ASYMPTOTICS II: LAPLACE'S METHOD FOR MULTIPLE INTEGRALS

W. FULKS AND J. O. SATHER

Laplace's method is a well known and important tool for studying the rate of growth of an integral of the form

$$I(h) = \int_a^b e^{-hf} g dx$$

as $h \to \infty$, where f has a single minimum in $[a, b]$. It's extension to multiple integrals has been studied by L. C. Hsu in a series of papers starting in 1948, and by P. G. Rooney (see bibliography). These authors establish what amount to a first term of an asymptotic expansion. All but one (see [7]) of these results are under fairly heavy smoothness conditions.

In this paper we examine multiple integrals of the form

$$I(h) = \int_R e^{-hf} g dx$$

where f and g are measurable functions defined on a set R in E_p. Without making any smoothness assumptions on f and g, and using only the existence of $I(h)$ and, of course, asymptotic expansions of f and g near the minimum point of f we obtain an asymptotic expansion of I. The special features of our procedure are the lack of smoothness assumptions and the fact that we get a complete expansion.

Without loss of generality we may assume that the essential infimum of f occurs at the origin, and that this minimal value is zero. We introduce polar coordinates: $x = (\rho, \Omega)$ where

$$\rho = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2}\ ,$$

and where $\Omega = x/|x|$ is a point on the surface, S_{p-1}, of the unit sphere.

Our hypothesis are the following:

(1) The origin is an interior point of R.

(2) For each $\rho_0 > 0$ there is an $A > 0$ such that $f(\rho, \Omega) \geqq A$ if $\rho \geqq \rho_0$. (This says that f can be close to zero only at the origin.)

(3) There is an $n \geqq 0$ and $n + 1$ continuous functions $f_k(\Omega)$, $k = 0, 1, 2, \cdots, n$, defined on S_{p-1} with $f_0 > 0$ for which

$$f(\rho, \Omega) = \rho^\nu \sum_{k=0}^{n} f_k(\Omega)\rho^k + o(\rho^{n+\nu}) \text{ as } \rho \to 0$$

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where $\nu > 0$. (This is meant in the following sense: for each $\varepsilon > 0$ there is a $\rho_0 > 0$ for which

$$|f(\rho, \Omega) - \rho^{\nu} \sum_{k=0}^{n} f_k(\Omega)\rho^k| < \varepsilon\rho^{n+\nu}$$

whenever $\rho \leqq \rho_0$. Besides giving the asymptotic behavior of f near the origin (3) implies that the infimum of f in R is indeed zero.)

(4) There are $n+1$ functions $g_k(\Omega), k = 0, 1, \cdots n$, for which

$$g = \rho^{\lambda-p} \sum_{k=0}^{n} g_k(\Omega)\rho^k + o(\rho^{n+\lambda-k}) \text{ as } \rho \to 0$$

where $\lambda > 0$. (Thus g is permitted a mild singularity at the origin. The expansion is meant in the same sense as the one in (3).)

Under these conditions we will prove that if there is a h_0 for which $I(h)$ exists then it exists for all $h \geqq h_0$ and

$$I(h) = \sum_{k=0}^{n} c_k h^{-(k+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu})$$

where the c_k's are constants depending only on the f_j's and g_j's for $j \leqq k$. Their evaluation will be described in the proof of this result. In particular

$$C_0 = \frac{\Gamma((\lambda+1)/\nu)}{\lambda} \int_{S_{p-1}} g_0(\Omega)/[f_0(\Omega)]^{\lambda/\nu} d\Omega$$

where $d\Omega$ is the element of $(p-1)$-dimensional measure on S_{p-1}.

In the course of the proof we will use the following lemmas, which are given now so as to not interrupt the main thread of the argument.

LEMMA 1. *Let f be a measurable function on a set R in E_p, and let $g \in L_1(R)$. Then the function $G(z)$ defined by*

$$G(z) = \int_{\{f \leqq z\}} g dx$$

has bounded variation on $\{-\infty < z < \infty\}$.

