ON THE DETERMINATION OF SETS BY THE SETS OF SUMS OF A CERTAIN ORDER

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1. Introduction. Let $X = \{x_1, \cdots, x_n\}$ be a set of (not necessarily distinct)[1] elements of a torsion free Abelian group. Define $P_s(X) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_s} \mid i_1 < i_2 < \cdots < i_s\}$. Thus $P_s(X)$ has $\binom{n}{s}$ (not necessarily distinct) elements. We introduce the equivalence relation $X \sim Y$ if and only if $P_s(X) = P_s(Y)$. Let $F_s(n)$ be the greatest number of sets X which can fall into one equivalence class. Our purpose in this paper is to study conditions under which $F_s(n) > 1$. Clearly $F_s(n) = \infty$ if $n \leqq s$ so that we may restrict our attention to $n > s$.

In [5] Selfridge and Straus studied this question, restricting attention to sets of elements of a field of characteristic 0. In § 2 we show that the numbers $F_s(n)$ remain the same even if we restrict ourselves to sets of positive integers. Thus the results in [5] remain valid in our case. These included a necessary condition for $F_s(n) > 1$ and the proof that $F_2(n) > 1$ (and hence $F_{n-2}(n) > 1$) if and only if n is a power of 2. Also $F_s(2s) > 1$.

In § 3 we give a simpler form of the necessary condition in [5].

In § 4 we examine this necessary condition and prove that for $s > 2$ we have $F_s(n) = 1$ for all but a finite number of n. This was conjectured in [5]. The method seems to be of independent interest since it can be applied to a class of Diophantine equations in two unknowns which are algebraic in one and exponential in the other variable.

In § 5 we apply the methods of [5] to show that $F_2(8) = 3$, $F_2(16) \leqq 3$, $F_3(6) \leqq 6$ and $F_4(12) \leqq 2$.

The fact that $F_2(8) = 3$ disproves the conjecture $F_2(n) \leqq 2$ made in [5]. Except for the corresponding result $F_6(8) = 3$ we have not found another nontrivial case in which we can prove $F_s(n) > 2$.

In the final section we adapt a method of Lambek and Moser [3] to the case $s = 2$ and get a partial characterization of those sets which are equivalent to other sets.

2. Reduction to sets of integers. In this section we demonstrate that there exist $F_s(n)$ distinct equivalent sets of positive integers so that in any effort to evaluate $F_s(n)$ we may restrict our attention to sets of integers.

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[1] Throughout this paper we use the word "set" to mean "set with multiplicities" in the sense in which one speaks of the set of zeros of a polynomial.

Let $N = F_s(n)$ and let $X_1 = \{x_{11}, \cdots, x_{n1}\}, \cdots, X_N = \{x_{1N}, \cdots, x_{nN}\}$ be the members of a maximal equivalence class. Since the x_{ij} form of a finite set of elements of a torsion-free Abelian group, they generate a group with basis $y_1, \cdots, y_m$ over the integers (see e.g. [2], Theorem 6). In other words every x_{ij} can be represented as an m-vector, $x_{ij} = (a_{ij}^1, \cdots, a_{ij}^m)$ with integral components. The addition of a fixed m-vector with sufficiently large components to all x_{ij} does not effect the equivalence of the X_j so that we may assume that every a_{ij}^k is non-negative. Now let A be an integer with $A > s \max a_{ij}^k$ and associate to each x_{ij} the number

$$y_{ij} = a_{ij}^1 + a_{ij}^2 A + \cdots + a_{ij}^m A^{m-1} .$$

It is now clear that two sums of s or fewer y_{ij} are same if and only if the corresponding sums of x_{ij} are the same. In other words the sets of integers $Y_j = \{y_{1j}, \cdots, y_{nj}\}$ $(j = 1, \cdots, N)$ form an equivalence class with $N = F_s(n)$ distinct members.

