CONVERGENCE OF INVERSE SYSTEMS

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1. Introduction. Let X_i be a metrizable continuum for each positive integer i, and let f_i be a mapping of X_{i+1} onto X_i. Let X be the inverse limit space of the inverse system $(\{X_i\}, \{f_i\})$; in notation $X = \lim (\{X_i\}, \{f_i\})$, and let π_i be the projection mapping of X onto X_i. In [2] it is proved that if metrics d_i for X_i are properly chosen then the inverse limit space $X = \lim (\{X_i\}, \{f_i\})$ is locally connected if and only if the collection $\{(X_i, d_i) \mid i \text{ a positive integer}\}$ of metric spaces is equi-uniformly locally connected. Also the X_i were embedded in their cartesian product in such a way that X is locally conndected if and only if the sequence $X_1, X_2, X_3, \cdots$ converges 0-regularly to X.

In this paper similar relations between semi-local connectedness, equi-uniform semi-local connectedness and 0-coregular convergence are established. These results are then combined with known results about 0-regular and 0-coregular convergence to obtain properties of certain inverse limit spaces. For example, if each X_i is a simple closed curve and X is semi-locally connected and cyclic (i.e., without cut points), then X is a simple closed curve. Similar theorms for 2-spheres and 2-cells are also obtained.

For definitions and results on semi-local connectedness see [5] or [6], equi-uniform local connectedness [2], inverse limits [1], 0-regular and 0-coregular convergence [3] and [4]. Throughout this paper $S_\varepsilon(p)$ will denote ε-neighborhood of p.

REMARK 1. Note the following relationship for the case where X is cyclic between semi-local connectedness and local connectedness:

If X is cyclic then it is $\begin{Bmatrix}\text{locally connected}\\ \text{semi-locally connected}\end{Bmatrix}$ at a point p if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that any two points $\begin{Bmatrix}\text{inside } S_\delta(p)\\ \text{outside } S_\varepsilon(P)\end{Bmatrix}$ can be joined by a connected set lying $\begin{Bmatrix}\text{inside } S_\varepsilon(p)\\ \text{outside } S_\delta(p)\end{Bmatrix}$. Note also that since X is compact it is locally connected (semi-locally connected) if and only if it is uniformly locally connected (uniformly semi-locally connected).

2. Equi-uniform semi-local connectedness.

DEFINITION 1. A collection $\{(Y_\alpha, \rho_\alpha) \mid \alpha \in A\}$ of metric spaces is *equi-uniformly locally connected* if and only if corresponding to each $\varepsilon > 0$

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there is a $\delta > 0$ such that if $\alpha \in A$ and s and t are members of Y_α for which $\rho_\alpha(s, t) < \delta$ then s and t lie in a connected subset of Y_α of diameter less than ε.

DEFINITION 2. A collection $\{(Y_\alpha, \rho_\alpha) | \alpha \in A\}$ of metric spaces is *equi-uniformly semi-locally connected* if and only if corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\alpha \in A$ then for any p in Y_α and s and t in $Y_\alpha - S_\varepsilon(p)$ we have s and t lie in a connected subset of $Y_\alpha - S_\delta(p)$.

DEFINITION 3. Let d_i be a metric for X_i for each positive integer i. The sequence $d_1, d_2, d_3, \cdots$ is *admissible* if there exists a metric d for X such that

$$\lim_{i\to\infty} d_i(\pi_i(u), \pi_i(v)) = d(u, v)$$

uniformly on $X \times X$.

REMARK 2. Let D_i be a metric for X_i such that $D_i(x, y) \leqq 1$ for all x and y in X_i. If $i > j$ let f_{ij} denote the composite mapping $f_j \cdots f_{i-2} f_{i-1}$.
Define

$$d_i(x, y) = \sum_{j=1}^{i} 2^{-j} D_j(f_{ij}(x), f_{ij}(y))$$

for each positive integer i and all x and y in X_i. Also define

$$d(u, v) = \sum_{j=1}^{\infty} 2^{-j} D_j(\pi_j(u), \pi_j(v)) .$$

In ([7], Theorem 1) it is shown that $\lim_{1-\infty} d_i(\pi_i(u), \pi_i(v)) = d(u, v)$ uniformly on $X \times X$ and hence the sequence $d_1, d_2 \cdots$ is admissible.

THEOREM 1. *Suppose d_i is a metric for X_i for each i and the sequence $d_1, d_2, d_3, \cdots$ is admissible. Then if X is cyclic it is semi-locally connected if and only if the collection $\{(X_i, d_i) | i \geqq n\}$ is equi-uniformly semi-locally connected for some positive integer n.*

Proof. Suppose $\{(X_i, d_i) | i \geqq n\}$ is equi-uniformly semi-locally connected. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any p in X_i and x and y in $X_i - S_{\varepsilon/3}(p)$ we have x and y are contained in a connected subset of $X_i - S_\delta(p)$. Now suppose u and v are in $X - S_{\varepsilon/3}(q)$ where $\pi_i(q) = p$. There exists a positive integer m such that if $i \geqq m$ then $\pi_i(u)$ and $\pi_i(v)$ are in $X_i - S_{\varepsilon/3}(p)$. For $i \geqq \max(m, n)$ let K_i be a connected subset of $X_i - S_\delta(p)$ containing $\pi_i(u)$ and $\pi_i(v)$. Let H_i denote the closure of $\bigcup_{j\geqq i} f_{ji}[K_j]$. Each H_i is a continuum, $f_{ji}[H_j] \subset H_i$ for $j > i$, so $H = \lim(\{H_i\}, \{f_i | H_{i+1}\})$ is a subcontinuum of $X - S_\delta(q)$ containing u and v. Hence since X is cyclic it is semi-locally connected.