Proof. Let $g = g_1 - g_2$, where

$$g_1(x) = \begin{cases} g(x), & g(x) \geqq 0 \\ 0, & g(x) < 0 \end{cases}; \qquad g_2(x) = \begin{cases} 0, & g(x) \geqq 0 \\ -g(x), & g(x) < 0, \end{cases}$$

and define G_1 and G_2 by

$$G_1(z) = \int_{\{f \leqq z\}} g_1 dx, \qquad G_2(z) = \int_{\{f \leqq z\}} g_2 dx .$$

Clearly G_1 and G_2 are increasing and bounded on $\{-\infty < z < \infty\}$, and $G = G_1 - G_2$.

LEMMA 2. *Let $F(t)$ be a continuous function defined on a possibly infinite interval $\{a < t < b\}$, and let f be a measurable function on a set R in E_p taking values in the interval $\{a < t < b\}$. If $g \in L_1(R)$, and $F(f)g \in L_1(R)$ and G is defined as in Lemma 1, then*

$$\int_R F(f)gdx = \int_a^b F(t)dG(t) .$$

Proof. Suppose first that a and b are finite, and that $g \geqq 0$. Form a partition: $a = t_0 < t_1 < \cdots < t_n = b$, and set

$$E_j = \{x \mid t_{j-1} < f \leqq t_j\} ,$$

and let $M_j = \sup_{\{t_{j-1} \leqq t \leqq t_j\}} F(t)$ and $m_j = \inf_{\{t_{j-1} \leqq t \leqq t_j\}} F(t)$.

Then

$$\int_R F(f)gdx = \sum_{j=1}^{n} \int_{E_j} F(f)gdx \leqq \sum_{j=1}^{n} M_j \int_{E_j} gdx$$
$$= \sum_{j=1}^{n} M_j[G(t_j) - G(t_{j-1})] .$$

Similarly

$$\int_R F(f)gdx \geqq \sum_{j=1}^{n} m_j[G(t_j) - G(t_{j-1})] .$$

If we let $n \to \infty$ so that $\max_{1 \leqq j \leqq n} (t_j - t_{j-1}) \to 0$ then both

$$\sum_{j=1}^{n} M_j[G(t_j) - G(t_{j-1})] \text{ and } \sum_{j=1}^{n} m_j[G(t_j) - G(t_{j-1})]$$

converge to $\int_a^b F(t)dG(t)$, since F is continuous and G monotone.

If g is not positive we can write $g = g_1 - g_2$ as in Lemma 1, apply the proof just completed to each of g_1 and g_2, and combine the results to complete the proof for the case where a and b are finite.

Suppose for example b is infinite. Then for any finite b',

$$\int_R F(f)gdx = \lim_{b' \to \infty} \int_{\{f \leqq b'\}} F(f)gdx = \lim_{b' \to \infty} \int_a^{b'} F(t)dG(t)$$
$$= \int_a^{\infty} F(t)dG(t) .$$

A similar argument applies if $a = -\infty$.

We now return to the proof of the main theorem. First we note that if $h \geqq h_0$ then $e^{-h_0 f}g$ forms a dominating function for $e^{-hf}g$, so that

$I(h)$ exists.

For each $\varepsilon > 0$ we define the two functions $f_+(\rho, \Omega)$ and $f_-(\rho, \Omega)$ by

$$f_\pm(\rho, \Omega) = \rho^\nu \sum_{k=0}^{n} f_k(\Omega)\rho^k \pm \varepsilon\rho^{n+\nu} .$$

These functions are defined in all of E_p. Now given an $\varepsilon > 0$ there is a ρ_0 so that

(i) $|f(\rho, \Omega) - \rho^\nu \sum_{k=0}^{n} f_k(\Omega)\rho^k | < \varepsilon\rho^{n+\nu}$

(ii) $| g(\rho, \Omega) - \rho^{\lambda-p} \sum_{k=0}^{n} g_k(\Omega)\rho^k | < \varepsilon\rho^{n+\lambda-p}$ for $\rho < \rho_0$,

and so that

(iii) both the functions $f_\pm(\rho, \Omega)$ are increasing in ρ for $\{0 \leqq \rho \leqq \rho_0\}$ for each $\Omega \in S_{p-1}$. This can easily be achieved since f_0 is positive (and therefore bounded away from zero) and the other f_k's are bounded.