3. Simplification of the necessary conditions for $F_s(n) > 1$.

In this section we show that the Diophantine equation $f(n, k) = 0$ of [5] can be writen in the form

$$(1) \quad \binom{n}{s-1} - \binom{n}{s-2} 2^{k-1} + \binom{n}{s-3} 3^{k-1} - \cdots + (-1)^{s-1} s^{k-1} = 0 .$$

To see this we start with the expression given in [5], namely

$$f(n, k) = \frac{1}{s} \sum_P (-1)^{st} n^{t-1} \sum_{i=1}^{r} a_i i^k ,$$

where P runs through all permutations on s letters, a_i is the number of cycles of length i in P, and $t = \Sigma a_i$ is the total number of cycles in P. Changing the order of summation we get

$$f(n, k) = \sum_i i^{k-1} (-1)^{s-1} \frac{i}{s} \sum_{t=1}^{s-i} (-n)^{t-1} N_{it} ,$$

where N_{it} is the number of permutations P which contain exactly t cycles, including at least one i-cycle. Since there are $\binom{s}{i} (i-1)!$ choices of the one i-cycle, and $[(s-i)!/(t-1)!] \sum_{\Sigma c_j = s-i} 1/(c_1 c_2 \cdots c_{t-1})$ choices of the other cycles of length $c_1, \cdots, c_{t-1}$, we have

$$f(n, k) = \sum_i i^{k-1} \frac{(-1)^{s-1} i}{s} \frac{s!}{(s-i)!\, i!} (i-1)!\, (s-i)!$$
$$\cdot \sum_{t=1}^{s-i} \frac{(-n)^{t-1}}{(t-1)!} \sum_{\Sigma c_j = s-i} \frac{1}{c_1 c_2 \cdots c_{t-1}}$$
$$= \sum_i i^{k-1} (-1)^{s-1} (s-1)! \sum_{t=1}^{s-i} \frac{(-n)^{t-1}}{(t-1)!} \sum_{\Sigma c_j = s-i} \frac{1}{c_1 c_2 \cdots c_{t-1}}$$

Now if $|x| < 1$, we have

$$-\log(1-x) = \sum_{c=1}^{\infty} \frac{x^c}{c} ,$$

$$(-1)^v \log^v (1-x) = \sum_{w=v}^{\infty} x^w \sum_{\Sigma c_j = w} \frac{1}{c_1 c_2 \cdots c_v} .$$

Multiplying by $(-n)^v/v!$ and summing over v we obtain

$$\begin{aligned}(1-x)^n = e^{n\log(1-x)} &= \sum_{v=0}^{\infty} \frac{n^v \log^v(1-x)}{v!} \\ &= \sum_{p=0}^{\infty} x^p \sum_{v=0}^{p} \frac{(-n)^v}{v!} \sum_{c_1+\cdots+c_v=p} \frac{1}{c_1 c_2 \cdots c_v}\end{aligned}$$

from which we deduce that

$$(-1)^w \binom{n}{w} = \sum_{v=0}^{w} \frac{(-n)^v}{v!} \sum_{c_1+\cdots+c_v=w} \frac{1}{c_1 c_2 \cdots c_v} .$$

Putting $v = t-1$, $w = s-i$, we obtain

$$\begin{aligned}f(n,k) &= \sum_i i^{k-1}(-1)^{s-1}(s-1)!\,(-1)^{s-i}\binom{n}{s-i} \\ &= \sum_i i^{k-1}(s-1)!\,(-1)^{i-1}\binom{n}{s-i} .\end{aligned}$$

Therefore the equation $f(n,k)=0$ is equivalent to $\sum_i i^{k-1}(-1)^{i-1}\binom{n}{s-i} = 0.$

4. Proof that for $s > 2$ we have $F_s(n) = 1$ for all but a finite number of n.

LEMMA. *For large values of k the equation* (1) *has $s-1$ real roots $n = n_1, \cdots, n_{s-1}$ where*

$$(2) \qquad n_j = (s-j)(1+1/j)^{k-1} + O((1+1/j)^{\delta k}), \qquad \delta < 1 .$$

