# ON MEASURABILITY OF STOCHASTIC PROCESSES IN PRODUCTS SPACE

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**1. Introduction.** Let $\mathscr{X}$ be a $\sigma$-algebra of subsets of $X$, and $T$ a set. Let $\Omega = X^T$, and let $\mathscr{C}$ be the $\sigma$-algebra of subsets of $\Omega$ generated by the finite cylinder sets, i.e., sets of the form $\Lambda = \{\omega \in \Omega \mid \omega(t_1) \in A_1, \cdots, \omega(t_n) \in A_n\}$, $A_1, \cdots, A_n \in \mathscr{X}$. Let $P_0$ be a probability measure on $\mathscr{C}$. Thus the coordinate variables $x_t(\omega) = \omega(t)$, $t \in T$, are the Kolmogorov version [5] of the stochastic process with joint distributions $F_{t_1}, \cdots, {}_{t_1}(A_1, \cdots, A_n) = P_0\{\Lambda\}$. For various purposes, it is appropriate to enlarge this $\sigma$-algebra and extend the measure. In the present paper two methods of doing this will be mentioned, and one of the methods will be studied.

[$A$] Suppose $X$ is a compact Hausdorff space and $\mathscr{X}$ the Borel sets. Then $\Omega$ is a compact Hausdorff space in the product topology. A straightforward application of the Stone-Weierstrass theorem and the Riesz-Markov theorem shows that there is a unique regular measure on the Borel subsets $\mathscr{B}$ of $\Omega$ which agrees with $P_0$ on $\mathscr{C}$, provided the finite-dimensional marginal measures are all regular. We call this measure $P$. This idea is due to S. Kakutani [3], and was discussed in detail by E. Nelson [8].

[$B$] By a *condition* is meant a set-valued function $k$ from $T$ to $\mathscr{X}$. For any condition $k$, we define

$$\Gamma(k) = \{\omega \mid \omega(t) \in k(t) \text{ for all } t \in T\}\,, \quad \text{and}$$
$$\Gamma(S, k) = \{\omega \mid \omega(t) \in k(t) \text{ for all } t \in S\}\,,$$

$S$ being a subset of $T$. It is possible to extend $P_0$ to a class of sets of the form $\Gamma(k)$, as follows.

The following lemma is a straightforward generalization of the separability lemma in [1], p. 56.

LEMMA 1.1. *For any condition $k$ $\exists$ a countable set $S \subset T$ such that $P_0\{\Gamma(S, k) - \Gamma(\{t\}, k)\} = 0$ for all $t \in T$.*

The proof is a simple exhaustion argument. Such a countable subset $S$ will be called *determining* for $k$.

Let $\mathscr{K}$ be a family of sets with the properties

(i) $X \in \mathscr{K}$

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(ii) any countable subfamily of $\mathscr{K}$ with the finite intersection property (F.I.P.) has nonnull intersection. Such a family will be called *countably compact*. If (ii) holds without the countability restriction, then $\mathscr{K}$ is called *compact*. If a condition $k$ has values in $\mathscr{K}$, it will be called a $\mathscr{K}$-condition.

The set of positive integers will be written $I$. Unions and intersections whose index set is $I$ will be written simply $\bigcup_j$, etc. rather than $\bigcup_{j\in I}$, etc. The following result can then be proven. It is stated in [7].

LEMMA 1.2. *Let $S_n$ be a determining set for the $\mathscr{K}$-condition $k_n$, $n \in I$. Let $\Delta = \bigcup_n\{\Gamma(S_n, k_n) - \Gamma(k_n)\}$. Then $\Delta$ has inner $P_0$ — measure* 0.

$\mathscr{C}_{\mathscr{K}}$ is now defined to be those subsets $\Gamma$ of $\Omega$ such that $\exists\, \Gamma'$ in $\mathscr{C}$ with $(\Gamma - \Gamma^1) \cup (\Gamma^1 - \Gamma)$ subset of a set of the form of $\Delta$ in the above lemma. These sets $\Gamma$ form a $\sigma$-algebra, and the assignment to $\Gamma$ of the same measure as the $P_0$-measure of $\Gamma^1$ determines unambiguously a measure $P_{\mathscr{K}}$ on $\mathscr{C}_{\mathscr{K}}$, which is an extension of $P_0$. This construction, based on ideas of Doob and Khintchine [4] is done by A. Mayer in [6], [7].

