{K_n} \langle u, y' \rangle d\mu .$$

By (I), $\int_{K_n} u d\mu$ belongs to F; so, since S separates points of F, we conclude that $\int_{K_n} u d\mu = s_n \in E$ and the s_n are $\mathscr{C}$-convergent to x (proved above). From (III) we see that necessarily $\mu = \mu_x$ and hence $s_n = s_n(x)$. Thus (3) reads

$$(5) \qquad p_{\alpha,r}(s_n(x_i) - s_n(x)) \leq \eta_{\alpha,r,i} .$$

So finally

$$\begin{aligned} N_{\alpha,r}(x_i - x) &= q_r(\mu_x - \mu_{x_i}) + \operatorname*{Sup}_n p_{\alpha,r}(s_n(x_i) - s_n(x)) \\ &\leq q_r(\mu - \mu_i) + \eta_{\alpha,r,i} \end{aligned}$$

converges to 0 as $i \to \infty$. Thus $x = \mathscr{G}_\alpha\text{-}\lim x_i$ and the completeness of G_α is thereby established.

(c) *To each α corresponds a β such that*

$$E_\alpha \subset G_\beta \text{ and } \mathscr{G}_\beta | E_\alpha \leq \mathscr{C}_\beta \; . \tag{6}$$

Define $H_\beta = E_\alpha \cap G_\beta (\beta = 1, 2, \cdots)$ and equip H_β with the topology induced by $\mathscr{G}_\beta$. We claim that H_β is complete and therefore a Fréchet space. Indeed, let $(x_i) \subset H_\beta$ be $\mathscr{G}_\beta$-Cauchy. By completeness of G_β, there exists x in G_β such that $x_i \to x$ for $\mathscr{G}_\beta$; a fortiori, $x_i \to x$ for $\mathscr{C}$ (since all the x_i and x belong to $G_\beta \subset E_\beta$ and $\mathscr{C} | E_\beta \leq \mathscr{C}_\beta$ and $\mathscr{C}_\beta | G_\beta \leq \mathscr{G}_\beta$). Since E_α is sequentially $\mathscr{C}$-closed, it follows that $x \in E_\alpha$ and hence $x \in E_\alpha \cap G_\beta = H_\beta$. Completeness of H_β follows.

Consider now the injection maps $\psi_\beta : H_\beta \to E_\alpha$. It is easily verified that ψ_β is closed and therefore continuous. By a theorem of Bourbaki ([5], p. 36, Ex. 13a)) applied to the Fréchet spaces $E_\alpha, H_1, H_2, \cdots$ and the maps $\psi_\beta : H_\beta \to E_\alpha$, we conclude the existence of β such that $E_\alpha = H_\beta$, i.e. $E_\alpha \subset G_\beta$. It remains to show that the injection map θ of E_α into G_β is continuous, i.e. that θ is closed. But this is very easy.

(d) *$\mathscr{C}$ is the inductive limit of the $\mathscr{G}_\alpha$.* Let $\mathscr{G}$ denote the inductive limit of the $\mathscr{G}_\alpha$. Recall that to say that $\mathscr{C}$ (resp. $\mathscr{G}$) is the inductive limit of the $\mathscr{C}_\alpha$ (resp. the $\mathscr{G}_\alpha$), signifies that $\mathscr{C}$ (resp. $\mathscr{G}$) is the finest locally convex topology on E such that for each α $\mathscr{C} | E_\alpha \leq \mathscr{C}_\alpha$ (resp. $\mathscr{G} | G_\alpha \leq \mathscr{G}_\alpha$). This being so, since $\mathscr{C} | G_\alpha \leq \mathscr{C}_\alpha | G_\alpha \leq \mathscr{G}_\alpha$, by (a_1), we see that $\mathscr{C} \leq \mathscr{G}$. On the other hand, given α we choose β as in (6). Then, since $\mathscr{G} | G_\beta \leq \mathscr{G}_\beta$, one has $\mathscr{G} | E_\alpha = (\mathscr{G} | G_\beta) | E_\alpha \leq \mathscr{G}_\beta | E_\alpha \leq \mathscr{C}_\alpha$ (by (6)), there follows $\mathscr{C} \geq \mathscr{G}$. Thus $\mathscr{C} = \mathscr{G}$, as was to be proved.

This completes the proof of Theorem 1.