Proof. Divide the left side of (1) by $(-1)^{j-1}\binom{n}{s-j-1} j^{k-1}$ and then consider its behavior in the neighborhood N_j of $n = n_j^* = (s-j)(1+1/j)^{k-1}$ say $N_j = \{n \mid n_j^*/2 \leqq n \leqq 2n_j^*\}$. We have

$$\binom{n}{s-i} i^{k-1} \Big/ \binom{n}{s-j-1} j^{k-1} < c_1 n^{j-i+1}(i/j)^k < c_2(i(j+1)^{j-i+1}/j^{i-j+2})^k = c_2 l_{ij}^k .$$

It remains to show that $l_{ij} < 1 + 1/j$ for all $i \leqq i < j$ and all $j+1 < i \leqq s-1$. For $i < j$ this leads to

$$1 + \frac{1}{j} < \left(1 - \frac{j-i}{j}\right)^{1/(i-j)} = 1 + \frac{1}{j} + \cdots$$

and for $i > j + 1$ to

$$i/j < (1 + 1/j)^{i-j} = 1 + (i - j)/j + \cdots .$$

Thus, if we set

$$\delta = \max_{\substack{1 \leqq i < j \\ j+1 < i < s}} \{j - i + 1 + \log(i/j)/\log(1 + 1/j)\}$$

Then $\delta < 1$ and (1) becomes

$$(3) \qquad (n - s + j + 1)/(s - j) - (1 + 1/j)^{k-1} + O((1 + 1/j)^{\delta k}) = 0$$

for $n \in N_j$. Thus (1) must have a root in N_j and according to (3) this is the real root given in (2).

THEOREM. *If $s > 2$ then there is only a finite number of n for which $F_s(n) > 1$.*

Proof. If the Diophantine equation (1) has solutions for arbitrarily large k then by the Lemma the solutions are of the form

$$n = (s - j)(1 + 1/j)^{k-1} + O(n^\delta) ,$$

where $1 \leqq j \leqq s - 1$ and $\delta < 1$.

On the other hand all solutions of (1) satisfy $n \mid (s - 1)!\, s^{k-1}$ so that all prime factors of n are less than or equal to s. The same holds for the prime factors which occur in the numerator and denominator of $(s - j)(1 + 1/j)^{k-1}$.

Now according to a Theorem of Ridout [4] for any $\varepsilon > 0$ there is at most a finite number of integers p, q whose prime divisors belong to fixed finite sets and which satisfy $0 < |1 - p/q| < 1/q^\varepsilon$; or, equivalently

$$0 < |q - p| < q^{1-\varepsilon} .$$

But

$$|nj^{k-1} - (s - j)(j + 1)^{k-1}| < cj^{k-1} n^\delta < c_1(nj^{k-1})^{1-\varepsilon} .$$

for some $\varepsilon > 0$, so that if there is an infinite number of solutions we must have $nj^{k-1} = (s - j)(j + 1)^{k-1}$ infinitely often. For large k, this implies $j = 1$ and $n = (s - 1)\cdot 2^{k-1}$. For $s = 2$ this does indeed give an infinite family of solutions, but for $s > 2$ we see that for $n = (s - 1)\cdot 2^{k-1}$

$$\binom{n}{s-1} - \binom{n}{s-2} 2^{k-1} = O(2^{(s-2)k})$$

$$\binom{n}{s-j} j^{k-1} = O(2^{(s-j)k} j^k)$$

so that the third term in (1) dominates the sum of the first two terms as well as all the subsequent terms and the equation cannot hold for large k.

Using a method of Davenport and Roth [1] we could obtain an upper bound on the number of n for which $F_s(n) > 1$, but this bound would probably be far from best possible.

