SUPERCOMPLETE SPACES

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A uniform space is *supercomplete* if its space of closed subsets is complete. The purpose of this note is to characterize the supercomplete spaces, a surprisingly small class, though it includes all complete metric spaces. A uniformisable space admits a supercomplete uniformity if and only if it is paracompact.

The idea and the name of supercomplete spaces were invented by S. Ginsburg and me in 1954. What is new is the simple definition given above, together with the application (of the metric case) in the following paper [5]. Ginsburg and I had an application, a proof that every locally fine uniformity finer than a complete metric uniformity is fine; but the proof which we published [4] is simpler.

If $\mathscr{U}$ is a uniform covering of a uniform space μX, two subsets A and B of X are said to be within $\mathscr{U}$ of each other if $St(A, \mathscr{U}) \supset B$ and $St(B, \mathscr{U}) \supset A$. The set of all closed subsets of μX is made into a uniform space $H(\mu X)$ by taking for a basis of uniform coverings the coverings $\{W(A, \mathscr{U}) : A \in H(\mu X)\}$, where $W(A, \mathscr{U})$ is the set of all closed subsets of μX within $\mathscr{U}$ of A and $\mathscr{U}$ runs through μ.

As was said above, μX is called supercomplete when $H(\mu X)$ is complete. At this point we could give a short direct proof that every complete metric space is supercomplete, and go on to the next paper. But to characterize the supercomplete spaces, the following preliminaries seem to be essential.

Recall from [4] that a *locally fine* space is one in which every uniformly locally uniform covering is uniform; that for every uniform space μX there is a next finer locally fine uniformity $\lambda\mu$; that $\lambda\mu$ may be gotten by transfinite iteration of the passage $\mu \rightarrow \mu^1$ from the family μ of uniform coverings to the family μ^1 of the uniformly locally uniform coverings; and that the intermediate constructs $\mu^1, \mu^{(2)}, \cdots, \mu^{(\alpha)}, \cdots$, while they are not known to be uniformities in the usual sense, are known to be uniformities in the weaker sense of [6].

A function f on a partially ordered set P to a uniform space μX is said to be *convergent* [3] if for every uniform covering $\{U_\alpha\}$ of μX there exists a family of residual sets S_α in P whose union is cofinal in P, such that $f(S_\alpha) \subset U_\alpha$ for each α. A point x of X is a *cluster point* of f if for every neighborhood U of x there is a nonempty residual set $S \subset f^{-1}(U)$.

A filter $\mathscr{F}$ of subsets of a uniform space μX is said to *converge*

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to a set if the closures of the elements of $\mathscr{F}$ form a convergent net (ordered by inclusion) in $H(\mu X)$. Equivalently, if K is the set of all cluster points of the filter $\mathscr{F}$, every uniform neighborhood of K must contain an element of $\mathscr{F}$. $\mathscr{F}$ is *stable* if for every uniform covering $\mathscr{U}$ there is $A \in \mathscr{F}$ such that for every $B \in \mathscr{F}$, $St(B, \mathscr{U}) \supset A$.

Evidently every filter converging to a set is stable. The converse is obvious for supercomplete spaces. It characterizes supercomplete spaces; and the theorem is

THEOREM. *The following conditions on a uniform space μX are equivalent*:

(a) *μX is supercomplete*;

(b) *X is paracompact and $\lambda\mu X$ is fine*;

(c) *every convergent function with values in μX has a cluster point*;

(d) *every stable filter in μX converges to a set.*

Proof. (a) $\Rightarrow$ (b). It is equivalent to (b) to say that every open covering of X is in $\lambda\mu$. Suppose this is not the case, $\mathscr{U} \notin \lambda\mu$. We construct a net N of points of $H(\mu X)$, ordered by inclusion; N consists of all those closed $S \subset X$ such that on some uniform neighborhood of $X - S$, $\mathscr{U}$ coincides with some covering in $\lambda\mu$. Because of the requirement of a uniform neighborhood, N is directed by inclusion. Moreover, N is Cauchy. To prove this we must show that for every $\mathscr{V}$ in μ, there is S in N such that every $T \subset S$ in N is within $\mathscr{V}$ of S. Let S be the closure of the union of all V in $\mathscr{V}$ such that $\mathscr{U}$ does not coincide on V with any covering in $\lambda\mu$. S is in N because the union of all other V in $\mathscr{V}$ is a uniform neighborhood W of $X - S$ (it contains $St(X - S, \mathscr{V})$); and on any intermediate uniform neighborhood Z of $X - S$ such that $St(X - S, \mathscr{V})$ is a uniform neighborhood of Z, $\mathscr{U}$ coincides uniformly locally with coverings in $\lambda\mu$ and therefore coincides with a covering in $\lambda\mu$. Next, for any $T \in N$, every point x of S is in the closure of $St(T, \mathscr{V})$. Before proving this we note that it will show that if $T \subset S$ then T is within $\mathscr{V}^*$ of S; since every covering in μ is refined by some $\mathscr{V}^*$, $\mathscr{V}$ in μ, this suffices to prove that N is Cauchy. Now $x \in S$ is certainly arbitrarily near to sets V on which $\mathscr{U}$ does not coincide with any covering in $\lambda\mu$; but $\mathscr{U}$ does do this on $X - T$, which means that $X - T$ contains no such set V and T meets all of them.