(iv) the sphere $\{\rho \leqq \rho_0\}$ is in R.

We denote $\{\rho \leqq \rho_0\}$ by R_0 and write $I(h)$ in the form

$$I(h) = \int_{R_0} e^{-hf} g dx + \int_{R-R_0} e^{-hf} g dx \equiv I_1(h) + I_2(h)$$

respectively. We proceed to estimate I_2: by hypothesis (2) there is an $A > 0$ so that $f \geqq A$ if $\rho \geqq \rho_0$. Thus

$$\begin{aligned} | I_2(h) | &\leqq \int_{R-R_0} e^{-hf} | g | dx \leqq e^{-(h-h_0)A}\int_{R-R_0} e^{-h_0 f} | g | dx \\ &= Ce^{-hA} \text{ where } C \text{ is a constant.} \end{aligned}$$

That is,

$$I_2(h) = O(e^{-hA}) \text{ as } h \to \infty ,$$

so it is clear that the dominant part of $I(h)$ must arise from $I_1(h)$. The remainder of the proof is largely concerned with estimating I_1.

In R_0 we define $r(\rho, \Omega)$ by

$$g(\rho, \Omega) = \rho^{\lambda-p} \sum_{0}^{n} g_k(\Omega)\rho^k + r(\rho, \Omega)\rho^{n+\lambda-p} .$$

Let

$$g_k^+(\Omega) = \begin{cases} g_k(\Omega), & g_k(\Omega) \geqq 0 \\ 0, & g_k(\Omega) < 0 \end{cases}, \qquad g_k^-(\Omega) = \begin{cases} 0, & g_k(\Omega) \geqq 0 \\ -g(\Omega), & g_k(\Omega) > 0 \end{cases}$$

and

$$r^+(\rho, \Omega) = \begin{cases} r(\rho, \Omega), & r(\rho, \Omega) \geqq 0 \\ 0, & r(\rho, \Omega) < 0 \end{cases}; \quad r^-(\rho, \Omega) = \begin{cases} 0, & r(\rho, \Omega) \geqq 0 \\ -r(\rho, \Omega), & r(\rho, \Omega) < 0 . \end{cases}$$

In R_0 we now define $g^+(\rho, \Omega)$ and $g^-(\rho, \Omega)$ by

$$g^+(\rho, \Omega) = \rho^{\lambda-p} \sum_{k=0}^{n} g_k^+(\Omega)\rho^k + r^+(\rho, \Omega)\rho^{n+\lambda-p}$$

and

$$g^-(\rho, \Omega) = \rho^{\lambda-p} \sum_{k=0}^{n} g^-(\Omega)\rho^k + r^-(\rho, \Omega)\rho^{n+\lambda-p} .$$

Then $g = g^+ - g^-$ and

$$I_1 = \int_{R_0} e^{-hf} g^+ dx - \int_{R_0} e^{-hf} g^- dx .$$

Thus we may assume that $g \geqq 0$ in R_0.

We recall the definition of f_+ and f_- and define $I_+(h)$ and $I_-(h)$ by

$$I_+(h) = \int_{R_0} e^{-hf_+} g dx, \; I_-(h) = \int_{R_0} e^{-hf_-} g dx .$$

Since $g \geqq 0$ we conclude

$$I_+(h) \leqq I_1(h) \leqq I_-(h) .$$

Next we turn our attention to I_+: Let $R_t = \{x \mid f_+ \leqq t\}$ and choose a so small that $R_a \subset R_0$. Then

$$I_+(h) = \int_{R_a} e^{-hf_+} g dx + \int_{R_0 - R_a} e^{-hf_+} g dx = I'_+ + I''_+ ,$$

respectively. Now f_+ is bounded away from zero in R_0 outside any neighborhood of the origin. Thus by the same argument used on I_2 we get

$$I''_+ = O(e^{-hA'}) .$$

Furthermore e^{-hf_+} is bounded away from zero in R_a, since f_+ is bounded there. Thus $e^{-hf_+} g \in L_1(R_a)$ and by Lemma 2,

$$I'_+ = \int_0^a e^{-ht} dG(t) ,$$

where $G(t) = \int_{R_t} g dx$. Integrating by parts we get

$$\begin{aligned} I'_+ &= e^{-ha} G(a) + h\int_0^a e^{-ht} G(t) dt \\ &= h\int_0^a e^{-ht} G(t) dt + O(e^{-ha}) . \end{aligned}$$

We next do some preliminary calculations, preparatory to estimating $G(t)$. For each $t, 0 \leqq t \leqq a$, the equation $t = f_+(\rho, \Omega)$ has a unique solution for ρ which is continuous in Ω, since f_+ is increasing in ρ.