REMARK 1.1. Notice that $\mathscr{C}_{\mathscr{K}}$ contains all sets of the form $\Gamma(k)$, for any $\mathscr{K}$-condition $k$, assigning to such a set the measure $P_0\{\Gamma(S, k)\}$, $S$ being any determining set for $k$.

REMARK 1.2. If $X$ is compact Hausdorff, $\mathscr{X}$ the Borel sets, $\mathscr{K}$ the compact sets, and $P_0$ satisfies the regularity condition of [$A$], then $\mathscr{C}_{\mathscr{K}} \subset \mathscr{B}$, and $P \mid \mathscr{C}_{\mathscr{K}} = P_{\mathscr{K}}$. This is a consequence of the following (under the hypotheses of the last sentence):

LEMMA 1.3. *If $S$ is determining for the condition $k$, and $k(t)$ is compact for all $t$, then $P\{\Gamma(k)\} = P\{\Gamma(S, k)\}$.*

*Proof.* By Theorem 2.2 of [8] there is some countable subset $S_1$ of $T$ such that $P\{\Gamma(S_1, k)\} = P\{\Gamma(k)\}$. Now, $\Gamma(S_1, k) \supset \Gamma(S \cup S_1, k) \supset \Gamma(k)$, so $P\{\Gamma(S \cup S_1, k)\} = P\{\Gamma(k)\}$. But

$$\Gamma(S, k) = \Gamma(S \cup S_1, k) \cap \bigcap_{s \in S}\{\Gamma(S_1, k) = \Gamma(\{s\}, k)\} .$$

Thus $P\{\Gamma(S, k)\} = P\{\Gamma(S \cup S_1, k)\}$.

We will deal mainly with the situation where $T$ is a topological space, and with a certain $\sigma$-subalgebra $\mathscr{D}_{\mathscr{K}}$ of $\mathscr{C}_{\mathscr{K}}$, where $\mathscr{D}_{\mathscr{K}}$ is defined like $\mathscr{C}_{\mathscr{K}}$, except that the only conditions $k$ used for $\mathscr{D}_{\mathscr{K}}$ will be those of the form

$$k(t) = \begin{cases} K \text{ for } t \in U \\ X \text{ for } t \notin U, \end{cases}$$

$U$ being an open set in $T$, and $K \in \mathscr{K}$. For such a $k$, we write $\Gamma(k)$ as $\Delta(U, K)$. The restriction of $P_{\mathscr{K}}$ to $\mathscr{D}_{\mathscr{K}}$ will be called $Q_{\mathscr{K}}$.

If $\mathscr{K}$ consists of closed sets in a metric space, $T$ is locally compact, and $\tau$ is a regular measure on $T$, then $(\mathscr{D}_{\mathscr{K}}, Q_{\mathscr{K}})$ has the convenient property that whenever the map $t \to x_t$ (where $x_t(\omega) = \omega(t)$) is measurable in probability, i.e. is continuous in probability outside of some $\tau$-null set, then the map $(\omega, t) \to \omega(t)$ can be made measurable the $\mu \times \tau$-completion of $\mathscr{A} \times \mathscr{T}$, where $\mathscr{T}$ is the Borel sets of $T$ and $(\mathscr{A}, \mu)$ some extension of $(\mathscr{D}_{\mathscr{K}}, Q_{\mathscr{K}})$. (See [7], Theorem 2.) This says, in a sense, that $\mathscr{D}_{\mathscr{K}}$ is "not too large." On the other hand, it is "not too small," in the sense that it contains many natural subsets which are not in $\mathscr{C}$; this will be shown.

In § 2 are given some examples and general remarks concerning compact and countably compact families.