5. Special cases. As in [5] we put $S_k = \sum_{i=1}^{n} x_i^k$ and $\Sigma_k = \Sigma(x_{i_1} + \cdots + x_{i_s})^k$, the summation being extended over all indices $i_1, \cdots, i_s$ with $1 \leqq i_1 < i_2 < \cdots < i_s \leqq n$. Then each Σ_k can be expressed as a polynomial in $S_1, \cdots, S_k$. Since all sets X of an equivalence class give rise to the same Σ_k's, and since the elements of X are uniquely determined by $S_1, \cdots, S_n$, we can obtain an upper bound for $F_s(n)$ by estimating the number of different n-tuples $(S_1, \cdots, S_n)$ corresponding to a given set of Σ's. Since $\Sigma_1 = \binom{n-1}{s-1} S_1$ we see that all members of an equivalence class have the same S_1. We can assume without loss of generality that $S_1 = 0$.

The case $s = 2$, $n = 8$.

In this case there are 28Σ's, and the first 12 of them are given by the following expressions (for $S_1 = 0$)

$$\Sigma_1 = 0 \tag{1}$$
$$\Sigma_2 = 6S_2 \tag{2}$$
$$\Sigma_3 = 4S_3 \tag{3}$$
$$\Sigma_4 = 3S_2^2 \tag{4}$$
$$\Sigma_5 = -8S_5 + 10S_2S_3 \tag{5}$$
$$\Sigma_6 = -24S_6 + 15_2S_4 + 10S_3^2 \tag{6}$$
$$\Sigma_7 = -56S_7 + 21S_2S_5 + 35S_3S_4 \tag{7}$$
$$\Sigma_8 = -120S_8 + 28S_2S_6 + 56S_3S_5 + 35S_4^2 \tag{8}$$
$$\Sigma_9 = -248S_9 + 36S_2S_7 + 84S_3S_6 + 126S_4S_5 \tag{9}$$
$$\Sigma_{10} = -504S_{10} + 45S_2S_8 + 120S_3S_7 + 210S_4S_6 + 126S_5^2 \tag{10}$$
$$\Sigma_{11} = -1016S_{11} + 55S_2S_9 + 165S_3S_8 + 330S_4S_7 + 462S_5S_6 \tag{11}$$
$$\Sigma_{12} = -2040S_{12} + 66S_2S_{10} + 220S_3S_9 + 495S_4S_8 + 792S_5S_7 + 462S_6^2 . \tag{12}$$

Equations (2), (3), and (5) show that S_2, S_3, and S_5 are uniquely determined by the Σ's. Furthermore (6), (7), and (8) imply that S_6, S_7, and S_8 are uniquely determined by the Σ's once S_4 is known. So to prove $F_2(8) \leqq 3$, it is sufficient to show that corresponding to a given set of Σ's, there can be at most 3 values of S_4. Now S_9, S_{10}, S_{11}, and S_{12} can be expressed in terms of $S_1, \cdots, S_8$ using the theory of symmetric functions. Since these in turn can be expressed in terms of S_4 and the Σ's, equations (9), (10), (11), and (12) give us four equations involving S_4 and the Σ's. Now (9) is linear in S_4, (10) and (11) are quadratic in S_4, while

(12) is cubic in S_4. We shall show that the coefficient of S_4^3 in (12) is not zero, which implies that S_4 can have at most 3 values. Then, in order that it actually can have 3 values, we must have the coefficients of S_4 in (9), (10), (11) and the coefficients of S_4^2 in (10), (11) equal to 0. This gives considerable information on the structure of the 3-member equivalence classes.