Now assume X is semi-locally connected and cyclic. Suppose $\varepsilon > 0$. Then there exists a $\delta^* > 0$ such that for any p in X if u and v are in $X - S_{\varepsilon/2}(p)$ they lie in a connected subset of $X - S_{\delta^*/2}(p)$. There is a positive integer n such that if $i \geqq n$ and u and v are in X then $|d_i(\pi_i(u), \pi_i(v)) - d(u, v)| < \min(\delta^*/6, \varepsilon/2)$. Now suppose $i \geqq n$ and x and y are in X_i for which x and y are in $X_i - S_\varepsilon(p_i)$ where $\pi_i(p) = p_i$. There exists u and v in X such that $\pi_i(u) = x$ and $\pi_i(v) = y$. It follows u and v are in $X - S_{\varepsilon/2}(p)$ and hence u and v lie in a connected subset K of $X - S_{\delta^*/2}(p)$. Since $\operatorname{diam}(\pi_i^{-1}(p_i)) < \delta^*/6$ we have $K \cap \pi_i^{-1}(p_i) = \phi$. Therefore $\pi_i(K)$ is a connected subset of X_i which contains x and y and is contained in $X_i - S_{\delta^*/3}(p_i)$. Let $\delta = \delta^*/3$. Then the collection $\{(X_i, d_i) \mid i \geqq n\}$ is equi-uniformly semi-locally connected.

3. 0-coregular convergence. Let P be the cartesian product of the sequence $X_1, X_2, \cdots$. Let D_i, d_i and d be metrics defined as in Remark 2. If we define a metric d^* for P by

$$d^*(a, b) = \sum_{i=1}^{\infty} 2^{-j} D_j(a_j, b_j) \quad \text{for } a = (a_1, a_2, \cdots) \text{ and } b = (b_1, b_2, \cdots),$$

then the inclusion map is an isometry of (X, d) into (P, d^*). Choose a point $p = (p_1, p_2, \cdots)$ in P, and define for each positive integer i an isometry h_i on (X_i, d_i) into (P, d^*) by letting

$$h_i(x) = (f_{i1}(x), \cdots, f_{ii}(x), \; p_{i+1}, p_{i+2}, \cdots)$$

where f_{ii} is the identity map. In the following denote $h_i[X_i]$ by X_i^*.

THEOREM 2. *The sequence $X_1^*, X_2^*, \cdots$ converges 0-coregularly to X if and only if X is semi-locally connected and cyclic.*

Proof. It is obvious that $X_1^*, X_2^*, \cdots$ converges to X. Suppose X is semi-locally connected and cyclic, then it follows from Theorem 1 that the collection $\{(X_i, d_i) \mid i \geqq n\}$ of metric spaces is equi-uniformly semi-locally connected for some positive integer n. Since each h_i is an isometry, it follows at once that $X_1^*, X_2^*, \cdots$ must converge 0-coregularly to X.

Now suppose the sequence $X_1^*, X_2^*, \cdots$ converges 0-coregularly to X. Then by ([4], Theorem 2.1) X must be semi-locally connected and cyclic. White proved the following lemma in [4].

LEMMA 1. *If $A_i \to A$ 0-coregularly and if each A_i is a*

$$\left\{\begin{array}{l}\textit{simple closed curve}\\ \textit{2-sphere and } A_i \to A \textit{ 0-regularly}\\ \textit{2-cell and } A_i \to A \textit{ 0-regularly}\end{array}\right\}$$

then A is a

$$\left\{\begin{array}{l}\textit{simple closed curve or a point}\\ \textit{2-sphere or a point}\\ \textit{2-cell, 2-sphere or a point}\end{array}\right\}.$$

THEOREM 3. *If X is semi-locally connected and cyclic and each X_1 is a simple closed curve, then X is a simple closed curve.*

Proof. Since X is semi-locally connected and cyclic, by Theorem 2 $X_i^* \to X$ 0-coregularly where X_i^* is isometric to X_i. So by Lemma 1 X is a simple closed curve.

LEMMA 2. ([2], Theorem 3) *The sequence $X_1^*, X_2^*, \cdots$ converges 0-regularly to X if and only if X is locally connected.*

THEOREM 4. *If X is locally connected and cyclic and each X_i is a 2-sphere then X is a 2-sphere.*

Proof. For compact spaces local connectedness implies semi-local connectedness. So X is semi-locally connected and cyclic. Therefore by Theorem 2 $X_i^* \to X$ 0-coregularly and by Lemma 2 $X_i^* \to X$ 0-regularly where X_i^* is isometric to X_i. So by Lemma 1, X is a 2-sphere or a point and since π_i maps X onto X_i, we have X is a 2-sphere.

THEOREM 5. *If X is locally connected and cyclic and each X_i is a 2-cell then X is a 2-cell.*

Proof. Since X is locally connected and cyclic $X_i^* \to X$ 0-coregularly and by Lemma 2 $X_i^* \to X$ 0-regularly where X_i^* is isometric to X_i. So by Lemma 1 X is a 2-cell, 2-sphere or a point. The third possibility is ruled out since π_i maps X onto X_i. Since the second Čech homology group of each X_i is zero so is that of X. Hence the second possibility is ruled out also. So X is a 2-cell.

REMARK 3. To see the need for requiring X to be cyclic in Theorem 5 consider Example 1 of [2]. Each X_i is a 2-cell but X is an arc. So since X is locally connected $X_i^* \to X$ 0-regularly but not 0-coregularly since X has a cut point.

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