4. Comments on Theorem 1.

(1) There is no difficulty in removing from (III) the demand that μ be uniquely determined by x, provided one assumes in return that the set M_0 of μ in M, for which

$$\mathscr{C}\text{-}\lim_{n \to \infty} \int_{K_n} u d\mu = 0 \; ,$$

is closed in M. One would then replace M by the quotient M/M_0, which would still be Fréchet space.

(2) If one is dealing with series bases, (ii) will normally be a trivial consequence of (i): for (see §7) (i) will normally imply the continuity of the associated coefficient functionals φ_n, and $s_n(x)$ is just the finite partial sum $\sum_{m \leq n} \varphi_m(x) u_m$. Continuity of each s_n is thus made

plain, and the rest of (ii) follows as before.

(3) In view of the direct proof of Theorem 2 to follow, and the fact that the Closed Graph Theorem is applicable to maps of E into M (see Appendix 4), it is natural to consider whether one could not simplify the proof of Theorem 1 by showing rapidly that X is closed. The difficulty in showing the closed character of X is, however, only removed *after* equicontinuity of the s_n has been established . . . though the fact that E is a t-space reduces this preliminary task to that of proving the continuity of each s_n individually.

(4) We append a few remarks concerning conditions (A^*).

(i) (A1)-(A3) are satisfied if E is the *strict* (internal) inductive limit of the E_α. (This category of spaces includes the (LF)-spaces of Dieudonné and Schwartz ([6], pp. 65-67).) For any such E, $\mathscr{C}_\alpha = \mathscr{C} | E_\alpha$ ([4], p. 64, Proposition 3), so that (A3) is clearly fulfilled. (A2) may be established by using the proof of Proposition 4 of [6]. Sequential completeness of E follows from (A2), combined with the relation $\mathscr{C} | E_\alpha = \mathscr{C}_\alpha$ and completeness of E_α.

(ii) (A2) is satisfied if E is the internal inductive limit of the *countable* family (E_α) of Fréchet spaces, provided that $E = \bigcup_\alpha E_\alpha$ and that the bounded, closed subsets of E are sequentially complete. See Appendix 1.

(iii) (A2) is satisfied if E is the internal inductive limit of an increasing sequence (E_α), each E_α being a Banach space, $\mathscr{C}_{\alpha+1} | E_\alpha \leq \mathscr{C}_\alpha$, and the closed unit ball B_α of E_α having a $\mathscr{C}$-closure which is a $\mathscr{E}_\alpha$-bounded subset of E_α. See Appendix 2.

5. The case of weakly absolutely convergent integrals. Hitherto we have expressly avoided the assumption that a representation $x = \int_T u d\mu$ holds for each x in E, the integral being a WF-integral. In the series case, this assumption leads to the investigation of weak absolute bases; see §7. We shall now impose this condition and correspondingly modify other hypotheses of the groups (A^*) and (D). The result will be a much more direct proof of the conclusion of Theorem 1 which, moreover, will apply to more general spaces E and M.

In place of (A^*), it will suffice to assume that E is a perfectly general external inductive limit of Fréchet spaces (as in the opening paragraph of §2). As for M, it will be enough to assume that it is an external inductive limit of a sequence of Fréchet spaces M_n and maps $\lambda_n : M_n \to M (n = 1, 2, \cdots)$ such that $M = \bigcup_n \lambda_n(M_n)$: we shall describe this situation by saying that M satisfies condition (B').

Conditions (I)-(III) are to be modified thus:

(I′) $\int_T u(t)d\mu(t) \in F$ *for each* μ *in* M.

(II′) *There exists a separating subset* S *of* F' *such that, for each* y' *in* S, *the linear form*

$$\mu \to \int_T \langle u(t), y' \rangle \, d\mu(t)$$

is continuous on M.

(III′) *To each* x *in* E *corresponds a unique* $\mu = \mu_x$ *in* M *for which* $x = \int_T u(t)d\mu_x(t)$.

THEOREM 2. *The hypotheses are that* E *is a perfectly general external inductive limit of Fréchet spaces, that* M *satisfies* (B'), *and that* (I′)-(III′) *are fulfilled. The conclusion is that the mapping* $X : x \to \mu_x$ *is continuous from* E *into* M.