First we compute the coefficient of S_4^2 in equation (11). It arises only from the terms $-1016S_{11}$, $165S_3S_8$ and $330S_4S_7$. The last term contributes $330((35/56)S_3)S_4^2 = (825/4)S_3S_4^2$, making use of (7). To compute the contribution of $-1016S_{11}$ we use the relation from the theory of symmetric functions

$$0 = \frac{1}{11}S_{11} - \frac{1}{18}S_2S_9 - \frac{1}{24}S_3S_8 + \frac{1}{96}S_3S_4^2 - \frac{1}{28}S_4S_7 + \cdots .$$

This, combined with equations (7) and (8) gives

$$\begin{aligned} S_{11} &= \frac{11}{24}S_3\left(\frac{35}{120}S_4^2\right) + \frac{11}{28}\left(\frac{35}{56}S_3S_4\right)S_4 - \frac{11}{96}S_3S_4^2 + \cdots \\ &= \frac{979}{96.35}S_3S_4^2 + \cdots . \end{aligned}$$

From (8), the term $165S_3S_8$ contributes $165\cdot(35/120)S_3S_4^2$. Hence the coefficient of S_4^2 in equation (11) is

$$\left(-1016\cdot\frac{979}{96.35} + \frac{825}{4} + \frac{165.35}{120}\right)S_3$$

where the number in parentheses is $\neq 0$. Thus in order for an equivalence class to contain 3 members, we must have $S_3 = 0$. Next consider the coefficient of S_4 in equation (9) (supposing from now on that $S_3 = 0$). It arises from the terms $-248S_9$ and $126S_4S_5$. But

$$0 = \frac{1}{9}S_9 - \frac{1}{20}S_5S_4 + \cdots ,$$

from which $S_9 = (9/20)S_5S_4 + \cdots$. So the coefficient of S_4 is

$$-248\left(\frac{9}{20}S_5\right) + 126S_5 = \frac{72}{5}S_5 .$$

Hence in order to have more than one member in such an equivalence class we must have $S_5 = 0$. Next consider the coefficient of S_4 in equation (11) (supposing $S_3 = S_5 = 0$). It arises from $-1016S_{11}$ and from $330S_4S_7$. Since $0 = (1/11)S_{11} - (1/28)S_4S_7 + \cdots$ the coefficient is

$$-1016\left(\frac{11}{28}\right)S_7 + 330S_7 = \frac{-584}{7}S_7 .$$

Hence we must have $S_7 = 0$ in order to have 3 sets in the same equivalence class. Finally the coefficient of S_4^3 in equations (12) arises from $-2040S_{12}$ and from $495S_4S_8$. Using the relation

$$0 = \frac{1}{12} S_{12} - \frac{1}{32} S_4S_8 + \frac{1}{6\cdot 64} S_4^3 - \cdots$$

and (8), we obtain a coefficient of

$$(-2040)\left(\frac{12}{32}\right)\left(\frac{35}{120}\right) - 2040\left(\frac{-12}{6\cdot 64}\right) + 495\left(\frac{35}{120}\right) \neq 0 ,$$

which completes the proof that $F_2(8) \leqq 3$. Moreover we see from the proof that if X, Y, Z form a 3-member equivalence class (with $S_1 = 0$), then X, Y, Z all have $S_k = 0$ for k odd, and hence each consists of 4 members and their negatives. In addition, there can be only one such equivalence class having a given value for Σ_6 and 3 given values for S_4. For the three given values of S_4 determine the coefficients of the cubic equation (12), and hence determine Σ_2, Σ_8, and Σ_{13}. But then all other Σ's are determined from these. Now if a, b, c, d are any 4 numbers, then the sets $X = X_1 \cup -X_1$, $Y = Y_1 \cup -Y_1$, and $Z = Z_1 \cup -Z_1$, where $X_1 = \{a, b, c, d\}$, $Y_1 = \{\frac{1}{2}(-a + b + c + d), \frac{1}{2}(a - b + c + d), \frac{1}{2}(a + b - c + d), \frac{1}{2}(a + b + c - d)\}$, and $Z_1 = \{\frac{1}{2}(a + b + c + d), \frac{1}{2}(a + b - c - d), \frac{1}{2}(a - b + c - d), \frac{1}{2}(a - b - c + d)\}$ are all equivalent. Furthermore if any 4 (complex) numbers $\Sigma_6, S_4', S_4'', S_4'''$ are given, it is possible to choose a, b, c, d so that $\Sigma_6(X) = \Sigma_6(Y) = \Sigma_6(Z) = \Sigma_6$, $S_4(X)S_4'$, $S_4(Y) = S_4''$, $S_4(Z) = S_4'''$. Indeed, it is easy to see that the prescribed conditions merely determine the symmetric functions of a, b, c, d, and of course one can always find complex a, b, c, d for which these have preassigned values. It follows that the sets X, Y, Z give a parametric representation of all 3-member equivalence classes (with $S_1 = 0$).