In [8], with $X$ and $T$ compact metrizable spaces, various natural subsets of $\Omega$ and $\Omega \times T$ were shown to be in $\mathscr{B}$, $\overline{\mathscr{B}}$, or product $\sigma$-algebras derived from them (the bar over a $\sigma$-algebra signifies completion with respect to the measure being considered on it). In § 3 and 4 we show (in a somewhat more general context) that these subsets are in $\mathscr{D}_{\mathscr{K}}$, $\overline{\mathscr{D}}_{\mathscr{K}}$, or the corresponding product $\sigma$-algebrans, where $\mathscr{K}$ is a countably compact family of closed subsets of $X$ which contains a complete system of neighborhoods for each point of $X$ (or, briefly, *generates the topology of* $X$).

## 2. Some topological considerations.

LEMMA 2.1. *Let $X$ be a* 1-st *countable Hausdorff space. Then any countable compact family $\mathscr{K}$ of subsets of $X$ which generates the topology of $X$ consists of closed sets only.*

*Proof.* Suppose $K \in \mathscr{K}$, and $x \notin K$. Choose a countable family $\{K_n \mid n \in I\}$ of neighborhoods of $x$ in $\mathscr{K}$, with $\bigcap_n K_n = \{x\}$. If $x \in \bar{K}$, then $K \cap K_1 \cap \cdots \cap K_n$ is never empty. Thus, $K \cap \bigcap_n K_n$ is nonempty, so $x \in K$.

REMARK 2.1. If we assume that $X$ actually has a countable base for its open sets, then clearly any intersection of sets of $\mathscr{K}$ can be reduced to a countable intersection. In particular, it follows that $\mathscr{K}$ is actually a *compact* family, not just countably compact.

LEMMA 2.2. (*Alexander*). *Let $\mathscr{K}$ be a compact family of subsets* *of a set X. Let $\tilde{\mathscr{K}}$ be the family of arbitrary intersections of finite unions of sets of then $\tilde{\mathscr{K}}$ is closed under arbitrary intersections and finite unions, and is again a compact family.*

*Proof.* See [9], p. 139.

COROLLARY 2.1. *The most general compact family of sets on a set X arises by choosing a subfamily of the closed sets, for some compact topology on X.*

*Proof.* Given a compact family $\mathscr{K}$ on a set $X$, use $\tilde{\mathscr{K}}$ as the family of closed sets for $X$; this gives a compact space.

REMARK 2.2. The property of *countable* compactness does *not* persist from $\mathscr{K}$ to $\tilde{\mathscr{K}}$. For example, let $A$ be all ordinals up to and including the first uncountable ordinal $\alpha_0$. Let $B$ be the rational numbers $\{0\,;1, \frac{1}{2}, 1/3, \cdots\}$. Let $X = A \times B - \{(\alpha_0, 0)\}$. Let $\mathscr{K}$ consist of all sets of the form $K_{\alpha,n} = \{(\alpha^1, x) \mid \alpha^1 > \alpha, x < 1/n\}$, where $\alpha$ is a countable ordinal and $n \in I$. Then *no* countable intersection of sets $K_{\alpha,n}$ is empty, so $\mathscr{K}$ is countably compact. But let $L_n = \bigcap_{\alpha<\alpha 0} K_{\alpha,n} = \{(\alpha_0, x) \mid x < 1/n\}$. Then the $L_n$ have the F.I.P., but $\bigcap_n L_n = \phi$.

In § 3 we shall be considering countably compact families $\mathscr{K}$ on separable metrizable spaces $X$, $\mathscr{K}$ generating the topology of $X$. Some examples follow.

(a) $X$ a Banach space which is separable and a dual, $\mathscr{K}$ the set of all closed spheres. This is mentioned in [6].