Proof. The spaces E and M are of such a type that the Closed Graph Theorem is applicable: regarding this point, see Appendix 4. Thus it suffices to show that X is closed. This is almost trivial: suppose that a directed family (x_i) converges to 0 in E, and that the directed family $(\mu_i) = (\mu_{x_i})$ converges to μ in M. It must be shown that μ is necessarily 0. But, for each y' in S, (II′) yields

$$0 = \lim \langle x_i, y' \rangle = \lim \int_T \langle u, y' \rangle d\mu_i = \int_T \langle u, y' \rangle d\mu \, .$$

By (I′), $\int_T u d\mu$ belongs to F; so, since S separates points of F, it follows that $\int_T u d\mu = 0$. The uniqueness part of (III′) then forces μ to be 0, which was to be proved.

6. Concerning weak Schauder integral bases. We shall now consider what can be said if we know in advance that $X : x \to \mu_x$ is continuous, whilst (III) is replaced by the requirement that $s_n(x) \to x$ *weakly* in E.

THEOREM 3. *Assume that* E *is a space of the type described by* (B') (§5), *that* M *is an arbitrary topological vector space of measures on* T, *and that one has a continuous linear mapping* $X : x \to \mu_x$ *of* E *into* M *such that for* x *in* E *the sequence of WF-integrals*

$$s_n(x) = \int_{K_n} u(t)d\mu_x(t)$$

is bounded in E. *Assume finally that* (II) *holds. The conclusion is that the maps* s_n: $E \to E$ *are equicontinuous, and that the set* E_0, *formed* *of those x in E for which the sequence* $(s_n(x))$ *is Cauchy in* E, *forms a closed vector subspace of* E.

Proof.[4] It is clear that E_0 is a vector subspace of E. The closed character of E_0 will follow from the equicontinuity of the maps s_n. Since E is a t-space, and since the s_n are given to be continuous and bounded at each point, equicontinuity follows from the continuity of each s_n ([5], p. 27, Théoremè 2). Finally, continuity of s_n will, thanks to our hypotheses on E, be ensured when it is known to be closed (Appendix 4 once again). Suppose that a directed family $x_i \to 0$ in E and $s_n(x_i) \to \bar{x}$ in E. Then $\mu_{x_i} \to 0$ in M and so, by (II), y' in S entails that

$$\langle \bar{x}, y' \rangle = \lim_i \langle s_n(x_i), y' \rangle = \lim_i \int_{K_n} \langle u, y' \rangle \, d\mu_{x_i} = 0 \, .$$

Consequently, $\bar{x} = 0$. This establishes the claim that s_n is closed, and so completes the proof.

From Theorem 3 we can derive a property of what may be termed weak Schauder integral bases.

COROLLARY. *Assume that E and M are as in Theorem* 3, *and that the continuous mapping $X: x \to \mu_x$ of E into M is such that* $\lim_{n\to\infty} s_n(x) = x$ *weakly in E. Assume finally that* (II) *holds, and that for each x in E and iach integer m,* $\chi_{K_m} \cdot \mu_x \in M$ (χ_{K_m} the characteristic function of K_m). *The conclusion is that* $\lim_{n\to\infty} s_n(x)$ *in the sense of the initial topology* $\mathscr{C}$ of E.

REMARK. The main hypothesis amounts to the requirement that the basis elements $u(t)\,(t \in T)$ form a Schauder integral basis for the weak topology, and the conclusion is that they form a basis of the same type for the initial topology.

Proof. Weak convergence of the $s_n(x)$ implies pointwise boundedness of the maps s_n. Moreover, if a sequence in E is weakly convergent and also $\mathscr{C}$-Cauchy, then it is $\mathscr{C}$-convergent to the same limit. (There exists always a base of $\mathscr{C}$-neighbourhoods of 0 which are weakly closed.) In view of Theorem 3, therefore, it suffices to show that E_0 is $\mathscr{C}$-dense in E. Since E_0 is a vector subspace of E, the $\mathscr{C}$-closure of E_0 coincides with its weak closure (Hahn-Banach Theorem), so that it is certainly enough to show that $x^* = s_m(x)$ belongs to E_0 whenever $x \in E$ and $m = 1, 2, \cdots$. However, if $\mu = \chi_{K_m} \cdot \mu_x$ and $n \geq m$, we have

[4] For the series case, cf. Dieudonné [**7**], proof of Proposition 5.

$$x^* = s_m(x) = \int_{K_m} u d\mu_x = \int_{K_n} u d\mu \, .$$

By hypothesis, $\mu \in M$; so uniqueness shows that $\mu = \mu_x$. Thus $s_n(x^*) = x^*$ for $n \geq m$ and, a fortiori, $s_n(x^*) \to x^*$ for $\mathscr{C}$. E_0 therefore contains x^*, as was to be proved.