Thus the solution defines a star-shaped curve (or surface) given by $\rho = \rho(t, \Omega)$. We proceed to estimate $\rho(t, \Omega)$. Set $t = U^\nu$ then $t = f_+(\rho, \Omega)$ can be written in the form

$$U^\nu = \rho^\nu\Big[\sum_0^n f_k(\Omega)\rho^k + \varepsilon\rho^n\Big]$$

or

$$U = \rho[f_0(\Omega) + f_1(\Omega)\rho + \cdots (f_n(\Omega) + \varepsilon)\rho^n]^{1/\nu} .$$

From here on we assume $n > 0$, for if $n = 0$, we can solve directly for ρ and the estimates are considerably simpler than those which follow.

Now the right hand side of the last equation is a monotone function of $\rho, 0 \leqq \rho \leqq a$, hence an inverse function exists. Since, for each fixed Ω, U is an $(n + 2)$-times differentiable (it's even analytic!) function of ρ, $0 \leqq \rho \leqq a$, then ρ is an $(n + 2)$-times differentiable function of U, and it can therefore be expanded in a Taylor series with remainder. Thus since $f_0(\Omega) > 0$ we get

$$\rho = \psi_1(\Omega)U + \psi_2(\Omega)U^2 + \cdots + \psi_{n+1}(\Omega, \varepsilon)U^{n+1} + \psi_{n+2}(\Omega, \varepsilon, U)U^{n+2}$$

where $\psi_1(\Omega) = 1/[f_0(\Omega)]^{1/\nu}$. Since the ψ_k's are expressible in terms of the f_k's it is easy to check that ψ_k depends only on f_j's for $j \leqq k$, that ψ_k is independent of ε for $k \leqq n$, that ψ_{n+1} depends only linearly on ε and finally that ψ_{n+2} is uniformly bounded for $\Omega \in S_{p-1}$, $0 \leqq \varepsilon \leqq 1$, and $0 \leqq U \leqq a^{1/\nu}$.

Since $U = t^{1/\nu}$ we express ρ in terms of t, Ω, and ε by

$$\begin{aligned}\rho(t, \Omega) = \psi_1(\Omega)t^{1/\nu} + \psi_2(\Omega)t^{2/\nu} + \cdots + \psi_{n+1}(\Omega, \varepsilon)t^{(n+1)/\nu} \\ + \psi_{n+2}(\Omega, \varepsilon, U)t^{(n+2)/\nu}\end{aligned}$$

By definition $G(t) = \int_{R_t} g dx$, which we can write as

$$G(t) = \int_{S_{p-1}}\int_0^{\rho(t,\Omega)} g(\rho, \Omega)\rho^{p-1}d\rho d\Omega ,$$

where $d\Omega$ represents the element of measure on the sphere $S_{p-1} : \{\rho = 1\}$. We proceed to compute:

$$\begin{aligned}G(t) &= \int_{S_{p-1}}\int_0^{\rho(t,\Omega)}\Big(\sum_0^n g_k(\Omega)\rho^{k+\lambda-1} + o(\rho^{n+\lambda-1})\Big)d\rho d\Omega \\ &= \int_{S_{p-1}}\Big[\rho^\lambda(t, \Omega)\Big(\sum_0^n \frac{g_k(\Omega)}{k + \lambda}\rho^k(t, \Omega)\Big) + o(\rho^{n+\lambda}(t, \Omega))\Big]d\Omega .\end{aligned}$$