Other values of $F_s(n)$. A similar treatment can be given for the other values of $F_s(n)$ mentioned in the introduction. We will omit the details and merely sketch the general method in these cases. If $s = 2$, $n = 4$, the first S_k not uniquely determined by the Σ's is S_3, and all other S_k are determined by S_3 and the Σ's. The equation for Σ_6 then becomes a quadratic equation in S_3 and the coefficient of S_3^2 in this equation is not 0. Hence, corresponding to a given set of Σ's there can be at most 2 values of S_3, and accordingly at most 2 sets X and Y. Thus $F_2(4) \leqq 2$. An argument similar to that given above shows that $F_2(4) = 2$ and that all 2-member equivalence classes are given by $X = \{a, b, c, d\}$, $Y = \frac{1}{2}(-a + b + c + d), \frac{1}{2}(a - b + c + d), \frac{1}{2}(a + b - c + d), \frac{1}{2}(a + b + c - d)\}$. In the case $s = 2, n = 16$, we find that S_5 is the first S_k not uniquely determined by the Σ's, and that all other S_k are uniquely determined by S_5 and the Σ's. The equation for Σ_{17} gives a cubic equation for S_5, the coefficient of S_5 being a nonzero multiple of S_2. By § 2 we can assume that the sets X are real, and hence $S_2 > 0$. This proves $F_2(16) \leqq 3$. On the other hand $F_2(16) \geqq 2$ as was shown in [4]. We do not know at present whether $F_2(16) = 2$ or 3. This type of reasoning can probably be made to yield the estimate $F_2(2^k) \leqq \alpha$, where α is the least integer such that $(k+1)\alpha > 2^k$; however, this seems to be far from the best possible result.

For $s = 4$, $n = 12$ the first S_k not uniquely determined by the Σ's is S_6, and all other S_k are uniquely determined once S_6 is known. The equation for Σ_{14} gives a quadratic equation for S_6, the coefficient of S_6^2 being a nonzero multiple of S_2. Hence $F_4(12) \leqq 2$. We do not know whether $F_4(12) = 1$ or 2.

Finally, if $s = 3$, $n = 6$, then the equations for the Σ's in terms of the S's show that S_2 and S_4 are uniquely determined by the Σ's, while S_6 is determined by the Σ's and by S_3. The equations for Σ_8 contains a term in S_3S_5 with nonvanishing coefficient. Hence it can be used to write $S_5 = (\alpha S_3^2 + \beta)/S_3$, where α, β depend on the Σ's.

Then the expression for Σ_{12} yields a sextic equation for S_3 and the coefficient of S_3^6 is nonzero. Hence $2 \leqq F_3(6) \leqq 6$.

6. Generating functions for the case $s = 2$. In this section we use a method suggested by Lambek and Moser [2] to obtain some results on equivalent sets in the case $s = 2$.