7. The series case. Since many of the hypotheses simplify in the series case, it merits separate attention.

T will now be the set N of natural numbers. We make the normal (but not obligatory) choice for K_n, namely the interval $\{1, 2, \cdots, n\}$ of N. u is simply a sequence $(u_n)_{n \in N}$ of elements of F, and M is a topological vector space of scalar sequences $\mu = (\mu(n))_{n \in N}$. For Theorems 1 and 3 (but not for Theorem 2), the normal choice of M will be the vector space f^N of all scalar sequences (f the real or complex field of scalars), and a natural topology will be that of pointwise convergence, which choice makes $M = f^N$ a Fréchet space. We shall in any case assume for simplicity that *the topology of M is at least as fine as that of pointwise convergence.*

Integrals are replaced by sums: if $y = (y_n)_{n \in N}$ is a sequence of elements of F, the WF-integral $\int_K y d\mu$ exists (qua element of F'^*) if and only if

$$\sum_{n \in K} |\langle y_n, y' \rangle| < +\infty$$

for each y' in F', and the value of the integral is then the sum $\sum_{n \in K} y_n$, this series being unconditionally convergent for the topology $\sigma(F'^*, F')$. If K is finite, $\sum_{n \in N} y_n$ always exists and belongs to F. Thus condition (I) is entirely ignorable. (II) is automatically fulfilled, thanks to the hypothesis made immediately above concerning the topology of M.

$s_n(x)$ is just the finite sum $\sum_{m \leq n} \mu(m) u_m$, where $\mu(m) = \mu_x(m) = \varphi_m(x)$ and $\varphi_m \in E^*$. The assumption that $s_n(x) \in E$ for all n entails that $\varphi_n(x) = 0$ whenever $u_n \notin E$. So there is no real loss of generality in assuming from the outset that each u_n belongs to E, and the role of F is really reduced to the part it plays in interpreting the convergence of infinite sums; in the applications of Theorems 1 and 3, the role is entirely negligible.

With there remarks in mind, we proceed to summarise the applications of Theorems 1, 2 and 3 to the series case.

Application of Theorem 1. We take $M = f^N$ with the aforesaid topology of pointwise convergence. We derive from Theorem 1 the conclusion: *If E satisfies* (A1)–(A3), *and if (u_n) is a basis in E, then the associated coefficient functionals φ_n are continuous,* i.e. $\varphi_n \in E'$ for each n. This result appears essentially as Theorem 12 of [**1**]; it constitutes an extension of the Banach-Newns Theorem.

Application of Theorem 2. We assume here that E is a perfectly general inductive limit of Fréchet spaces, and that M satisfies (B') of §5. It is supposed in addition that the series $\sum_{n\in N}|\mu(n)\langle u_n, y'\rangle| < +\infty$, and that the series $\sum_{n\in N}\mu(n)u_n$ is $\sigma(F, F')$-convergent, whenever $\mu\in M$ and $y'\in F'$; and that the coefficient functionals $\varphi_n\in E^*$ are such that the sequences $n\to\varphi_n(x)\,(x\in E)$ belong to M, the series $\sum_{n\in N}\varphi_n(x)u_n$ being $\sigma(F, F')$-convergent to x for each x in E. *The conclusion is again that each* $\varphi_n\in E'$.

When F is taken to be identical with E (as a topological vector space), we are dealing with the situation in which (u_n) is a weak absolute basis in E.

Application of Theorem 3, *Corollary*. Here we take again $M=f^N$, and suppose that E satisfies (B') of §5. The coefficient functionals φ_n are assumed at the outset to be continuous. The main hypothesis is that

$$s_n(x)=\sum_{m\leq n}\varphi_m(x)u_m\to x$$

weakly in E, so that (u_n) is a weak Schauder basis in E. The conclusion is that (u_n) is a Schauder basis for the initial topology of E. However, for series bases this conclusion is in fact valid whenever E is a t-space, and the proof may be effected much more directly in this case ([**1**], Theorem 11; and the footnote to Theorem 3).