In this connection, however, notice that the separable Banach space C of all continuous functions on, say, the closed interval $[-1, 1]$, is not a dual; and, in fact, the family of all closed spheres in this Banach space is *not* a countably compact family. To see this, let

$$f_n(\lambda) = \begin{cases} 1 \text{ if } -1 \leqq \lambda \leqq 0 \\ 1 - n\lambda \text{ if } 0 < \lambda < \dfrac{1}{n} \\ 0 \text{ if } \dfrac{1}{n} \leqq \lambda \leqq 1\,, \end{cases}$$

and let $f'_n(\lambda) = -f_n(-\lambda)$. Let $K_n$ be the closed sphere of radius 2 about $f_n - 2$, and $K'_n$ the closed sphere of radius 2 about $f'_n + 2$. Then

$$K_n \cap K'_n = \{g \mid f'_n \leqq g \leqq f_n\} \neq \phi\ .$$

Since $f_1 \geqq f_2 \geqq \cdots$ and $f'_1 \leqq f'_2 \leqq \cdots$, we have $K_1 \cap K'_1 \supset K_2 \cap K'_2 \supset \cdots$. Thus, the spheres $\{K_n, K'_n \mid n = 1, 2, \cdots\}$ have the F.I.P., but there is no continuous function in their intersection. The author does not know, however, whether *some* $\mathscr{K}$ does not exist for $\mathbf{C}$.

(b) An example where the metric space is not complete: let $X$ be the nondyadic numbers in the unit interval. $\mathscr{K}$ will be defined as follows. Let $S_n$ be the set of dyadics of the form $k/2^n$, $k = 0, \cdots, 2^n$. Then $X = [0, 1] - \bigcup_n S_n$. Let $\mathscr{K}_n$ be the intersection with $X$ of intervals [a, b], where $a = (k + 1/8)1/2^n$, $b = (k + 7/8)1/2^n$, $k = 0, 1, \cdots, 2^n - 1$. Let $\mathscr{K} = \bigcup_n \mathscr{K}_n$.

To see that $\mathscr{K}$ generates the topology of $X$, we must show that any $x \in X$ is an interior point of some interval in $\mathscr{K}_n$, for arbitrarily large $n$. But a nondyadic number $x$ is characterized by the property that a zero followed by a one occurs arbitrarily far out in its dyadic expansion. Thus, for arbitrarily large $n$, we can get $k/2^n + 1/2^{n+2} < x < k/2^n + 1/2^{n+1}$, so that $x$ is interior to an interval of $\mathscr{K}_n$.

To see that $\mathscr{K}$ is countably compact, suppose we have a sequence $K_1, K_2, \cdots$ with the F.I.P. Assume repetitions have been eliminated. Then no two can come from the same $\mathscr{K}_n$, since two members of $\mathscr{K}_n$ are either identical or disjoint. Consider now the closed intervals $\bar{K}_n$ in $[0, 1]$. These have the F.I.P., and are closed in $[0, 1]$. Thus their intersection is nonempty. Further, let $K_n \in \mathscr{K}_{i_n}$. Then $\bar{K}_n \cap S_{i_n} = \phi$, so $(\bigcap_n \bar{K}_n) \cap (\bigcup_m S_{i_m}) = \phi$. Since $i_m$ does not repeat itself, and since $S_1 \subset S_2 \subset \cdots$, we have $\bigcup_m S_{i_m} = \bigcup_n S_n$. Thus, $(\bigcap_n \bar{K}_n) \cap X \neq \phi$. But this is the same as $\bigcap_n K_n$.

(c) A metric space for which *no* countably compact family can generate the topology: let $X$ be the dyadic numbers in $[0, 1]$. Suppose, in fact, we had such a family $\mathscr{K}$. Let $x_1, x_2, \cdots$ be an enumeration of $X$. Then one could choose a sequence $K_j^n$ of neighborhoods of $x_j, K_j^n \in \mathscr{K}$, and with the length of $K_j^n$ less than $1/n + j$. Let $U_j^n$ be the interior of $\overline{K_j^n}$. Then $x_j \in U_j^n$. Consider now the set $\bigcap_n \bigcup_j U_j^n$. This is a $G_\delta$ in the reals, and contains all the dyadics. Then it must contain some non-dyadics, since the dyadics are not a $G_\delta$. On the other hand, if $\xi$ is a nondyadic in $\bigcap_n \bigcup_j U_j^n$, then $\xi$ is in some $\bigcap_n U_{j_n}^n$. Thus $\{K_{j_n}^n \mid n \in I\}$ has the F.I.P. But $\bigcap_n \overline{K_{j_n}^n} = \{\xi\}$, since the lengths of the $K_{j_n}^n$ go to zero as $n \to \infty$. Thus $\bigcap_n K_{j_n}^n = \bigcap_n (\overline{K_{j_n}^n} \cap X) = \phi$.