If we substitute for $\rho(t, \Omega)$ the expression previously computed for it, the preceding integral can be written in the form

$$G(t) = \int_{S_{p-1}} \left[t^{\lambda/\nu} \sum_0^{n-1} \gamma_k(\Omega) t^{k/\nu} + \gamma_n(\Omega, \varepsilon) t^{(n+\lambda)/\nu} + o(t^{(n+\lambda)/\nu}) \right] d\Omega$$

where γ_k is independent of ε for $k = 0, 1, 2, \cdots, n-1$, and γ_n is linear in ε. We may also note that each of the g_j's enter the γ_k's linearly. In particular

$$\gamma_0 = g_0(\Omega)/[f_0(\Omega)]^{\lambda/\nu} .$$

Now if we write $\gamma_n(\Omega, \varepsilon) = \gamma_n(\Omega) - \varepsilon \gamma_n'(\Omega)$ we have

$$\begin{aligned} G(t) &= \int_{S_{p-1}} \left(\sum_0^n \gamma_k(\Omega) t^{(k+\lambda)/\nu} - \varepsilon \gamma_n'(\Omega) t^{(n+\lambda)/\nu} \right) d\Omega + o(t^{(n+\lambda)/\nu}) , \\ &= \sum_0^n \eta_k t^{(k+\lambda)/\nu} - \varepsilon \eta_n'^{(n+\lambda)/\nu} + o(t^{(n+\lambda)/\nu}) \end{aligned}$$

where $\eta_k = \int_{S_{p-1}} \gamma_k(\Omega) d\Omega$. In particular $\eta_0 = (1/\lambda) \int_{S_{p-1}} [g_0(\Omega)/[f_0(\Omega)]^{\lambda/\nu}] d\Omega$.

Now by Watson's lemma we can multiply this asymptotic formula for G by e^{-ht} and integrate termwise to get

$$I_+' = \sum_0^n c_k h^{-(k+\lambda)/\nu} - \varepsilon c_n' h^{(n+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu})$$

where $c_k = \eta_k \Gamma((k + \lambda + 1)/\nu)$. In particular $c_0 = \eta_0 \Gamma((\lambda + 1)/\nu)$. Since $I_+ = I_+' + I_+'' = I_+' + o(e^{-hA'})$, we have also

$$I_+ = \sum_0^n c_k h^{-(k+\lambda)/\nu} - \varepsilon c_n' h^{-(n+\lambda)\nu} + o(h^{-(n+\lambda)/\nu}) .$$

By the same argument, since I_- differs from I_+ only in the sign of ε, we get

$$I_- = \sum_0^n c_k h^{-(k+\lambda)/\nu} + \varepsilon c_n' h^{-(n+\lambda)/\nu} + o(h^{-(n+\lambda)/\nu}) .$$

Now as we have shown before

$$I_+(h) \leqq I_1(h) \leqq I_-(h) .$$

Thus

$$I_+ - \sum_0^n c_k h^{-(k+\lambda)/\nu} \leqq I_1(h) - \sum_0^n c_k h^{-(k+\lambda)/\nu} \leqq I_- - \sum_0^n c_k h^{-(k+\lambda)/\nu} .$$

If we multiply through by $h^{(n+\lambda)/\nu}$ and let $h \to \infty$ we get

$$-\varepsilon c_n' \leqq \overline{\underline{\lim}} \left[(I_1(h) - \sum_0^n c_k h^{-(k+\lambda)/\nu}) h^{(n+\lambda)/\nu} \right] \leqq \varepsilon c_n' .$$

But $I(h) = I_1(h) + o(e^{-hA})$ so that we have also

$$-\varepsilon c_n' \leqq \overline{\underline{\lim}} \left[\left(I(h) - \sum_0^n c_k h^{-(k+\lambda)/\nu} \right) h^{(n+\lambda)/\nu} \right] \leqq \varepsilon c_n' ,$$

for every $\varepsilon > 0$. Let $\varepsilon \to 0$ to complete the proof for $g \geqq 0$.

If g may change sign near the origin we can decompose g with g^+ and g^- as described earlier. The proof just completed applies to each of these. We can then subtract the results to obtain the result for g. Also since g_j''s enter into the c_k''s linearly, the same formula for the c's applies whether g is one signed or has a variable sign near the origin.

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OREGON STATE COLLEGE
AND
INSTITUTE OF TECHNOLOGY
UNIVERSITY OF MINNESOTA