8. Dual bases. In the series case, if (u_n) is a Schauder basis (or merely a weak Schauder basis), we can choose a system of coefficient functionals $\varphi_n\in E'$ such that (u_n) and (φ_n) are biorthogonal, whilst the expansion

$$x=\sum_n\varphi_n(x)u_n$$

shows that (φ_n) is a weak Schauder basis in E':

$$x'=\sum\langle u_n, x'\rangle\varphi_n\ ,$$

the series being weakly convergent for each x' in E'. It is natural to consider the analogous situation for integral bases. Here, of course, the situation is inevitably more complicated.

Given that u defines a weak Schauder integral basis, one can under suitable conditions define an E'-valued measure u' on T by the requirement that

$$(7) \qquad \langle x, u'(P)\rangle = \mu_x(P)$$

for those subsets P of T for which the right hand side is continuous in x: alternatively, one may seek to define $u'(f) \in E'$ by

$$(7') \qquad \langle x, u'(f)\rangle = \int f \, d\mu_x$$

for those scalar-valued functions f for which the right hand side is continuous in x. In view of the assumed continuity of $X : x \to \mu_x$, the class of admissible P (or f) depends primarily upon the topology of M. Under favourable conditions the restriction of the measure u' to compact subsets of T will be a countably additive Borel measure (with values in E'). Moreover, for suitable y' in F', from

$$x = \text{weak-}\lim_{n\to\infty} \int_{K_n} u(t) d\mu_x(t) ,$$

the integrals existing in the WF-sense, there will follow

$$(8) \qquad \langle x, j^t y'\rangle = \lim_{n\to\infty} \int_{K_n} \langle u(t), y'\rangle \, d\langle x, u'(t)\rangle ,$$

the integrals being taken with respect to the scalar-valued measure $P \to \langle x, u'(P)\rangle$; and so

$$(9) \qquad j^t y' = \lim_{n\to\infty} \int_{K_n} \langle u(t), y'\rangle du'(t)$$

weakly in E'. (For clarity we are introducing explicitly the injection map j of E into F, and its transposed map j^t of F' into E'.) Formula (9), whenever justified, expresses the role of the E'-valued measure u' as the basis in E' dual to the basis u in E.

It is necessary to examine more closely several steps in the preceding story. It turns out that the following conditions are sufficient to justify the argument:

(1°) *u is a weak integral basis in E, in the sense that there is a continuous linear map $X : x \to \mu_x$ such that*

$$s_n(x) = WF - \int_{K_n} u(t) d\mu_x(t) \in E$$

for $n = 1, 2, \cdots$, and

$$\lim_{n\to\infty} s_n(x) = x \textit{ weakly in } E;$$

(2°) *if P is a relatively compact Borel set in T, $\mu \to \mu(P)$ is continuous on M;*

(3°) *for each y' in F', and each n,*

$$\mu \to \int_{K_n} \langle u(t), y' \rangle d\mu(t)$$

is continuous on M.
Indeed, (1°) and (2°) together guarantee that the definition (7) makes sense and yields an E'-valued Borel measure on T. Moreover,

$$s_n'(y') = \int_{K_n} \langle u(t), y' \rangle du'(t)$$

exists as an element of E^* determined by

$$\begin{aligned}\langle x, s_n'(y') \rangle &= \int_{K_n} \langle u(t), y' \rangle d\langle x, u'(t) \rangle \\ &= \int_{K_n} \langle u(t), y' \rangle d\mu_x(t)\end{aligned}$$

holding for all x in E. Then (3°) shows that $s_n'(y') \in E' \subset E^*$. The last displayed formula can be written, thanks to (1°), in the form

$$\langle x, s_n'(y') \rangle = \langle s_n(x), y' \rangle = \langle s_n(x), j^t y' \rangle .$$

Consequently, by the last clause of (1°), we see that

$$\lim_{n \to \infty} s_n'(y') = j^t y' \; (= y' \,|\, E = y' \circ j) \tag{10}$$

in the sense of $\sigma(E', E)$. This is entirely equivalent to (9).