The question remains open whether, for example, every complete separable metric space has a countably compact family which generates its topology.

**3. Measurability of various classes of functions.** Throughout this section, let $X$ be a separable metric space; $\mathscr{X}$ the Borel sets. Let $\mathscr{K}$ be a collection of sets in $\mathscr{X}$ such that

(a) $\mathscr{K}$ is a countably compact family,

(b) $\mathscr{K}$ generates the topology of $X$.

Let $T$ be a compact metric space, and consider $\mathscr{D}_{\mathscr{K}}$, $Q_{\mathscr{K}}$, as defined in § 1. For brevity, we write simply $\mathscr{D}$, $Q$. We remark that the results of this section extend immediately to the case where $T$ is locally compact metrizable, and separable, since the classes of functions discussed are defined by their local properties in $T$.

Let $\mathscr{K}_0$ be a countable subset of $\mathscr{K}$ which still contains a complete system of neighborhoods at each point. Also, let $K_{\varepsilon,n}$ be an enumeration of the sets of $\mathscr{K}_0$ of diameter $\leqq \varepsilon$. Let $\Delta(\varepsilon, S) = \bigcap_{s \in S}\{\omega \mid \exists$ some open neighborhood $U$ of $s$ and some $n$ such that $\omega$ sends $U$ into $K_{\varepsilon,n}\}$. Finally, let $\Phi(\varepsilon, S) = \{\omega \mid \exists$ some open $U \supset S$ and $n$ such that $\omega$ sends $U$ into $K_{\varepsilon,n}\}$.

LEMMA 3.1. *$\Delta(\varepsilon, S)$ and $\Phi(\varepsilon, S)$ are in $\mathscr{D}$ for any closed set $S$ and any $\varepsilon > 0$.*

*Proof.* Let $\mathscr{U}$ be a countable base for the open sets of $T$. Let $\mathscr{U}_1, \mathscr{U}_2, \cdots$ be an enumeration of the finite coverings of $S$ by sets in $\mathscr{U}$. Then $\Delta(\varepsilon, S) = \bigcup_n \bigcup_m \bigcap_{U \in \mathscr{U}_n} \Delta(U, K_{\varepsilon,m})$, and

$$\Phi(\varepsilon, S) = \bigcup_m \bigcup_n \Delta(\bigcap_{U \in \mathscr{U}_n} U, K_{\varepsilon,m}) .$$

THEOREM 3.1. *The set of all functions which are continuous at all points of the closed set $S \subset T$ is in $\mathscr{D}$.*

*Proof.* This set is precisely $\bigcap_m \Delta(1/m, S)$.

THEOREM 3.2. *For any regular measure $\nu$ on $T$, the set of $\nu$-almost everywhere continuous functions is in $\mathscr{D}$.*

*Proof.* Let $V_{n,m}$, $n, m \in I$, be an enumeration of those finite unions of sets $\mathscr{U}$ such that $\nu(V_{n,m}) < 1/n$. A function $\omega$ is $\nu$-almost everywhere continuous if and only if for arbitrary small $\varepsilon > 0$ there is a closed set $S$ whose complement has arbitrarily small measure, such that $\omega \in \Delta(\varepsilon, S)$. But $\omega \in \Delta(\varepsilon, S) \Rightarrow \omega \in \Delta(\varepsilon, \bar{U})$ for some open set $U \supset S$. New, $S^\perp$ is a union of sets in $\mathscr{U}$. Since $S^\perp \supset U^\perp$, and $U^\perp$ is compact, $U^\perp$ is covered by a finite union of sets of $\mathscr{U}$ which does not intersect $S$, and thus has $\nu$-measure no greater than that of $S$. Hence, the set of $\nu$-almost everywhere continuous functions is contained in $\bigcap_j \bigcap_n \bigcup_m \Delta(1/j, V_{n,m})$. The converse inclusion is obvious.