Suppose $A = \{a_1, \cdots, a_n\}$ (where $0 = a_1 \leqq a_2 \leqq \cdots \leqq a_n$) and $B = \{b_1, \cdots, b_n\}$ (with $0 \leqq b_1 \leqq \cdots \leqq b_n$) are equivalent sets of nonnegative integers. Construct the generating polynomials $f(x) = \Sigma x^{a_i}$, $g(x) = \Sigma x^{b_i}$. Then the generating polynomial for the set of sums is $\frac{1}{2}(f^2(x) - f(x^2)) = \frac{1}{2}(g^2(x) - g(x^2))$. Hence $f^2(x) - g^2(x) = f(x^2) - g(x^2)$. Let $F = f + g$, $G = f - g$; then $F(x)G(x) = G(x^2)$, so that $G(x) \mid G(x^2)$. This is possible only if every zero of G has a square which is itself a zero of G, in other words only if

$$G(x) = cx^\alpha \prod_i \varphi_i(x) ,$$

where the φ_i are cyclotomic polynomials. We can write this, in the customary way, as

$$G(x) = cx^\alpha \prod(1 - x^{\beta_i}) / \prod(1 - x^{\gamma_j}) \tag{13}$$

where the β_i and γ_j are positive integers, and hence

$$F(x) = \frac{G(x^2)}{G(x)} = x^\alpha \prod(1 + x^{\beta_i}) / \prod(1 + x^{\gamma_j}) . \tag{14}$$

Since $F(1) = 2n$ is a power of 2 we have here a new and simple proof of the fact that $F_2(n) > 1$ only when n is a power of 2.

The problem of finding equivalent sets of integers now reduces to that of determining the β_i and γ_j (we must clearly set $\alpha = 0$) for which the polynomials F and G have nonnegative coefficients. This makes $|c| = |G(0)| \leqq F(0) = 1$ in (13) necessary so that $c = 1$. We certainly get nonnegative coefficients if there are no denominators (no γ_j) which proves $F_2(2^k) > 1$ and permits a simple construction of equivalent classes of order 2^k:

Given $k + 1$ numbers $\alpha_0, \cdots, \alpha_k$ let X be the set of sums of an even number of α's and Y the set of sums of an odd number of α's. Clearly $P_2(X) = P_2(Y)$. These are the sets which were obtained in [5].

However there do exist cases in which the γ_j are not absent, for example

$$G(x) = (1-x)^4(1-x^2)^2(1-x^3)^3(1-x^6)^3(1-x^{12})/(1-x^4) .$$

This cyclotomic polynomial leads to the following two sets A and B with 2^{11} elements each:

element	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
multiplicity in A	1	0	6	8	14	10	19	41	26	29	35	78	69	37	58	104	114	22	56	129	120	48
multiplicity in B	0	4	2	7	11	22	23	14	34	45	59	22	52	105	78	47	47	122	108	46	40	136
element	43	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	26	24	23	22

The symmetry in the multiplicities is typical since the cyclotomic polynomials are reciprocal. We have no example with non-trivial denominators in (14) which leads to two sets without multiple elements.

A complete characterization of the possible functions F, G seems therefore difficult.

The fact that $F_2(8) = 3$ in the notation of the introduction and the characterization of the classes containing three equivalent sets can now be understood from this point of view by noting that f need not determine F uniquely. Namely if we write $F = (1 + x^{\alpha_1})(1 + x^{\alpha_2})(1 + x^{\alpha_3})(1 + x^{\alpha_4})$ and $F^* = (1 + x^{\beta_1})(1 + x^{\beta_2})(1 + x^{\beta_3})(1 + x^{\beta_4})$ then F and F^* give rise to the same f whenever the set of sums of an even number of α's is the same as the set of sums of an even number of β's. In other words, whenever $\beta_i = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_i$ (after suitable reordering). The generating functions $f, g_1 = F - f$ and $g_2 = F^* - f$ then describe the three equivalent sets given in § 5.

The question whether $F_2(n) \leqq 2$ for $n > 8$ reduces to that of whether two different $F(x)$ and $F^*(x)$ can give rise to the same f when $F(1) > 16$.

References

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