REMARK. In many cases, (3°) will be a consequence of (2°). In fact, this will be so *if M is a t-space, and if, for each y' in F', the function $t \to \langle u(t), y' \rangle$ is bounded and Borel measurable on each compact subset of T.* To see this, it suffices to show that, granted (2°) and the hypotheses just stated, $\mu \to \int_K f d\mu$ is continuous on M for each compact $K \subset T$ and each Borel measurable function f on K satisfying $0 \leq f(t) < 1$ for t in K. However, using the Lebesgue "ladder construction", for each $\varepsilon > 0$ we can choose numbers $c_0, \cdots, c_r$ such that $0 = c_0 < c_1 < \cdots < c_r = 1$, and Borel subsets P_k of K, for which

$$\left| \int_K f d\mu - \sum_{k=1}^{r} c_k \mu(P_k) \right| \leq \varepsilon . \quad |\mu|(K) .$$

It therefore suffices to show that

$$|\mu|(K) = \operatorname{Sup} \sum_k |\mu(P_k)| ,$$

the supremum being taken over all partitions of K into Borel sets $P_1, \cdots, P_r$, is a continuous function of μ. As a function of μ, $|\mu|(K)$ is clearly a seminorm on M. The way in which it is defined, together with (2°), shows that it is a lower semicontinuous function of μ. But then, since M is a t-space, it is continuous.

9. Similar bases. In a recent note [2] Arsove has shown that if each of two Fréchet spaces admits a series basis, and if these bases are "similar" in the sense that a series $\sum\alpha_n u_n$ in one of them is convergent if and only if the series $\sum\alpha_n v_n$ in the other is convergent, then the two Fréchet spaces concerned are isomorphic. We may refer to this as the "Similar Bases Theorem". I am grateful to Professor Arsove for raising the question which asks to what extent this theorem extends to integral bases. Using the notation of the present paper, it is clear that the answer depends to some extent on the precise nature of the basis concept which is used. We illustrate one possibility which starts from Theorem 1.

Suppose that E and M satisfy the hypotheses in Theorem 1, except that we strengthen condition (I) in the following way:

(I′) *For each* $\mu \in M$, $\int_{K_n} u d\mu \in E$ $(n = 1, 2, \cdots)$, *and*

$$\mathscr{C}\text{-}\lim_{n\to\infty}\int_{K_n} u d\mu$$

exists in E.

This means that the mapping $x \to \mu_x$, which is clearly biunique and which (Theorem 1, (i)) is continuous, maps E *onto* M. We aim to show that this mapping is a topological isomorphism, i.e. that the inverse mapping, $\mu \to x_\mu$, is continuous from M into E.

Reference to Appendix 4 (with E and M interchanged therein) shows that we have only to prove that $\mu \to x_\mu$ has a closed graph. Suppose therefore that $\mu_i \to 0$ in M and $x_i = x_{\mu i} = \lim_{n\to\infty}\int_{K_n} u d\mu_i$ converges to x: we have to show that $x = 0$. Assertion (ii) of Theorem 1 comes to our aid here. Given any neighbourhood U of 0 in E, we shall have by (ii)

$$\int_{K_n} u d\mu_i - \int_{K_n} u d\mu \in U \text{ (all } n, i \geq i_0) ,$$

where $\mu = \mu_x$. Taking $y \in S'$, this yields

$$\int_{K_n} \langle u, y'\rangle d\mu_i - \int_{K_n} \langle u, y'\rangle d\mu \in y'(U) ,$$

again for all n and all $i \geq i_0$. If we let $i \to \infty$ and use (II), there follows

$$\int_{K_n} \langle u, y'\rangle d\mu \in \overline{y'(U)} \qquad (n = 1, 2, \cdots) .$$

Letting $n \to \infty$, it follows that $\langle x, y' \rangle \in \overline{y'(U)}$. Hence, U being arbitrary, $\langle x, y' \rangle = 0$, and this for each $y' \in S$. Therefore $x = 0$, as we had to show, and the following theorem is established.

THEOREM 4. *Suppose that E satisfies* (A1)-(A3) *and* (I′), (II) *and* (III), *and suppose that M is a Fréchet space. Then $x \to \mu_x$ is an isomorphism of E onto M* (the isomorphism being, of course, vectorial and topological).

The genesis of an analogue of the Similar Bases Theorem is now clear. Suppose that one has two spaces of the same nature as E, say E_1 and E_2, which admit integral bases $t \to u_1(t)$ and $t \to u_2(t)(t \in T)$ respectively such that the hypotheses of Theorems 4 are satisfied *with the same choice of M* (this latter requirement being the analogue of Arsove's concept of similarity for series bases). The immediate conclusion from Theorem 4 is that *E_1 and E_2 are both Fréchet spaces, and are isomorpic.*

APPENDICES

Appendix 1. Suppose that E is the external inductive limit of a sequence E_α of Fréchet spaces relative to maps $\varphi_\alpha : E_\alpha \to E$, that $E = \bigcup_\alpha \varphi_\alpha(E_\alpha)$, and that in E the bounded, closed subsets are sequentially complete. Then to any bounded subset B of E corresponds an index α with the property that, if U_α is any neighbourhood of 0 in E_α, B is absorbed by $\varphi_\alpha(U_\alpha)$. The proof is a slight elaboration of arguments proposed by Bourbaki ([5], p. 36, Exercise 13b).