THEOREM 3.3. *The set of functions whose points of discontinuity form a first category set, is in $\mathscr{D}$,* *Proof*. Let $O_\varepsilon(\omega) = \{s \mid \text{for every open } U \ni s \ \exists \ r, t \in U \text{ with } d(\omega(r), \omega(t)) > \varepsilon\}$. $O_\varepsilon(\omega)$ is a closed set, and increases as $\varepsilon$ decreases. Thus, the set $\bigcup_{\varepsilon \searrow 0} O_\varepsilon(\omega)$ is of first category if and only if each $O_\varepsilon(\omega)$ is nowhere dense. Let $D$ be a countable dense subset of $T$, and let $D_{n,m}$ be an enumeration of the finite $1/m$-dense subsets of $D$(i.e. every point of $T$ is within $1/m$ of some point of $D_{n,m}$, for every $n, m$). Then following Nelson in Theorem 3.3 of [8], $O_\varepsilon(\omega)$ is nowhere dense if and only if, for every $m \in I$, $O_\varepsilon(\omega) \subset$ some $D^{\perp}_{n,m}$. Thus, $\omega$ has a first category set of discontinuities if and only if

$$\omega \in \bigcap_j \bigcap_m \bigcup_n \angle(\tfrac{\perp}{j}, D_{n,m}) .$$

THEOREM 3.4. *Let $T$ be a compact interval. Then the set of all $\omega$ with discontinuities of the first kind only, is in $\mathscr{D}$.*

*Proof*. If $\omega$ has only discontinuities of the first kind, then for any $\varepsilon > 0$ one can choose, for each $t \in T$, an open interval $R_t$ such that there are some fixed integers $n_+$ and $n_-$ for which $\omega(s) \in K_{\varepsilon, n_+}$ for all $s$ in $(R_t - \{t\})_+ \cap T$ and $\omega(s) \in K_{\varepsilon, n_-}$ for all $s$ in $(R_t - \{t\})_- \cap T$. (Note: $(R_t - t)_{(\pm)}$ denotes the $\binom{\text{upper}}{\text{lower}}$ of the two intervals into which $R_t - \{t\}$ splits.)

Let $S_t$ be a *rational* open interval with $t \in S_t \subset \bar{S}_t \subset R_t$, and, for given $\delta > 0$, let $U_t$ be another rational interval, of length $< \delta$, with $t \in U_t \cup S_t$. Then $\omega \in \Phi(\varepsilon, (\bar{S}_t - U_t)_+ \cap T)$, and $\omega \in \Phi(\varepsilon, (\bar{S}_t - U_t) \cap T)$. Since $T$ can be covered by finitely many of the $S_t$, we finally get the following: let $\mathscr{S}_1, \mathscr{S}_2, \cdots$ be an enumeration of the finite coverings of $T$ by rational open intervals. For any rational open interval $S$, let $\mathscr{U}_k(S)$ be the set of all open rational subintervals of $S$ having length $< 1/k$. Then if $\omega$ has only discontinuities of the first kind, we have $\omega \in \bigcap_n \bigcup_m \bigcap_k \bigcap_{S \in \mathscr{S}_m} \bigcup_{U \in \mathscr{U}_k(S)} \{\Phi(1/m, (\bar{S} - U)_+ \cap T) \cap \Phi(1/n, (\bar{S} - U)_- \cap T)\}$. And conversely, if $\omega$ has a discontinuity of the second kind at $t_0$, then there is some integer $n$ such that no matter what open rational interval $S$ one chooses about $t_0$, $\omega$ will oscillate by more that $1/n$ either in $(\bar{S} - U)_+ \cap T$ or $(\bar{S} - U)_- \cap T$, provided $U$ is a sufficiently short interval. Thus, the inclusion is an equality.