It remains to note that N is not convergent; in fact, the filter base N has no cluster points, since every point has a neighborhood on which $\mathscr{U}$ is uniform.

(b) $\Rightarrow$ (c). Suppose μX satisfies (b), and $f\colon\ P \to \mu X$ is convergent. We show next that $f\colon\ P \to \lambda\mu X$ is also convergent, applying transfinite induction on the Morita uniform spaces $\mu^{(\alpha)}X$. Suppose $f\colon\ P \to \mu^{(\alpha)}X$ is nonconvergent for some α, and pick the least such α. It cannot be a limit ordinal, for then each uniform covering of $\mu^{(\alpha)}X$ is in $\mu^{(\beta)}$ for some $\beta < \alpha$. Hence α must be of the form $\beta + 1$; and $\mu^{(\alpha)}$ has a basis consisting of coverings of the form $\{U_i \cap V_j^i\}$, where $\{U_i\}$ and each $\mathscr{V}^i = \{V_j^i\}$ are in $\mu^{(\beta)}$. We must show that every p in P has a successor in a residual set Q such that $f(Q) \subset U_i \cap V_j^i$ for some i and j. For this, p has a successor r in a residual set mapped into some U_i; and r has a successor q in a residual set mapped into some V_j^i. Let Q be the set of successors of q.

Now if f had no cluster point, there would be an open covering consisting of sets whose inverse images contain no nonempty residual sets. Since every open covering is in $\lambda\mu$, this is absurd.

(c) $\Rightarrow$ (d). Suppose $\mathscr{F}$ is a stable filter in μX which fails to converge. Thus the set K of cluster points of $\mathscr{F}$ has a uniform neighborhood U which fails to contain any element of $\mathscr{F}$. Let P be the partially ordered set of all subsets of $X - U$ which contain a uniform neighborhood of a set compatible with $\mathscr{F}$ (meeting every element of $\mathscr{F}$); order P by reverse inclusion. Let $f: P \to \mu X$ be any choice function. Obviously f has no cluster point; but we shall see that f is convergent. Let $\{U_\alpha\}$ be any uniform covering of μX. For any $S \in P$, let S be a uniform neighborhood of compatible T, and let $\mathscr{V}$ be a uniform covering so fine that $St(T, \mathscr{V}^*) \subset S$ and $\mathscr{V}^{**}$ is a refinement of $\{U_\alpha\}$. In the stable filter $\mathscr{F}$ let A be a member which is contained in every $St(B, \mathscr{V})$, $B \in \mathscr{F}$. Let x be a point of $T \cap A$. Some U_α contains $R = St(x, \mathscr{V}^*)$, which is a subset of S also; moreover, since $x \in A$, every B in $\mathscr{F}$ meets $St(x, \mathscr{V})$. Thus R is a uniform neighborhood of a compatible set, and R is a successor of S all of whose successors are mapped into U_α by f.

The proof that (d) implies (a) amounts to no more than verifying that if $\{S_\lambda\}$ is a Cauchy net in $H(\mu X)$ then the sets which contain almost all S_λ form a stable filter; and we omit it.

COROLLARY. *Every complete metric space is supercomplete.*

Proof. It satisfies (b) [4].

Finally we note the connection with H. H. Corson's *weakly Cauchy* filters, which have cluster points in an undetermined class of paracompact uniform spaces including the fine paracompact spaces [2]. A stable filter is weakly Cauchy. Since we exhibited above (in (a) $\Rightarrow$ (b)) a stable filter without cluster points in an arbitrary nonsupercomplete space, Corson's class is a subclass of the supercomplete spaces. Corson and I satisfied ourselves in 1958 that it is a proper subclass.

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