Appendix 2. By changing the norm on E_α into an equivalent one, we may arrange that $B_\alpha \subset B_{\alpha+1}$. A linear form f on E is continuous (i.e. $f \in E'$) if and only if its restriction to E_α is continuous on E_α. Accordingly we define the seminorms n_α on E' by

$$n_\alpha(f) = \operatorname{Sup} |f(B_\alpha)| ,$$

which is just the usual norm of $f|E_\alpha$. Then $n_\alpha(f) \leq n_{\alpha+1}(f)$. It is easily verified that E' is complete for the structure defined by the n_α, hence is Fréchet space. Given a bounded set B in E, let $n(f) = \operatorname{Sup} |f(B)| (f \in E)$. It is clear that n is a lower semicontinuous seminorm on E'. Since E' is a Fréchet space, n is actually continuous on E'. This signifies that there exists an index α and a number k such that $n(f) \leq k.n_\alpha(f)$ for all f. The Bipolar Theorem then shows that B is contained in the closure in E of $k.B_\alpha$.

Appendix 3. Thanks to (C_2), we may assume from the outset that F is separable. (C_3) and (C_4) combine to show that the *WF*-integral exists. To show that its value, z, belongs to F it suffices, thanks to (C_1) and a theorem of Grothendieck [8], to verify that $z|Q$ is continuous for the induced weak topology whenever $Q \subset F'$ is equicontinuous. Since F is separable, this induced weak topology on Q is metrisable. Thus we have merely to show that if a sequence $(y'_i) \subset Q$ converges weakly to $y' \in Q$, then

$$\lim_i z(y'_i) = \lim_i \int_K \langle y(t), y'_i \rangle \, d\mu(t)$$

is equal to

$$z(y') = \int_K \langle y(t), y' \rangle d\mu(t) .$$

Now, Q being equicontinuous, there is a continuous seminorm p on F such that $|\langle y, y' \rangle| \leq p(y)$ for y in F and y' in Q. Hence $|\langle y(t), y'_i \rangle| \leq p(y(t))$ for all t in T and all i. In addition, weak convergence entails that $\lim_i \langle y(t), y'_i \rangle = \langle y(t), y' \rangle$ for each t in T. Thus (C_4) combines with Lebesgue's convergence theorem to yield the desired result.

Appendix 4. The validity of the Closed Graph Theorem for the case in which E is a Fréchet space and M satisfies (B'), stems from Bourbaki's results ([5], p. 36, Exercise 13c). The extension to the case in which E is a general inductive limit of Fréchet spaces is almost immediate: if a linear mapping $X : E \rightarrow M$ has a closed graph, each $X \circ \varphi_\alpha : E_\alpha \rightarrow M$ has a closed graph; since E_α is a Fréchet space, $X \circ \varphi_\alpha$ is continuous. Hence X is continuous, q.e.d.

References

1. M. G. Arsove and R. E. Edwards, *Generalized bases in topological linear spaces*, to appear Studia Math..

2. ————, *Similar bases and isomorphisms in Fréchet spaces*, Math. Annalen, **135** (1958), 283–293.

3. S. Banach, *Théorie des opérations linéaires*. Varsovie, 1932.

4. N. Bourbaki, *Espaces Vectoriels Topologiques*: Ch. I, II. Paris, 1953.

5. ————, *Espaces Vectoriels Topologiques*: Ch. III–V. Paris, 1955.

6. J. Dieudonné, and L. Schwartz, *La dualité dans les espaces* (F) *et* (LF), Annales Inst. Fourier I (1949–50), 61–101.

7. ————, *On biorthogonal systems*, Michigan Math. Journal II(1) (1953), 7–20.

8. A. Grothendieck, Sur la complétion du d'un espace localement convexe, C.R. Acad. Sci. (Paris), **230** (1950), 605–6.

9. W. F. Newns, *On the representation of analytic functions by infinite series*, Phil. Trans. Royal Soc. London (A), **245** (1953), 429–68.

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