THEOREM 3.5. *The set $\Theta$ of pairs $(\omega, t)$ in $\Omega \times T$ such that $\omega$ is discontinuous at $t$, is in $\mathscr{D} \times \mathscr{P}_T$($\mathscr{P}_T$ being the Borel sets in $T$). The function $(\omega, t) \to \omega(t)$ $\mathscr{C} \times \mathscr{B}_T \mid \Theta^{\perp}$-measurable, and a fortiori $\mathscr{D} \times \mathscr{B}_T$-measurable.*

(*Note: for a $\sigma$-algebra $\alpha$ on a set $Z$, and a set $Z_0 \subset Z$, we denote by $\mathscr{A} \mid Z_0$ the $\sigma$-algebra $\{A \cap Z_0 \mid A \in \mathscr{A}\}$. In case $Z_0 \in \mathscr{A}$, we get*

$$\mathscr{A} \mid Z_0 = \{A \in \mathscr{A} \mid A \subset Z_0\} .)$$

*Proof of Theorem* 3.5. $\mathscr{U}$ is again a countable basis for the open sets of $T$. Then we have $\Theta^{\perp} = \bigcap_n \bigcup_{U \in \mathscr{U}} \bigcup_m [\Delta(U, K_{1/n,m}) \times U]$. As for measurability of the function $(\omega, t) \rightarrow \omega(t)$: let $T_0$ be a countable dense subset of $T$. Let $\mathscr{V}_k$ be a finite covering of $T$ by sets of diameter $< 1/k$. Let $\{g_{k,V} \mid V \in \mathscr{V}_k\}$ be a partititon of unity for $\mathscr{V}_k$. Let $f$ be a continuous function on $X$. Let $\tilde{f}_k(\omega, t) = \sum_{V \in \mathscr{U}_k} g_{k,V}(t) \sup_{s \in T_0 \cap V} f(\omega(s))$. Then $\tilde{f}_k$ is $\mathscr{C} \times \mathscr{B}_T$-measurable, and, for fixed $\omega$, $f_k(t, \omega)$ is continuous in $t$. Furthermore, at all points $(\omega, t)$ in $\Theta^{\perp}$, we have $\tilde{f}_k(\omega, t) \rightarrow f(\omega(t))$. Thus, $f(\omega(t))$ is $\mathscr{C} \times \mathscr{B}_T \mid \Theta^{\perp}$-measurable for each continuous $f$. Now: for any closed set $K$ in $X$ there is a continuous function $f_K$ which is 1 only on that set. Then $\{(\omega, t) \mid \omega(t) \in K\} \cap \Theta^{\perp} = \{(\omega, t) \mid f_K(\omega(t)) = 1\} \cap \Theta^{\perp}$, which is in $\mathscr{C} \times \mathscr{B}_T \mid \Theta^{\perp}$. This completes the proof.

The generalization of Theorem 4.1 of [8] now goes through exactly as done there, by applying Fubini's theorem. Namely, if $\nu$ is a regular measure on $T$, then $\{\omega \mid \omega$ continuous at $t\}$ has $Q$-measure 1 for $\nu$-almost every $t \Longleftrightarrow \{t \mid \omega$ continuous at $t\}$ has $\nu$-measure 1 for $Q$-almost every $t \Longleftrightarrow \Theta$ has $Q \times \nu$-measure 0. Similarly, Theorem 4.2 of [8] generalizes to the present context: if $\{\omega \mid \omega$ continuous at $t\}$ has $Q$-measure 0 for *each* $t \in T$, then $\{\omega \mid$ the discontinuities of $\omega$ form a cat $I$ set in $T\}$ has $Q$-measure 1. The proof is gotten in the same way, but substituting $\tilde{f}$ of Theorem 3.5 above for Nelson's $f^{+}$. The details will be omitted.

## References

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